Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXV, s. I a, 1979, f. 1

FIXED POINTS FOR A SUM OF PROJECTIVE CONTRACTIONS

BY

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1. The aim of this paper is to give a partial improvement of a fixed point result established in [2], as well as some applications to Volterra functional equations. Throughout this note, the following general notations will be used :

i) $P(X)=\{Y\subset X\,;\,Y\neq\varnothing\}$, $F(X,Y)=\{f\,;\,f:X\to Y\}$

ii) $N=\{0,1,\ldots\}$, $N'=\{1,2,\ldots\}$, $X^n=X\times X\times\ldots\times X$, ($n$ times), $\forall n\in N'$

iii) $C(X,Y)=\{f\in F(X,Y)\,;\,f$ is continuous$\}$

iv) $R=]-\infty,+\infty[$, $R_+=[0,+\infty[$, $R_+^0=]0,+\infty[$, $\bar{R}_+=[0,+\infty]$

v) $(R^n,\|\cdot\|)=$ the euclidean n-dimensional space with the euclidean norm $\|\cdot\|$, $\forall n\in N'$.

2. Let $m\in N'$ be a fixed element. The order relation induced in R^m by R_+^m (respectively $(R_+^0)^m$ will be indicated by $\leqslant$ (respectively $<$).

Let us denote :

i) $\min(x)=\min(x_1,\ldots,x_m)$, $\max(x)=\max(x_1,\ldots,x_m)$, $\forall x=(x_1,\ldots,x_m)\in R^m$,

ii) $x/y=(x_1/y_1,\ldots,x_m/y_m)$, $\forall x=(x_1,\ldots,x_m)\in R^m$, $\forall y=(y_1,\ldots,y_m)\in(R_+^0)^m$,

iii) $[a,b]=\{x\in R^m\,;\,a\leqslant x\leqslant b\}$, $\forall a,b\in R^m$, $a<b$.

Hibert's projective metric of R^m is the mapping $d\in F(((R_+^0)^m)^2,R_+)$, given by

iv) $d(x,y)=\ln[(\max(x/y))/\min(x/y))]$, $\forall x,y\in(R_+^0)^m$.

The following lemma will be useful in this note

Lemma 2.1. *Let $a,b\in(R_+^0)^m$, $a<b$. Then,*

(2.1) $\exp(d(x,y))-1\leqslant 2((\max(b/a))/\min(a)))\,\|x-y\|$, $\forall x,y\in[a,b]$,

Proof. Firstly, we remark that $a/b\leqslant x/y\leqslant b/a$, $\forall x,y\in[a,b]$. Now, if $x,y\in[a,b]$ are given, we obviously have $x-y\leqslant\|x-y\|\,(y/\min)(a))$, $y-x\leqslant\|x-y\|\,(y/\min(a))$, and this gives $\max(x/y)\leqslant 1+\|x-y\|/\min(a)$, $\min(x/y)\geqslant 1-\|x-y\|/\min(a)$. Therefore, $\max(x/y)-\min(x/y\leqslant 2(\|x-y\|/\min(a))$, which implies $\exp(d(x,y))-1\leqslant 2((\max(y/x))/\min(a)))\,\|x-y\|\leqslant 2((\max(b/a))/\min(a)))\,\|x-y\|$. Q.E.D.

3. Let $n \in N'$ be a fixed element. Denote:

i) $X = C(R_+, R)$, $X_+ = C(R_+, R_+)$, $Y = \{f \in X; f(R_+^0) \subset R_+^0\}$.

Clearly, $X^n = C(R_+, R^n)$, $X_+^n = C(R_+, R_+^n)$, $Y^n = \{f \in X^n; f(R_+^0) \subset (R_+^0)^n\}$. The order relation induced in X^n by X_+^n (resp. Y^n) will be denoted by $\leqslant$ (resp. $<$). Let us define $m \in F(X_+^n \times Y^n, R_+)$, $M \in F(X_+^n \times Y^n, \overline{R}_+)$ by

ii) $m(x, y) = \sup\{\mu \in R_+; \mu y \leqslant x\}$.

iii) $M(x, y) = \inf\{\lambda \in R_+; x \leqslant \lambda y\}$, if $\{\lambda \in R_+; x \leqslant \lambda y\} \neq \varnothing$.

$M(x, y) = +\infty$, if $\{\lambda \in R_+; x \leqslant \lambda y\} = \varnothing$.

for every $(x, y) \in X_+^n \times Y^n$. In this case, Hilbert's projective metric of X^n is the mapping $d \in F((Y^n)^2, \overline{R}_+)$, expressed by

iv) $d(x, y) = \ln(M(x, y)/m(x, y))$, if $0 < m(x, y) \leqslant M(x, y) < +\infty$,

$d(x, y) = +\infty$, if $m(x, y) = 0$ or $M(x, y) = +\infty$,

for every $(x, y) \in (Y^n)^2$. The fundamental properties of this metric may be found in [1], [2]. On the other hand, let us denote

v) $|x| = (|x_1|, \ldots, |x_n|)$, if $x = (x_1, \ldots, x_n) \in X^n$,

vi) $\Gamma_n = \{(x, y) \in (Y^n)^2; 0 < m(x, y) \leqslant 1 \leqslant M(x, y) < +\infty\}$,

and for every $g \in Y^n$ let us put,

vii) $\|x\|_g = M(|x|, g)$, $\forall x \in X^n$,

viii) $X_g^n = \{x \in X^n; \|x\|_g < +\infty\}$, $\widetilde{X}_g^n = \{x \in X^n; \|x\|_g = 1\}$,

ix) $Y_g^n = \{x \in Y^n; 0 < m(x, g) \leqslant M(x, g) < +\infty\}$, $\widetilde{Y}_g^n = Y_g^n \cap \widetilde{X}_g^n$,

x) $Y_g^n(\alpha) = \{x \in Y^n; m(x, g) \geqslant \alpha\}$, $\widetilde{Y}_g^n(\alpha) = Y_g^n(\alpha) \cap \widetilde{X}_g^n$, $\forall \alpha \in R_+^0$.

The following results were proved in [3]

Lemma 3.1. [3, *theorem* 2.1]. *For every fixed* $g \in Y^n$, $(X^n, \|\cdot\|_g)$ *is a generalized Banach space and* X_g^n, $\widetilde{X}_g^n$, $Y_g^n(\alpha)$, $(\forall \alpha \in R_+^0)$ *are closed subsets in* $(X^n, \|\cdot\|_g)$.

Lemma 3.2. [3, *theorem* 2.2]. *Let* $g \in Y^n$ *be fixed. Then,*

(3.1) $\|x - y\|_g \leqslant (\exp(d(x, y)) - 1)\min\{\|x\|_g, \|y\|_g\}$, $\forall (x, y) \in \Gamma_n \cap (Y_g^n)^2$,

(3.2) $M(x, y) - m(x, y) \leqslant (2/m(y, g))\|x - y\|_g$, $\forall (x, y) \in (Y_g^n)^2$,

(3.3) $\|x - y\|_g \geqslant (1/2)(1 - \exp(-d(x, y)))\max\{m(x, g), m(y, g)\}$,

$$\forall (x, y) \in (Y_g^n)^2$$

Lemma 3.3. [3, *theorem* 2.3]. *Let* $g \in Y^n$ *be given. Then,* (Y_g^n, d) *is a complete metric space.*

4. A mapping $T \in F(Y^n, Y^n)$ is said to be a) monotone increasing on Y^n iff $x, y \in Y^n$, $x \leqslant y \Rightarrow Tx \leqslant Ty$; b) p-homogeneous on Y^n (where $p \in R$) iff $T(\lambda x) = \lambda^p Tx$, $\forall \lambda \in R_+^0$, $\forall x \in Y^n$.

The main result of this note is the following

Theorem 4.1. *Let* $\{T_1, \ldots, T_m\} \subset F(Y^n, Y^n)$ *and* $\{p_1, \ldots, p_m\} \subset [0, 1[$, $0 \leqslant p_1 < p_2 \ldots < p_m < 1$, *be such that*

(4.1) T_i *is monotone increasing on* Y^n, $\forall i \in \{1, \ldots, m\}$,

(4.2) T_i *is* p_i*-homogeneous on* Y^n, $\forall i \in \{1, \ldots, m\}$,

(4.3) *the set* $Y^n(T_1, \ldots, T_m) = \{g \in Y^n; \sum_{i=1}^m T_i(g) \in Y_g^n\}$ *is not empty*,

(4.4) $1 - 2q + pq > 0$ (*where*, $p = p_1$, $q = p_m$).

Then, there exists a map

(4.5) $g \to \bar{g} = \varphi(g) \in Y_g^n$, $\forall g \in Y^n(T_1, \ldots, T_m)$,

such that

(4.6) $\bar{g} = \sum_{i=1}^m T_i(\bar{g})$, $\forall g \in Y^n(T_1, \ldots, T_m)$.

Proof. Let $g \in Y^n(T_1, \ldots, T_m)$ be a fixed element. Denote $\gamma = m\left(\sum_{i=1}^m T_i(g), g\right)$, $\delta = M\left(\sum_{i=1}^m T_i(g), g\right)$ and let us define a family

$$\{S(a);\ a \in (R_+^0)^m\} \subset F(Y^n, Y^n) \text{ by}$$

(4.7) $S(a)(x) = \sum_{i=1}^m a_i T_i(x)$, $\forall a = (a_1, \ldots, a_m) \in (R_+^0)^m$, $\forall x \in Y^n$.

From (4.1) and 4.2), it easily follows

(4.8) $d(S(a)(x), S(b)(y)) \leqslant d(a, b) + qd(x, y)$, $\forall a, b \in (R_+^0)^m$, $\forall x, y \in Y_g^n$.

Moreover, $S(a)(Y_g^n) \subseteq Y_g^n$, $\forall a \in (R_+^0)^m$. So, we can define a family $\{U(a); a \in (R_+^0)^m\} \subset F(\widetilde{Y}_g^n, \widetilde{Y}_g^n)$, by

(4.9) $U(a)(x) = S(a)(x)/\|S(a)(x)\|_g$, $\forall a \in (R_+^0)^m$, $\forall x \in \widetilde{Y}_g^n$.

From (4.8), $d(U(a)(x), U(a)(y)) \leqslant q \cdot d(x, y)$, $\forall a \in (R_+^0)^m$, $\forall x, y \in \widetilde{Y}_g^n$, i.e., $U(a) \in F(\widetilde{Y}_g^n, \widetilde{Y}_g^n)$ is a contraction mapping, $\forall a \in (R_+^0)^m$. Therefore, (from Banach's fixed point theorem) there exists a family $\{z(a); a \in (R_+^0)^m\} \subset \widetilde{Y}_g^n$, such that

(4.10) $z(a) = U(a) z(a)$, $\forall a \in (R_+^0)^m$.

Now, let us define $h \in F((R_+^0)^m, R_+^0)$, $H \in F((R_+^0)^m, (R_+^0)^m)$ by

(4.11) $h(a) = \|S(a) z(a)\|_g$, $\forall a \in (R_+^0)^m$,

(4.12) $H(a) = ((h(a))^{p_1}, \ldots, (h(a))^{p_m})$, $\forall a \in (R_+^0)^m$.

Lemma 4.1. *If* (4.1) + (4.2) *are accepted, then,* $\forall i \in \{1, \ldots, m\}$,

(4.13) $\|T_i x\|_g \leqslant \|\sum_{i=1}^m T_i x\|_g \leqslant \delta$, $\forall x \in \widetilde{Y}_g^n$, (4.14) $\| T_i x - T_i y \|_g \leqslant 2(\delta/\alpha) \| x - y \|_g , \quad \forall x, y \in \tilde{Y}^n_g(\alpha), \quad \forall \alpha \in R^0_+ .$

Proof. The first part of the lemma is obvious. For the secon part, let $\alpha \in R^0_+$, $x, y \in \tilde{Y}^n_g(\alpha)$, $i \in \{1, \ldots, m\}$. Clearly, $(x, y) \in \Gamma_n$. As $m(x, y) y \leqslant x \leqslant M(x, y) y$, we have $m(x, y) T_i(y) \leqslant (m(x, y))^{p_i} T_i(y) \leqslant T_i(x) \leqslant (M(x, y))^{p_i} T_i(y) \leqslant M(x, y) T_i(y)$. Therefore, $T_i(x) - T_i(y) \leqslant (M(x, y) - m(x, y)) T_i(y)$, $T_i(y) - T_i(x) \leqslant (M(x, y) - m(x, y)) T_i(y)$. Taking int account the following simple property

(4.15) $$x, y \in X^n, \quad x \leqslant y, \quad -x \leqslant y \Rightarrow | x | \leqslant | y | .$$

we get $| T_i(x) - T_i(y) | \leqslant (M(x, y) - m(x, y)) | T_i(y) |$. Now, from (3.2 and (4.13) we obtain $\| T_i(x) - T_i(y) \|_g \leqslant (M(x, y) - m(x, y)) \| T_i(y) \|_g \leqslant 2 (\| T_i(y) \|_g / m(y, g)) \| x - y \|_g \leqslant 2(\delta/\alpha) \| x - y \|_g$. Q.E.D.

Lemma 4.2. *Let* $b, c \in (R^0_+)^m$, $b < c$, *and denote* $\alpha = ((\gamma \min (b)) /(\delta \max (c)))$, $\beta = \alpha^{1/(1-q)}$. *Then,*

(4.16) $$U(a) (\tilde{Y}^n_g(\beta)) \subset \tilde{Y}^n_g(\beta), \quad \forall a \in [b, c].$$

Proof. If $a \in [b, c]$, $x \in \tilde{Y}^n_g(\beta)$ are given, then,

$$M(S(a)(x), g) \leqslant \Big(M\Big(\sum_{i=1}^{m} T_i (g), g\Big)\max (c) = \delta \max (c), \text{ and } m(S(a) (x), g) \geqslant$$

$$\geqslant \Big(m\Big(\sum_{i=1}^{m} T_i(x), g\Big) \min(b) \geqslant \Big(m\Big(\sum_{i=1}^{m} T_i(\beta g), g\Big) \min (b) =$$

$$= \Big(m\Big(\sum_{i=1}^{m} \beta^{p_i} T_i(g), g\Big) \min (b) \geqslant \Big(m\Big(\sum_{i=1}^{m} T_i(\beta g), g\Big) \beta^q \min (b) = \gamma \beta^q (\min (b$$

Therefore, $m(U(a)(x), g) \geqslant (\gamma/\delta) ((\min (b))/\max(c))) \beta^q = \alpha\beta^q = \beta$ Q.E.D.

Lemma 4.3. *The mappings h and H are continuous on* $(R^0_+)^m$.

Proof Let $b, c \in (R^0_+)^m$, $b < c$. If $a', a'' \in [b, c]$ are given, then, b (4.8) and (4.10), $d(z(a'), z(a'')) = d (U(a') z (a'), U(a'') z(a'')) = d (S(a') (a'), S(a'') z (a'')) \leqslant d(a', a'') + q\, d(z(a'), z(a''))$, which gives $d(z(a'), z(a'')) \leqslant (1/(1 - q)) d(a', a'')$. Now, from (2.1) and (3.1)

$$\| z(a') - z(a'') \|_g \leqslant \exp (d(z(a'), z(a'')) - 1 \leqslant (\exp (d(a', a''))) -$$
$$- 1 \leqslant (1 + 2((\max (c/b))/(\min (b))) \| a' - a'' \|)^{1/(1-q)} - 1$$

and this gives (from (4.14) and lemma 4.1)

$$| h(a') - h(a'') | \leqslant \| S(a') z(a') - S(a'') z(a'') \|_g \leqslant$$
$$\leqslant \| S(a') z(a') - S(a'') z(a') \|_g + \| S(a'') z(a') - S(a'') z(a'') \|_g \leqslant$$
$$\leqslant (\| \sum_{i=1}^{m} T_i z(a') \|_g) \| a' - a'' \| + (\max (c)) \sum_{i=1}^{m} \| T_i z(a') - T_i z(a'') \|_g \leqslant$$
$$\leqslant \delta \| a' - a'' \| + 2m (\delta/\beta) (\max (c)) \| z(a') - z(a'') \|_g \leqslant \delta \| a' - a'' \| +$$
$$+ 2m (\delta/\beta) (\max (c)) ((1 + 2 ((\max (c/b))/(\min (b))) \delta a' - a'' \|)^{1/(1-q)} - 1)$$

Q.E.D.

Proof of theorem 4.1, *continued*. If, for a given $a \in (R^0_+)^m$ we denote $\alpha = ((\gamma \min (a))/(\delta \max (a))$, $\beta = \alpha^{1/(1-q)}$, then, from lemma 4.2 we get $z(a) \geqslant \beta g$, and this gives

(4.17) $$\beta^q \gamma \min (a) \leqslant h(a) \leqslant \delta \max (a).$$

Furthermore, let us consider the equation

(4.18) $$a = H(a).$$

Clearly, (4.18) is equivalent with

(4.19) $a_i = (a_m)^{p_i/q}, \ i = 1, \ldots, m - 1; \ a_m = (h((a_m)^{p/q}, \ldots, (a_m)^{p_{m-1}/q}, a_m))^q.$

Let us define $\{k, K\} \subset C(R^0_+, R^0_+)$ by

(4.20) $$k(t) = (h(t^{p/q}, \ldots, t^{p_{m-1}/q}, t))^q, \quad \forall t \in R^0_+.$$

(4.21) $$K(t) = t/k(t), \quad \forall t \in R^+_0.$$

From (4.17) we obtain the evaluations

(4.22) $t \in]0, 1[\Rightarrow k(t) \geqslant (\gamma^{q/(1-q)}/\delta^{q^2/(1-q)})\, t^{(q-pq)(1-q)}; \ t \in]1, +\infty[\Rightarrow k(t) \leqslant \delta^q t^q$

and this implies

(4.23) $$t \in]0, 1[\Rightarrow K(t) \leqslant (\delta^{q^2/(1-q)}/\gamma^{q/(1-q)})\, t^{(1-2q+pq)/(1-q)}$$
$$t \in]1, +\infty[\Rightarrow K(t) \geqslant (1/\delta)^q t^{1-q} .$$

Now, from (4.4) + (4.23) we get $\lim_{t \to 0} K(t) = 0$, $\lim_{t \to +\infty} K(t) = +\infty$, and thus, there exists $t' \in R^0_+$, such that $K(t') = 1$. From this fact, it easily follows that $a' = ((t')^{p/q}, \ldots, (t')^{p_{m-1}/q}, t') \in (R^0_+)^m$ is a solution of (4.18). Define $g = h(a') z(a') \in Y^n_g$. A simple evaluation shows that (4.6) holds and this completes the proof of our theorem. Q.E.D.

Corollary 4.1. *Suppose that* (4.3) *is replaced by the following restrictive condition*

(4.3)' *the set* $\bar{Y}^n (T_1, \ldots, T_m)$ *of all* $g \in Y^n$, *such that* a) $M(\sum_{i=1}^{m} T_i(g), g) < +\infty$; b) $\inf \{m(\sum_{i=1}^{m} T_i (x), g) ; x \in \tilde{Y}^n_g\} > 0$, *is not empty.*

Then, if (4.1) + (4.2) + (4.3)' *hold, the conclusion of the main result is still valid.*

Proof. Let $g \in \bar{Y}^n (T_1, \ldots, T_m)$, and denote $\gamma = \inf \{m(\sum_{i=1}^{m} T_i (x) ; x \in \tilde{Y}^n_g\}$ In this case, (4.17) may be replaced with

(4.17)' $$\gamma \min (a) \leqslant h(a) \leqslant \delta \max (a), \quad \forall a \in (R)^{m}{}^0_+$$

and, on the other hand, (4.22) and (4.23) become respectively

(4.22)' $t \in]0, 1[\Rightarrow k(t) \geqslant \gamma^q t^q ; \ t \in]1, +\infty[\Rightarrow k(t) \leqslant \delta^q t^q$, (4.23)' $t \in [0, 1[\Rightarrow K(t) \leqslant (1/\gamma)^q t^{1-q}$; $t \in]1, +\infty[\Rightarrow K(t) \geqslant (1/\delta)^q t^{1-q}$

and this implies $\lim\limits_{t\to 0} K(t) = 0$, $\lim\limits_{t\to+\infty} K(t) = +\infty$, proving our assertion. Q.E.D.

Corollary 4.2. *Suppose that*

(4.24) *T_1 is a constant map* (i.e., $T_1(x) = y_1$, $\forall x \in Y^n$),

(4.25) *the set $\tilde{Y}^n(T_1, \dots, T_m)$ of all $g \in Y^n$, such that* a) $M(y_1 + \sum\limits_{i=2}^{m} T_i(g), g) < +\infty$; b) $m(y_1, g) > 0$ *is not empty.*

Then, if (4.1) + (4.2) + (4.24) + (4.25) *hold, the conclusion of the main result is again valid.* Q.E.D.

Proof. Clearly, (4.24) + (4.25) $\Rightarrow$ (4.3)'

5. In this paragraph, as an application, we present an existence result concerning a class of Volterra functional equations. Firstly, let us denote

i) $\mathbf{J} = \{\alpha \in \mathrm{P}(R_+);\ \alpha = \text{interval}\}$; $D = \{(t, s);\ 0 \leqslant s \leqslant t < +\infty\}$,

ii) $\mathbf{M}_n = C(D, (R_+)^{n^2})$, $\mathbf{M}_n^0 = C(D, (R_+^0)^{n^2})$, (i.e., the set of all positive (resp. strictly positive) continuous $n \times n$ matrix functions),

iii) $x^p = (x_1^p, \dots, x_n^p)$, $\forall x = (x_1, \dots, x_n) \in R_+^n$, $\forall p \in R_+^0$.

Let $\varphi \in F(R_+, \mathbf{J})$ be a given mapping. For the sake of simplicity, we shall denote $\varphi(t) = \tilde{t}$, $\forall t \in R_+$, and $\varphi(R_+) = \tilde{R}_+$. Let us indicate by **H** the class of all $h \in F(Y^n \times \tilde{R}_+, R)$, such that

(5.1) $\forall x \in Y^n$ the map $t \to h(x, \tilde{t})$ is in Y^n,

(5.2) $x, y \in Y^n,\ x \leqslant y,\ t \in R_+ \Rightarrow h(x, \tilde{t}) \leqslant h(y, \tilde{t})$,

(5.3) $x \in Y^n,\ t \in R_+,\ \lambda \in R_+^0 \Rightarrow h(\lambda x, \tilde{t}) = \lambda h(x, \tilde{t})$.

Now, let $\{A_1, \dots, A_m\} \subset \mathbf{M}_n^0$, $\{h_1, \dots, h_m\} \subset \mathbf{H}$, $\{p_1, \dots, p_m\} \subset [0, 1]$; $0 \leqslant p = p_1 < p_2 < \dots < p_m = q < 1$. Consider the following Volterra functional equation

(5.4) $$x(t) = \sum_{i=1}^{m} \int_0^t A_i(t, s)(h_i(x, \tilde{s}))^{p_i} ds, \qquad \forall t \in R_+$$

The main result about (5.4) can be stated as follows.

Theorem 5.1. *Suppose there exist* $\{B_1, \dots, B_m\} \subset \mathbf{M}_n^0$, α, β, λ, $\mu \in R_+^0$, *and* $g \in Y^n$, *such that*

(5.5) $$\alpha B_i \leqslant A_i \leqslant \beta B_i, \qquad \forall i \in \{1, \dots, m\},$$

(5.6) $$\lambda g(t) \leqslant \sum_{i=1}^{m} \int_t^0 B_i(t, s)(h_i(g, \tilde{s}))^{p_i} ds \leqslant \mu g(t), \qquad \forall t \in R_+.$$

(5.7) $$1 - 2q + pq > 0.$$

Then, (5.4) *has at least a solution* $\bar{g} \in Y_0^n$.

Proof. For every $i \in \{1, \dots, m\}$ define $T_i \in F(Y^n, Y^n)$ by

(5.8) $$T_i(x)(t) = \int_0^t A_i(t, s)(h_i(x, \tilde{s}))^{p_i} ds, \qquad \forall x \in Y^n, \quad \forall t \in R_+.$$

Clearly, T_i is monotone increasing and p_i-homogeneous on Y^n, $\forall i \in \{1, \dots, m\}$. Furthermore, from (5.5) + (5.6),

$$\alpha\lambda g(t) \leqslant \sum_{i=1}^{m} \int_0^t \alpha B_i(t, s)(h_i(g, \tilde{s}))^{p_i} ds \leqslant \sum_{i=1}^{m} \int_0^t A_i(t, s)(h_i(g, \tilde{s}))^{p_i} ds =$$

$$= \sum_{i=1}^{m} T_i(g)(t) \leqslant \sum_{i=1}^{m} \int_0^t \beta B_i(t, s)(h_i(g, \tilde{s}))^{p_i} ds \leqslant \beta\mu g(t), \quad \forall t \in R_+.$$

Therefore, theorem 4.1 is applicable, and this completes the proof. Q.E.D.

Remark 5.1. *Suppose that* $\tilde{t} = [t, t]$, $\forall t \in R_+$. *Then, we may take* $h(x, \tilde{t}) = x(t)$, $\forall x \in Y^n$, $\forall t \in R_+$. *In this case we obtain theorems* 2.1 *and* 2.2 *of* [2].

Remark 5.2. *Suppose that* $\tilde{t} = [0, t]$, $\forall t \in R_+$. *Then, we may take*

$$h(x, \tilde{t}) = x(0), \qquad \text{if } t = 0,\ x \in Y^n,$$

$$h(x, \tilde{t}) = (1/t)\int_0^t x(s)\, ds, \qquad \text{if } t > 0,\ x \in Y^n,$$

(i.e., *$h(x, \tilde{t})$ is the mean value of $x \in Y^n$ on the interval $\tilde{t}$). In this case we obtain a result which is comparable with theorem* 6.1 *of* [3].

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Received 13.XI.1977

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