et qui satisfait aux conditions du théorème d'existence et d'unicité de la solution ; on observe qu'une solution de ce système est

(6) $(g_{ij}\,\xi^i_a\,\xi^j_a)(s) = (g_{ij}\,n^i\,n^j)(s) = 1,\ (g_{ij}\,\xi^i_a\,\xi^j_b)(s) = (g_{ij}\,\xi^i_a\,n^j)(s) = 0\ (a \neq b)$

qui satisfait aussi à la condition (4). De même, la solution unique $x^i(s)$, $\xi^i_a(s)$, $n^i(s)$ du système (3) satisfait aux conditions (4). Avec cette solution unique du système (3), on forme les fonctions (5) qui, évidemment satisfont à (6). Maintenant, en substituant cette solution dans la dernière équation (3) nous obtenons par quadratures, les équations de la courbe (C).

Le paramètre s est naturel, comme il résulte des egalités

$$\int_{s_0}^{s_1} \sqrt{\dot{x}^2(s)}\,ds = \int_{s_0}^{s_1} \sqrt{g_{ij}t^i t^j}\,ds = \int_{s^0}^{s_1} ds = s_1 - s_0$$

Donc, dans les conditions 1) et 2) du théorème nous avons déterminé une configuration $\mathfrak{M}(C;\ \xi^i_1,\ \xi^i_2,\ T^{n-1})$ unique, pour laquelle les vecteurs orthogonaux $\xi^i_1, \xi^i_2, \ldots, \xi^i_{n-1}$ déterminent pour tous $s \in (s_0 - a,\ s_0 + a)$ l'espace T^{n-1}; le vecteur unitaire n^i est normal à cet espace ; $(\xi^i_1, \xi^i_2, \ldots, \xi^i_{n-1}, n^i)$ forment un repère positivement orienté, et les fonctions G, T et α sont invariantes.

Ainsi, le théorème est demontré et montre que les invariants G, T, α forment un système complet.

Dans la première situation, quand la courbe (C) est donnée, le système (3) devient linéaire.

Cas particulier, $n = 3$. Les équations (1) et (2) prennent la forme

$$\frac{\nabla \xi^i_1}{ds} = k_g\,\xi^i_2 + k_n\,n^i,$$

$$(6)\qquad \frac{\nabla \xi^i_2}{ds} = -\,k_g\,\xi^i_1 + T_g\,n^i,$$

$$\frac{\nabla h^i}{ds} = -\,k_n\,\xi^i_1 - T_g\,\xi^i_2\ ,$$

et respectivement

$$\frac{dx^i}{ds} = \alpha^1\,\xi^i_1 + \alpha^2\,\xi^i_2 + \alpha n^i$$

avec $(\alpha^1)^2 + (\alpha^2)^2 + \alpha^2 = 1$. Les équations (1') sont de type Darboux-Ribaucour pour une courbe située sur une surface.

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Reçu le 1.XII.1977

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Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXV, s. I a, f. 1

SOME PROPERTIES OF THE ALMOST PERIODIC FUNCTIONS IN PROBABILITY

BY

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In the present note we shall study some properties of almost periodic functions in probability, suggested in a paper by O. Onicescu and V. Istrățescu [7].

Generally, we shall use the terminology and the notations of [7]. The reader is also refered to [6], chap. XVII, for some general results concerning the convergence, continuity, differentiability and integrability in probability.

Let $(\Omega,\ \mathbf{A},\ P)$ be a probability space and let R be the real line.

By a *random function* on R we mean any function defined on R, with values in the space of all random variables defined on $(\Omega, \mathbf{A}, P)$.

We shall denote by $C_P(R)$ the set of all continuous in probability random functions on R.

Definition 1. *A function $f \in C_P(R)$ is called almost periodic in probability if to any $\varepsilon > 0$, $\eta > 0$, there corresponds a number $l = l(\varepsilon, \eta) > 0$ such that in any interval of length l, there exists τ such that*

$$P(\{\omega\,;\ |f(t+\tau, \omega) - f(t, \omega)| \geqslant \varepsilon) < \eta,\ \textit{for all } t \in R.$$

The number l is called the *inclusion length* and τ is called an *$\varepsilon - \eta$ — period* of f.

Denote by $AP_P(R)$ the set of all almost periodic functions in probability on R.

Theorem 2. *If f is almost periodic in probability, then f is uniformly continuous in probability on R.*

Proof. If $\varepsilon > 0$ and $\eta > 0$, let $l = l(\varepsilon/3,\ \eta/3)$ be the inclusion length corresponding to $\varepsilon/3$, $\eta/3$, according to definition 1.

Since f is uniformly continuous in probability on the interval $[-1, -1 + l]$ ([6], chap. XVII) there exists $\delta = \eta(\varepsilon/3,\ \eta/3)$, $0 < \delta < 1$, such that $|t' - t''| < \delta$ and $t', t'' \in [-1, -1 + l]$ imply

$$(1)\qquad P(\{\omega\,;\ |f(t', \omega) - f(t'', \omega)| \geqslant \varepsilon/3) < \eta/3.$$

Let t_1, t_2 be real numbers so that $|t_1 - t_2| < \delta$ and let $t \in R$.

Since f is almost periodic in probability, there exists $\tau \in [-t_1, -t_1 + l]$ such that

$$(2)\qquad P(\{\omega\,;\ |f(t+\tau, \omega) - f(t, \omega)| \geqslant \varepsilon/3\}) < \eta/3.$$

Then $t_1 + \tau \in [0, l]$ and hence $t_2 + \tau \in [-1, l+1]$.

But

$$\{\omega\,;\ |f(t_2,\omega) - f(t_1,\omega)| \geq \varepsilon\} \leq \{\omega\,;\ |f(t_2,\omega) - f(t_2+\tau,\omega)| \geq \varepsilon/3\} \cup \{\omega\,;\ |f(t_2+\tau,\omega) - f(t_1+\tau,\omega)| \geq \varepsilon/3\} \cup \{\omega\,;\ |f(t_1+\tau,\omega) - f(t_1,\omega)| \geq \varepsilon/3\},$$

and then, according to (1) and (2), we get

$$P(\{\omega\,;\ |f(t_2,\omega) - f(t_1,\omega)| \geq \varepsilon\}) < \eta, \text{ for all } t_1, t_2 \in R$$

such that $|t_1 - t_2| < \delta$, which says that f is uniformly continuous in probability on R.

Theorem 3. *A function $f \in C_P(R)$ is almost periodic in probability if and only if to every $\varepsilon > 0$, $\eta > 0$, it corresponds a finite set $A = \{a_i\}_{i=1}^n$ of real numbers such that for every $a \in R$, there exists an $a_j \in A$ for which*

(3) $$P(\{\omega\,;\ |f(t+a,\omega) - f(t+a_j,\omega)| \geq \varepsilon\}) < \eta, \text{ for all } t \in R.$$

Proof. Necessity. Assume $\varepsilon > 0$, $\eta > 0$ and let $l = l(\varepsilon/2, \eta/2)$ be the inclusion length corresponding to $\varepsilon/2$, $\eta/2$, according to the almost periodicity in probability of f.

Since, by virtue of the theorem 2, f is uniformly continuous in probability, it follows that in the interval $[0, l]$, it can be chosen n points $\{a_i\}_{i=1}^n$ such that for every $x \in [0, l]$, there exists $a_j \in [0, l]$ with the property

(4) $$P(\{\omega\,;\ |f(t+x,\omega) - f(t+a_j,\omega)| \geq \varepsilon/2\}) < \eta/2, \text{ for all } t \in R.$$

Now assume that $a \in R$ and let τ be an $\varepsilon/2 - \eta/2$-period from the interval $[-a, -a+l]$; hence we have

(5) $$P(\{\omega\,;\ |f(t+a+\tau,\omega) - f(t+a,\omega)| \geq \varepsilon/2\}) < \eta/2, \text{ for all } t \in R.$$

Because $a + \tau \in [0, l]$, from (4) it follows that there exists $a_j \in [0, l]$ such that

(6) $$P(\{\omega\,;\ |f(t+a+\tau,\omega) - f(t+a_j,\omega)| \geq \varepsilon/2\}) < \eta/2, \text{ for all } t \in R.$$

Since

$$\{\omega\,;\ |f(t+a,\omega) - f(t+a_j,\omega)| \geq \varepsilon\} \subseteq \{\omega\,;\ |f(t+a,\omega) - f(t+a+\tau,\omega)| \geq \varepsilon/2\} \cup \{\omega\,;\ |f(t+a+\tau,\omega) - f(t+a_j,\omega)| \geq \varepsilon/2\},$$

for all $t \in R$,

from (5) and (6), we get

$$P(\{\omega\,;\ |f(t+a,\omega) - f(t+a_j,\omega)| \geq \varepsilon\}) < \eta, \text{ for all } t \in R,$$ which completes the necessity proof.

Sufficiency. If $\varepsilon > 0$, $\eta > 0$, let $A = \{a_i\}_{i=1}^n$ be such that for every $a \in R$ there exists an $a_j \in A$ for which (3) holds.

But it is easy to see that (3) is equivalent to

(7) $$P(\{\omega\,;\ |f(t,\omega) - f(t+a-a_j,\omega)| \geq \varepsilon\}) < \eta, \text{ for all } t \in R.$$

Let $l = \max\limits_{1 \leq i \leq n} |a_i|$. We observe that $a - a_j \in [a-l, a+l]$, $a - a_j$ is an $\varepsilon - \eta$-period for f and $L = 2l$ is an inclusion lenght for f.

Consequently the relation (7) says that f is almost periodic in probability and thus the proof of theorem 3 is complete.

Remark 4. In the probability theory it is wellknown that the convergence in probability is equivalent to the convergence in the semi-metric defined by

(8) $$d(\alpha, \beta) = \int_\Omega \frac{|\alpha(\omega) - \beta(\omega)|}{1 + |\alpha(\omega) - \beta(\omega)|}\, dP(\omega),$$

where α, β are random variables.

Using this remark, we can define the following semi-metric in the space of the random functions on R:

(8′) $$\tilde{d}(f, g) = \sup_{t \in R} d(f(t,\omega), g(t,\omega)),$$

for every two random function f, g on R.

It is easy to see that the uniform convergence in probability of a sequence of random functions on R is equivalent to the convergence in the semi-metric (8′).

If we denote by L the linear space of all random functions which are uniformly continuous in probability on R, and by $\tilde{L}$ the separated space associated with L, $\tilde{L}$ being endowed with the metric

(8″) $$\rho(\hat{f}, \hat{g}) = \tilde{d}(f, g)$$

for every $\hat{f}, \hat{g} \in \tilde{L}$, we can observe that $\tilde{L}$ is a Fréchet space [3].

Using this remark ($\tilde{L}$ is a complete metric space), the theorem 3 can be stated under the following equivalent form:

Theorem 5. *The function $f \in C_P(R)$ is almost periodic in probability if and only if $\{a_n\}$, being an arbitrary real sequence, there exists a subsequence $\{b_n\}$ such that the sequence $\{f(t+b_n)\}$ converges uniformly in probability on R (following Amerio [1], chap. I we shall say that $\{b_n\}$ is a regular sequence in probability).*

Using the theorem 5, one can easily prove the following result.

Theorem 6. *$AP_P(R)$ is an algebra over R, which is closed under uniform limits in probability.*

Now we shall establish a criterion for almost periodicity of the derivative of an almost periodic function in probability.

Theorem 7. *If $f \in AP_P(R)$, f is differentiable in probability on R and its derivative in probability f' is uniformly continuous in probability on R, then $f' \in AP_P(R)$.*

Proof. Because f' uniformly continuous in probability, for every $\varepsilon > 0$, $\eta > 0$ there exists $\delta = \delta(\varepsilon, \eta) > 0$ such that for all t_1, t_2 with $|t_1 - t_2| < \delta$, we have

$$(9) \qquad P(\{\omega\,;\ |f'(t_1,\omega) - f'(t_2,\omega)| \geq \varepsilon\}) < \eta.$$

Denote by $\varphi_n(t,\omega) = n[f(t+1/n,\omega) - f(t,\omega)]$, for all $t \in R$, $\omega \in \Omega$. We observe that the random functions φ_n are integrable in probability on the interval $[0, 1/n]$, where $n \geq \delta^{-1}$.

Moreover, according to [6], chap. XVII, we have

$$\{\omega\,;\ |\varphi_n(t,\omega) - f'(t,\omega)| \geq \varepsilon\} = \{\omega\,;\ |n.(P)\int_0^{1/n} [f'(t+y) - f'(t)]\,dy| \geq$$

$$\geq \varepsilon\} \subseteq \{\omega\,;\ n.(P)\int_0^{1/n} |f'(t+y) - f'(t)|\,dy \geq \varepsilon\},$$

where $(P)\int_0^{n/1} g(y)\,dy$ means the integral in probability of g on the interval $[0, 1/n]$.

From the above inclusion, taking into account the relation (9), we get $P(\{\omega\,;\ |\varphi_n(t,\omega) - f'(t,\omega)| \geq \varepsilon\}) < \eta$, for $n \geq \delta^{-1}$ and for all $t \in R$, i.e. the sequence $\{\varphi_n\}$ converges uniformly in probability on R to f'. Since, obviously $\varphi_n \in AP_P(R)$ for every $n \in N$ and, by virtue of the theorem 6, $AP_P(R)$ is closed under uniform limits in probability, it follows that $f' \in AP_P(R)$.

Definition 8. *Let* F *be a family of random functions on* R. I) F *is said to be equicontinuous in probability at* $t_0 \in R$, *if to each* $\varepsilon > 0$, $\eta > 0$, *it corresponds* $\delta = \delta(\varepsilon, \eta) > 0$ *such that*

$$P(\{\omega\,;\ |f(t,\omega) - f(t_0,\omega)| \geq \varepsilon\}) < \eta,$$

whenever $|t - t_0| < \delta$ *and for all* $f \in$ F.

II) F *is said to be equicontinuous in probability on* R *if it is equicontinuous in probability at every* $t \in R$.

Using the remark 4 we can easily prove (see [4], [5]) the following theorem which contains a compactness criterion in probability for families of almost periodic functions in probability.

Theorem 9. *A set* E *of almost periodic functions in probability on* R. *is relatively compact in the sense of the metric* (8″), *if and only if*

I) E *is equicontinuous in probability on* R;

II) *for every* $\varepsilon > 0$, $\eta > 0$ *there corresponds an inclusion length* l *such that in any interval of length* l *there exists an* $\varepsilon - \eta$ *— period* τ *common to all functions of the set* E (E *is called uniformly almost periodic in probability*);

III) *the family of random variables* $E_t = \{f(t,.)\,;\ f \in E\}$ *is relatively compact in the sense of semi-metric* (8) *(relatively compact in probability), for every* $t \in R$.

Remark. Since the almost sure convergence implies the convergence in probability and every sequence of random variables converging in probability contains a subsequence converging almost sure it follows that the compacity in probability is equivalent to the compacity in the topology of the almost sure convergence. Consequently the condition III may be replaced by condition:

III′) E_t *is relatively compact in the topology of almost sure convergence, for every* $t \in R$.

Taking into account the relationship between the p^{th}-mean convergence ($p \geq 1$) and the convergence in probability (see [8], chap. III) it follows that every almost periodic function in the mean of order $p (p \geq 1)$ (see [2], Chap. VI, §4) is also almost periodic in probability. Because there exist sequences of random variables converging in probability, but not converging in mean of order $p (p \geq 1)$, by virtue of the theorem 5 and of the well-known results concerning almost periodic functions in the mean, of order $p (p \geq 1)$, we can conclude that the almost periodicity in probability is more general than the almost periodicity in mean of order $p (p \geq 1)$.

On the other hand, since the Stepanov almost periodic functions are almost periodic functions in the mean of order p (see [2], Chap. VI, §4), we observe that almost periodic functions in the Stepanov sense are always periodic in probability.

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Received 22.IV.1978

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