$\leqslant |\Delta(t_0)| \exp \int_{t_0}^{t} \psi(s)ds$, où $t_0 > t^*$, $t \in R$, $t_0 > t$. De l'inégalité ci-dessus il résulte $\Delta(t) = 0$, c'est à dire la solution $x(t)$ est ω-périodique.

Remarque 4. Si nous remplaçons la condition 3° par l'hypothèse que

$\int_t^\theta \psi(s)ds \geqslant \alpha(\theta - t) + k$, $\alpha > 0$, $k \leqslant 0$, alors l'affirmation du théorème 3 est évidemment vraie. En remplaçant 3° par $\lim_{\theta \to -\infty} \inf \left[\exp \int_\theta^t \psi(s)ds\right] = 0$, alors nous pouvons obtenir un résultat concernant la solution bornée dans le passé.

Enfin, nous allons donner un exemple dans lequel nous ne pouvons pas utiliser ni le théorème de Z. Opial, ni le théorème de N. Gheorghiu, mais on peut utiliser le théorème 1. Soit l'équation $\dot{x} = x\,\psi(t) + h(t)$, où $\psi(t)$ et $h(t)$ sont presque-périodique, et $\psi(t)$ est une fonction à valeurs positives et negatives, telle que l'équation (5) a une solution positive et bornée (par exemple $\psi(t) = 1 + 2\cos t$). On voit que $f(t, x) = x\psi(t) + h(t)$ n'est pas monotone au sens large par rapport à x, et que $f_x(t, x) = \psi(t)$ ne peut pas satisfaire au théorème de N. Gheorghiu.

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ON LORENTZ INVARIANT SOLUTIONS OF TWO-VARIABLES QUASI-LINEAR WAVE EQUATION

BY

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§1. Introduction. It is well known that the wave operator

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}$$

is invariant at the Lorentz transformation group

$$x' = \alpha(x - vt), \quad t' = \alpha\left(t - \frac{v}{c^2}x\right), \quad \alpha = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \tag{1}$$

$$v = \text{const},$$

v being the speed of the moving system.

Our aim is to investigate sufficient conditions under which there exist solutions of the equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x, t, u), \tag{2}$$

remaining invariant at the Lorentz group (1). To come to such conditions we transform first equation (2) in an integral equation, more adequate to our purposes.

§2. Transformation of the problem. The main result of this paragraph is stated in the following:

Theorem 1. *If D is a domain in R^2 and f is a function satisfying conditions:*

a) $f \in C(D \times R)$, $\qquad f : D \times R \to R$,

b) $f(x, t, u) = f\left(\alpha(x - vt), \ \alpha\left(t - \frac{v}{c^2}x\right), u\right)$,

a Lorentz invariant solution of (2) satisfies the integral equation

$$u(x, t) = u_0 + \int_0^{x+ct} \int_0^{x-ct} f(\xi, \tau, u(\xi, \tau))\, d\xi\, d\tau, \tag{3}$$

u_0 *being an arbitrary constant.*

Proof. Suppose it exists a solution of (2) satisfying

$$u(x,t) = v\left(\alpha(x - vt),\quad \alpha\left(t - \frac{v}{c^2}x\right)\right). \tag{4}$$

Performing the substitution:

$$y_1 = x - ct, \quad y_2 = x + ct, \quad (x,t) \in D, \tag{5}$$

and denoting

$$\begin{aligned} u(x,t) &= u\left(\frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{2c}\right) = v(y_1, y_2), \\ f(x,t,u) &= f\left(\frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{2c}, u\right) = -g(y_1, y_2, v), \end{aligned} \tag{6}$$

equation (2) becomes:

$$4\frac{\partial^2 v}{\partial y_1 \partial y_2} = g(y_1, y_2, v), \tag{7}$$

and this leads to

$$\begin{aligned} v(y_1, y_2) &= F(y_1) + G(y_2) - F(0) + \frac{1}{4}\int_0^{y_1}\int_0^{y_2} g(\eta_1, \eta_2, v(\eta_1, \eta_2))\, d\eta_1\, d\eta_2 \\ F(0) &= G(0). \end{aligned} \tag{8}$$

F and G being two arbitrary functions $R \to R$, belonging to C^1.

On the other hand, the Lorentz group, acting on (x, t) becomes, in the variables y_1, y_2:

$$y_1' = \beta y_1, \quad y_2' = \frac{1}{\beta} y_2, \quad \beta = \left(\frac{c + v}{c - v}\right)^{1/2}. \tag{9}$$

The invariantness of u and f with respect to (1), becomes invariantness of v and g with respect to (9), i.e.

$$\begin{aligned} v(y_1, y_2) &= v\left(\beta y_1, \frac{1}{\beta} y_2\right), \\ g(y_1, y_2, v) &= g\left(\beta y_1, \frac{1}{\beta} y_2, v\right), \end{aligned} \tag{10}$$

which leads to

$$v(y_1, y_2) - v\left(\beta y_1, \frac{1}{\beta} y_2\right) = F(y_1) - F(\beta y_1) + G(y_2) -$$

$$- G\left(\frac{1}{\beta} y_2\right) + \frac{1}{4}\int_0^{y_1}\int_0^{y_2} g(\eta_1, \eta_2, v(\eta_1, \eta_2))\, d\eta_1\, d\eta_2 -$$

$$- \frac{1}{4}\int_0^{\beta y_1}\int_0^{\frac{1}{\beta} y_2} g(\eta_1, \eta_2, v(\eta_1, \eta_2))\, d\eta_1\, d\eta_2 .$$

If now in the second integral one performs the substitution $\eta_1 = \beta\eta_1'$, $\eta_2 = 1/\beta\eta_2'$, and takes into account the invariantness conditions (10), the two integrals become equal, and the above relation becomes

$$F(y_1) + G(y_2) = F(\beta y_1) + G\left(\frac{1}{\beta} y_2\right).$$

Putting

$$F(y_1) = \log F_1(y_1), \qquad G(y_2) = \log G_1(y_2),$$

it results that the function $\Phi(y_1, y_2) = F_1(y_1) G_1(y_2)$ is invariant at transformations which leave invariant the product $y_1 y_2$. It follows

$$F_1(y_1) G_1(y_2) = \Phi(y_1 y_2), \tag{11}$$

and

$$\begin{aligned} F_1(y_1) &= \Phi(y_1)/G_1(1), \quad G_1(y_2) = \Phi(y_2)/F_1(1), \\ F_1(1) G_1(1) &= \Phi(1). \end{aligned} \tag{12}$$

Denote

$$F_1(y_1)/F_1(1) = F_2(y_1) \qquad G_1(y_2)/G_1(1) = G_2(y_2),$$
$$\Phi(y_1 y_2)/\Phi(1) = \Psi(y_1 y_2);$$

it follows than from (12): $F_2(y_1) = \Psi(y_1)$, $G_2(y_2) = \Psi(y_2)$, and (11) gives $\Psi(y_1)\Psi(y_2) = \Psi(y_1 y_2)$.

If we denote

$$y_1' = \log y_1, \quad y_2' = \log y_2, \quad \Psi_1(y_1') = \log \Psi(e^{y_1'}),$$

it follows

$$\Psi_1(y_1') + \Psi(y_2') = \Psi(y_1' + y_2').$$

It results that Ψ_1 is a linear function:

$$\Psi_1(y_1') = a y_1' = a \log y_1,$$

a being a constant. Comming back to F and G, we arrive to:

$$F(y_1) = \log b + a \log y, \quad G(y_2) = \log c + a \log y_2 \qquad b, c = \text{const}.$$

But, as F and G must be regular functions for $y_1 = 0$, $y_2 = 0$, it follows $a = 0$, and as $F(0) = G(0)$, it results $\log b = \log c = u_0$ — a new constant and so we obtain for v the integral equation

$$v(y_1, y_2) = u_0 + \int_0^{v_1}\int_0^{v_2} g(\eta_1, \eta_2, v(\eta_1, \eta_2))\, d\eta_1\, d\eta_2 .$$

If we come back now to the function u, we arrive to the integral equation (3).

§3. Existence and uniqueness of a solution of equation (3).

The answer to the existence and uniqueness problem, is given by

Theorem 2. *If the function f satisfies the conditions*

a) *f is continuous on $D \times R$,*

b) *f satisfies the boundedness condition*

$$|f(x, t, u)\, e^{2\sigma x}| \leqslant A = \text{const.} \qquad \sigma = \text{const.}$$

c) *f satisfies the Lipschitz condition*

$$|f(x, t, u) - f(x, t, u')| \leqslant L\, |u - u'|,$$

d) *the constants L, σ satisfiy $L/4\sigma^2 < 1$,*

the equation (3) *has a solution $u(x, t)$ satisfying*

$$|u(x, t) - u_0|\, e^{2\sigma x} \leqslant \frac{1}{16\sigma^2} A.$$

Proof. Performing the substitution (5), we obtain equation (13); obviously, if this equation will admit a unique solution, then, the same will be true also for (5).

The boundedness and the Lipschitz condition for f, become

$$|g(y_1, y_2, v)\, e^{\sigma(v_1+v_2)}| \leqslant A, \tag{14}$$

$$|g(y_1, y_2, v) - g(y_1, y_2, v')| \leqslant L\, |v_1 - v_2| . \tag{15}$$

We will consider now the operator

$$\tilde{v} = \mathcal{T} v = u_0 + \frac{1}{4}\int_0^{v_1}\int_0^{v_2} g(\eta_1, \eta_2, v(\eta_1, \eta_2))\, d\eta_1\, d\eta_2 , \tag{16}$$

and prove the existence and uniqueness of a fixed point of the equation

$$v = \mathcal{T} v \qquad (\text{with fixed } u_0)$$

in the space of functions v, satisfying

$$|v(y_1, y_2) - u_0|\, e^{\sigma(v_1 v_2)} \leqslant \frac{A}{16\,\sigma^2} = \text{const}, \tag{17}$$

metrized by

$$\rho(v, v') = \sup|v(\eta_1, \eta_2) - v'(\eta_1, \eta_2)|\, e^{\sigma(v_1+v_2)} .$$

Indeed, we have for each v satisfying (17):

$$|\tilde{v} - u_0| \leqslant \frac{1}{4}\int_0^{v_1}\int_0^{v_2} g(\eta_1, \eta_2\, v(\eta_1, \eta_2))\, e^{\sigma(\eta_1+\eta_2)}\, e^{-\sigma(\eta_1+\eta_2)}\, d\eta_1\, d\eta_2 \leqslant$$

$$\leqslant \frac{1}{16\,\sigma^2} A\, e^{-\sigma(v_1+v_2)} ,$$

and finally:

$$|\tilde{v} - u_0|\, e^{\sigma(v_1+v_2)} \leqslant \frac{1}{16\sigma^2} A,$$

i.e. the operator $\mathcal{T}$ transforms our space into itself.

The operator $\mathcal{T}$ is also a contraction. Indeed we have

$$|\tilde{v}(y_1, y_2) - \tilde{v}'(y_1, y_2)| \leqslant \int_0^{v_1}\int_0^{v_2} L\, |v - v'|\, e^{\sigma(\eta_1+\eta_2)}\, e^{-\sigma(\eta_1+\eta_2)}\, d\eta_1\, d\eta_2 \leqslant$$

$$\leqslant \frac{L}{4\sigma^2}\, \rho(v, v')\, e^{-\sigma(v_1+v_2)} ,$$

and also

$$\rho(\tilde{v}, \tilde{v}') \leqslant \frac{L}{4\sigma^2}\, \rho(v, v') ,$$

and as $L/4\sigma^2 < 1$, the conclusion follows.

The next theorem proves that in the conditions of theorem 2, the hypothesis b) of theorem 1, is sufficient for the existence of a Lorentz invariant solution.

Theorem 3. *In the hypotheses of theorem 2, the condition b) of theorem 1 is sufficient for the existence of a Lorentz invariant solution of (2).*

Proof. As we have seen, (2) and our invariantness condition lead to (3) and after substitution (5), to (13), the solution of which must be invariant at transformations (9). From (13) we obtain:

$$v\left(\beta y_1, \frac{1}{\beta} y_2\right) = u_0 + \int_0^{\beta v_1}\int_0^{\frac{1}{\beta} v_2} g(\eta_1, \eta_2, v(\eta_1, \eta_2))\, d\eta_1\, d\eta_2 ,$$

and after the substitution

$$\beta\eta_1' = \eta_1 , \quad \frac{1}{\beta}\eta_2' = \eta_2 ,$$

taking into account condition b) of theorem 1 :

$$v\left(\beta y_1, \frac{1}{\beta} y_2\right) = u_0 + \int_0^{y_1}\int_0^{y_2} g\left(\eta_1, \eta_2, v\left(\beta\eta_1, \frac{1}{\beta}\eta_2\right)\right) d\eta_1\, d\eta_2 .$$

But equation (13) has a unique solution (by given u_0) so that

$$v(y_1, y_2) = v\left(\beta y_1, \frac{1}{\beta} y_2\right)$$

i.e.

$$u\left(\alpha(x - vt),\quad \alpha\left(t - \frac{v}{c^2}x\right)\right) = u(x, y).$$

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POSITIVE SOLUTIONS OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

BY

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Introduction. The existence of positive solutions of Cauchy Problem for a system of ordinary differential equations has a practical value. Such a problem occurs when studying some processes in physics, technique, biology a.s.o. Actually, we shall treat a problem more general than the problem of the existence of positive solutions. Namely, we give necessary and sufficient conditions such that each component $u_i(t)$ of the solution $u(t) = (u_1(t), \ldots, u_m(t))$, belongs to a given interval $[a_i, b_i]$, $i = 1, 2, \ldots, m$ (i.e. $a_i \leqslant u_i(t) \leqslant b_i$, $i = 1\ldots, m$, for all t in the domain of u). Here a_i and b_i are real numbers, with $a_i < b_i$.

We prove that the above problems can be derived from a result of C r a n d a l l [3] on the existence of solution on a locally closed subset F of m - dimensional euclidian space R^m.

Let us consider the Cauchy problem:

$$u'(t) = f(t, u(t)), \qquad u(a) = x,\ x \in F, \tag{1}$$

where $f: [a, b) \times F \to R^m$ is a continuous function, $a < b \leqslant +\infty$.

We say that F is forward invariant for (1) (or flow invariant set) if whenever u is a solution of (1) on $[a, a + T]$, $T > 0$ (in classical sense) and $x \in F$, then $u(t) \in F$ for $a \leqslant t \leqslant a + T$. Note that F-locally closed, means that for each $x \in F$, there is $r > 0$, such that $F_r = F \cap B_r(x)$ is closed in R^m, where $B_r(x)$ denotes the closed ball of radius r and center $x \in R^m$. Denote by $d(y; F)$ the distance from the point $y \in R^m$ to F.

The basic result on the Cauchy problem (1) (generalizing Peano's existence theorem) is the following :

Theorem 1. (Crandall [3]). *Let $f : [a, b) \times F \to R^m$ be a continuous function. A necessary and sufficient condition for the existence of a local solution $u : [a, a + T(x)] \to F$ of the problem (1) is the following :*

$$\lim_{h \downarrow 0} \frac{1}{h} d(x + hf(t, x); F) = 0, \textit{ for all } t \in [a, b) \text{ and } x \in F, \tag{2}$$

where $T = T(x) > 0$, depends on $x \in F$.

This theorem is given by C r a n d a l l [3] and is generalized by M a r t i n [5], P a v e l [6], and U r s e s c u [7].