Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXV, s. I a, 1979, f 1

ON A RANDOM INTEGRAL EQUATION OF THE MIXED VOLTERRA-FREDHOLM TYPE

BY

A. I. DALE

1. Introduction. The random integral equation of mixed type

$$x(t;\omega)=f(t;\omega)+\int_0^t a(t,s;\omega)\,g(s,x(s;\omega))\,ds+ \int_0^\infty b(t,s;\omega)\,h(s,x(s;\omega))\,ds, \tag{1.1}$$

$t \geqslant 0$, has been considered by various authors (e.g. Prakasa Rao and Rama Mohana Rao [4], Tsokos and Padgett [5]). However, no entirely satisfactory and sufficiently general proof of the existence and uniqueness (under appropriate conditions) of the solution has been given.

Thus, while Tsokos and Padgett [5] have undertaken a detailed investigation of (1.1), they have failed fully to exploit the random nature of the pertinent integral operators. On the other hand, Prakasa Rao and Rama Mohana Rao [4], while taking due cognisance of this component of randomness, have only managed to obtain existence theorems for the solution of (1.1), being unable to show, in one case, the measurability of the solution. Prakasa Rao [3] has remedied this to some extent, providing a proof of the existence and uniqueness of the *random* solution to

$$x(t;\omega)=f(t;\omega)+\int_0^t q(t,s,x(s;\omega);\omega)\,ds+\int_0^T r(t,s,x(s;\omega);\omega)\,ds$$

for $t\in[0,T]$.

It is our aim here to examine the equation

$$x(t;\omega)=f(t;\omega)+\int_0^t h_V(t,s,x(s;\omega);\omega)\,ds+ \int_0^\infty h_F(t,s,x(s;\omega);\omega)\,ds, \tag{1.2}$$

$t \in [0, \infty)$, and to prove an existence and uniqueness result which e tends the work detailed above. All such results, of course, generalis the usual deterministic situation, as exemplified in M i l l e r [2], for in tance. It will be noticed that while the methods of proof are often an logous to these used in deterministic situations, much emphasis is la on the *random* nature of the operators involved, and much use is mad of a *random* fixed point theorem.

In Section 2 we present definitions and results pertinent to o later sections. Section 3 contains an existence and uniqueness theore which differs from that presented by P r a k a s a R a o [3] both in co sidering different kernels and in providing a solution on $[0, \infty]$. Secti 4 contains some related results, providing a solution to (1.1) (thus e tending the work of T s o k o s and P a d g e t t [5]), and discussing t stochastic stability of (1.1) (a result analogous to that obtained by P r k a s a R a o [3]).

2. Preliminaries. Let $(\Omega, \mathcal{A}, P)$ denote a complete, atomic prob bility space, in which P has a countable basis (i.e. there exists a sequen $\{E_n\}$ of sets in $\mathcal{A}$ such that, for any $\varepsilon > 0$ and $A \in \mathcal{A}$, there is a E_k of the sequence for which $P(A \Delta E_k) < \varepsilon$). The random process $R^+ \times \Omega \to R$ is *regular* if, for each $t \in R^+$,

$$E \mid x(t;\omega) \mid^2 \doteq \int_\Omega \mid x(t;\omega) \mid^2 dP(\omega) < \infty.$$

($\doteq$ denotes equality by definition). $L_2 \doteq L_2(\Omega, \mathcal{A}, P)$ is the separable B nach space of all (equivalence classes of) regular random processes, w norm

$$\| x(t;\omega) \|_2 \doteq \left\{ \int_\Omega \mid x(t;\omega) \mid^2 dP(\omega) \right\}^{\frac{1}{2}}.$$

A real-valued random variable $x(\cdot)$ defined on $(\Omega, \mathcal{A}, P)$ is said to *P*-essentially bounded if

$$(\exists \alpha > 0)\ P(\{\omega : \mid x(\omega) \mid > \alpha\}) = 0.$$

The space of all such function will be denoted by $L_\infty \doteq L_\infty(\Omega, \mathcal{A}, P)$, t norm in this space being given by

$$\| x(\omega) \|_\infty = \inf_{\Omega_0} \sup_{\Omega - \Omega_0} \mid x(\omega) \mid.$$

where $P(\Omega_0) = 0$.

Let $C \doteq C(R^+, L_2)$ denote the space of all continuous, bounded fu tions on R^+ with values in L_2 — that is, the space of all bounded, m square continuous, regular random processes on R^+. The norm in t space is defined by

$$\| x(t;\omega) \|_C = \sup_{t \geq 0} \delta \| x(t;\omega) \|_2 < \infty.$$

Let $C_C(R^+, L_2)$ denote the space of all continuous functions on R^+ with values in L_2. The topology on this space is defined by the pseudo-distances $d_\alpha(x, y) = \| x - y \|_\alpha$, where $\{\| \cdot \|_\alpha : \alpha \in I\}$ is any family of semi-norms. In particular, we shall consider

$$\| x(t;\omega) \|_n = \sup_{\alpha \leq t \leq n} \left\{ \int_\Omega \mid x(t;\omega) \mid^2 dP(\omega) \right\}^{\frac{1}{2}}, \quad n = 1, 2, 3, \ldots$$

With this topology, $C_C(R^+, L_2)$ becomes a separable Fréchet space. Hence $C(R^+, L_2)$, being a subspace of $C_C(R^+, L_2)$, is a separable Banach space.

Definition 2.1. *The random process $x(t;\omega)$ is a random solution of (1.2) if, for every fixed $t \in R^+$, $x(t;.) L_2$ and $x(t;.)$ satisfies (1.2) almost surely.*

Definition 2.2. *Let E, F be two normed linear spaces. A linear random operator $T : \Omega \times E \to F$ is compact if, for any bounded sequence $\{x_n\}$ in E, there is a subsequence $\{x_{n_k}\}$ such that the sequence $\{T(\omega)\, x_{n_k}\}$ is almost surely convergent in F.*

Definition 2.3. *The ordered pair (B, D) of spaces is admissible with respect to the random operator $T : \Omega \times C_C(R^+, L_2) \to C_C(R^+, L_2)$ if $T(\omega) B \subset D$ almost surely.*

Definition 2.4. *The random operator $T : \Omega \times X \to X$ (where X is a Banach space) satisfies a Lipschitz condition, if there exists a positive real-valued random variable $k(\cdot)$ such that $\| T(\omega)\, x - T(\omega, y \| \leq k(\omega) \| x - y \|$ almost surely for all x, y in X. If $k(\omega) < 1$ almost surely, then T is a contraction random operator. If $k(\cdot)$ is constant, T is a uniform contraction random operator.*

The following random analogue of Krasnoselskii's fixed point theorem is due to P r a k a s a R a o [3]: it will be used extensively in the sequel.

Theorem 2.1. *Let $(\Omega, \mathcal{A}, P)$ be an atomic probability space and let E be a closed, bounded and convex subset of a separable Banach space X. Let T, S be continuous random operators from $\Omega \times X$ into X such that*

i) *$[S(\omega)\, x + T(\omega)\, y] \in E$ for all $x, y \in E$ and $\omega \in \Omega$,*

ii) *there exists a real-valued random variable $K(\cdot)$ such that $\| S(\omega)\, x - S(\omega)\, y \| \leq K(\omega) \| x - y \|$ for all $x, y \in E$ where $0 \leq K(\omega) < 1$ for all $\omega \in \Omega$, and*

iii) *$T(\omega)$ is a compact operator for each $\omega \in \Omega$.*

Then there exists an E-valued random variable $\xi(\omega)$ such that

$$S(\omega)\, \xi(\omega) + T(\omega)\, \xi(\omega) = \xi(\omega) \text{ for all } \omega \in \Omega.$$

3. Existence and uniqueness. Consider the equation.

$$x(t;\omega)=f(t;\omega)+\int_0^t h_V(t,s,x(s;\omega);\omega)\,ds+ \tag{3.1}$$
$$+\int_0^\infty h_F(t,s,x(s;\omega);\omega)\,ds.$$

$t\in R^+$, under the following assumptions:

A1. B, D are Banach spaces stronger than $C_C(R^+, L_2)$.

A2. $f\in D$ and h_V, h_F satisfy the following conditions:

i) for $0\leqslant s\leqslant t<\infty$ and $x(s;\omega)\in D$, $h_V(t,s,x(s;\omega);\omega)\in B$,

ii) for $0\leqslant s, t<\infty$ and $x(s;\omega)\in D$, $h_F(t,s,x(s;\omega);\omega)\in B$,

iii) for $s>t$, $h_V(t,s,x(s;\omega);\omega)=0$.

iv) $h_V(t,s,0;\omega)\equiv h_F(t,s,0;\omega)=0$ almost surely,

A3. (B,D) is admissible with respect to the random operators

$$U_V(\omega)\,x(t;\omega)=\int_0^t h_V(t,s,x(s;\omega);\omega)\,ds,\quad t\in R^+,$$

$$T_F(\omega)\,x(t;\omega)=\int_0^\infty h_F(t,s,x(s,\omega);\omega)\,ds,\quad t\in R^+,$$

the norms of these operators being denoted by the random variabl[es] $k_V(\cdot)$ and $k_F(\cdot)$ respectively.

A4. For every $\varepsilon>0$, $t\in R^+$ and $\omega\in\Omega$,

$$\| h_F(t+\tau,s,x(s;\omega);\omega)-h_F(t,s,x(s;\omega);\omega)\|_B$$

tends to zero as $|\tau|\to 0$ uniformly for all $x\in D$ with $\|x(t;\omega)\|_D\leqslant$

A5.i) For every $\varepsilon_1>0$ there exists a $\delta_1\equiv\delta(\varepsilon_1)>0$ such that for [all] x, y in D with $\|x(t;\omega)\|_D\leqslant\delta_1$ and $\|y(t;\omega)\|_D\leqslant\delta_1$, $\|h_V(t,s,x(s;\omega);\omega)$ $-h_V(t,s,y(s;\omega);\omega)\|_B\leqslant\varepsilon_1\|x(t;\omega)-y(t;\omega)\|_D$ P-almost surely for all t, s in R^+.

ii) For every $\varepsilon_2>0$, there exists a $\delta_2\equiv\delta(\varepsilon_2)>0$, such that for [all] x, y in D with $\|x(t;\omega)\|_D<\delta_2$ and $\|y(t;\omega)\|_D<\delta_2$, $\|h_F(t,s,x(s,\omega);\omega)$ $-h_F(t,s,y(s,\omega);\omega)\|_B\geqslant\varepsilon_2\|x(t;\omega)-y(t,\omega)\|_D$, P-almost surely [for] all $t,s\in R^+$.

Theorem 3.1. *Under the above assumptions there exists $\delta_0>0$ su[ch] that if $\delta\in(0,\delta_0]$, then there is an $\eta>0$ such that there is a unique ra[ndom] solution $x(t;\omega)$ of (3.1) for which $\|x(t;\omega)\|_D\leqslant\delta$ a.s. provided that* $\|f(t;\omega)\|_D\leqslant\eta$ a.s.

Proof. In view of the "almost sure" nature of assumptions *A2* and *A5*, there exists $N\in\mathcal{A}$ such that $P(N)=0$ and *A2*, *A5* are satisfied every-where on $\Omega_0=\Omega-N$. Henceforth we shall restrict our attention to such Ω_0. Let $\varepsilon_1, \varepsilon_2>0$ be given such that $\varepsilon_1 k_V(\omega)+\varepsilon_2 k_F(\omega)<1$ a.s., and choose δ_1,δ_2 so that *A5* is satisfied. Let $\delta_0=\min\{\delta_1,\delta_2\}$. For any positive random variable $p(\cdot)$ let

$$S(p)=\{x(t;\omega)\in D:\|x(t;\omega)\|_D\leqslant p(\omega)\}.$$

Clearly this is a closed, bounded and convex subset of D. We show firstly that, for any $\delta\in(0,\delta_0]$, there exists $\eta>0$ such that, for all $x, y\in S(p)$, where $p(\omega)\leqslant\delta$ a.s., and $\|f(t;\omega)\|_D\leqslant\eta$,

$$T_V(\omega)\,x(t;\omega)+T_F(\omega)\,y(t;\omega)\in S(p),$$

where $T_V(\omega)\,x(t;\omega)=f(t;\omega)+U_V(\omega)\,x(t;\omega)$. To this end, notice that, for $t\in R^+$,

$$\|T_V(\omega)\,x(t;\omega)+T_F(\omega)\,y(t;\omega)\|_D\leqslant\|f(t;\omega)\|_D+$$
$$+\Big\|\int_0^t h_V(t,s,x(s;\omega);\omega)\,ds\Big\|_D+\Big\|\int_0^\infty h_F(t,s,y(s;\omega);\omega)\,ds\Big\|_D\leqslant$$
$$\leqslant\|f(t;\omega)\|_D+k_V(\omega)\,\varepsilon_1\|x(t;\omega)\|_D+k_F(\omega)\,\varepsilon_2\|y(t;\omega)\|_D\leqslant$$
$$\leqslant\eta+k_V(\omega)\,\varepsilon_1 p(\omega)+k_F(\omega)\,\varepsilon_2 p(\omega)\leqslant p(\omega),$$

provided that $\eta\leqslant p(\omega)\,[1-\varepsilon k_V(\omega)-\varepsilon_2 k_F(\omega)]$. Thus

$$T_V(\omega)\,x(t;\omega)+T_F(\omega)\,y(t;\omega)\in S(p).$$

Furthermore, since for any $x, y\in S(p)$ and every $\omega\in\Omega_0$,

$$\|T_V(\omega)\,x(t;\omega)-T_V(\omega)\,y(t;\omega)\|_D\leqslant k_V(\omega)\,\varepsilon_1\|x(t;\omega)-y(t\,\omega)\|_D,$$

where $k_V(\omega)\,\varepsilon_1<1$ a.s., it follows that T_V is a contraction random operator on $S(p)$. To show that T_F is a compact operator on $\Omega_0\times S(p)$, notice firstly that, for $\omega\in\Omega_0$ and $t\in R^+$,

$$\|T_F(\omega)\,x(t+\tau;\omega)-T_F(\omega)\,x(t;\omega)\|_D\leqslant k_F(\omega)\,\|h_F(t+$$
$$+\tau,s,x(s;\omega);\omega)-h_F(t,s,x\,h_F(t,s,x(s;\omega);\omega)\|_B$$

which tends to zero as $|\tau|\to 0$ uniformly for $x\in S(p)$. Moreover, $\|T_F(\omega)\,x(t;\omega)\|_D\leqslant k_F(\omega)\,\varepsilon_2\,p(\omega)$ for all $x\in S(p)$. Hence, the random operator T_F is uniformly bounded on $S(p)$, and thus is compact. It follows by Theorem 2.1 that there exists at least one $x\in S(p)$ such that

$$T_V(\omega)\,x(t;\omega)+T_F(\omega)\,x(t;\omega)=x(t;\omega)\ \text{a.s.}$$

Finally, to show the uniqueness of the solution in $S(p)$, suppos that there exist two random solutions x_1 and x_2. Let $z = x_1 - x_2$. Then, on

$$\| z(t;\omega) \|_D \leqslant \| \int_0^t h_V(t, s, x_1(s;\omega);\omega) - h_V(t, s, x_2(s;\omega);\omega)\, ds \|_D +$$

$$+ \| \int_0^t h_F(t, s, x_1(s;\omega);\omega) - h_F(t, s, x_2(s;\omega);\omega)\, ds \|_D \leqslant$$

$$\leqslant (k_V(\omega)\, \varepsilon_1 + k_F(\omega)\, \varepsilon_2) \| z(t;\omega) \|_D.$$

Since $k_V(\omega)\, \varepsilon_1 + k_F(\omega)\, \varepsilon_2 < 1$ a.s., we obtain a contradiction $\| z(t;\omega) \|_D \neq 0$. Thus $x_1(t;\omega) = x_2(t;\omega)$ a.s., which completes th proof.

Suppose next that $B = D = C$, and replace assumption $A5$ by $A5^*$. There exists non-negative functions $k_V(t, s)$ and $k_V(t, s)$ such tha

i) for every $\varepsilon_1 > 0$, there exists $\delta_1 \equiv \delta(\varepsilon_1)$ such that for all x, in C with $\| x(t;\omega) \|_C \leqslant \delta_1$ and $\| y(t;\omega) \|_C \leqslant \delta_1$,

$$\| h_V(t, s, x(s;\omega);\omega) - h_V(t, s, y(s;\omega);\omega) \|_2 \leqslant$$
$$\leqslant \varepsilon_1 k_V(t, s) \| x(t;\omega) - y(t;\omega) \|_2.$$

(ii) for every $\varepsilon_2 > 0$, there exists $\delta \equiv \delta(\varepsilon_2)$ such that for all x, in C with $\| x(t;\omega) \|_C \leqslant \delta_2$ and $\| y(t;\omega) \|_C \leqslant \delta_2$,

$$\| h_F(t, s, x(s;\omega);\omega) - h_F(t, s, y(s;\omega);\omega) \|_2 \leqslant \varepsilon_2 k_F(t, s) \| x(t;\omega) - y(t;\omega) \|$$

iii) $$\sup_{0 \leqslant t < \infty} \int_0^t k_V(t, s)\, ds \equiv K_V < \infty,$$

iv) $$\sup_{0 \leqslant t < \infty} \int_0^t k_F(t, s)\, ds \equiv K_F < \infty.$$

Defining the random operators T_V, T_F as before, and noting th (C, C) is admissible with respect to these operators (cf. Tsokos a Padgett [5], Corollary 2.3.2), we have the following result:

Theorem 3.2. *Under assumptions $A1 - A4$, $A5^*$, there exists $\delta_0 >$ such that if $\delta \in (0, \delta_0]$, then there is an $\eta > 0$, such that there is a u que random solution $x(t;\omega)$ of (3.1) for which $\| x(t;\omega) \|_C \leqslant \delta$, provi that $\| f(t;\omega) \|_C \leqslant \eta$.*

Remark: The proof of this result parallels the methods used Miller [2], Chapter II, for solving analogous deterministic integ equations, and is accordingly omitted. A distinguishing feature of result, of course, is the use of *random* operators and a *random* fi point theorem, and we have already indicated, in the preceding theore the role that these concepts play. It should be noted, further, that t

set $S(p)$ of Theorem 3.1 may be taken to be $\{x(t;\omega) \in C : \| x(t;\omega) \|_C \leqslant \leqslant p\}$, where $p > 0$, while T_V turns out to be a *uniform* contraction random operator.

4. Related results. In this section we specialize the abstract random operators of equation (3.1) to obtain the random integral equation of Volterra-Fredholm type

$$x(t;\omega) = f(t;\omega) + \int_0^t k_V(t, s;\omega)\, g_V(s, x(s;\omega))\, ds + \tag{4.1}$$

$$+ \int_0^\infty k_F(t, s;\omega).\, g_F(s, x(s;\omega))\, ds$$

for $t \in R^+$.

As we have already mentioned, this equation has been studied by Tsokos and Padgett [5]. However, due to their failure fully to utilise the random nature of the pertinent operators, we propose to state certain results here which can be proved by methods similar to those of Theorem 3.1. Proofs of these results will not be given: we shall content ourselves by merely indicating aspects in which such proofs differ from those of Tsokos and Padgett, op. cit.

Consider equation (4.1) under the following conditions:

B1. B, D are Banach spaces stronger than $C_C(R^+, L_2)$.

B2. $f \in D$ and k_V, k_F satisfy the following conditions:

i) for $0 \leqslant s \leqslant t < \infty$, $k_V(t, s;\ ) \in L_\infty$.

ii) for $0 \leqslant s, t < \infty$, $k_F(t, s;\cdot) \in L_\infty$,

iii) for $s > t$, $k_V(t, s;\omega) \equiv 0$,

iv) k_V and k_F, as mappings from the sets $\{(t, s) : 0 \leqslant s \leqslant t < \infty\}$ and $\{(t, s) : 0 \leqslant s, t < \infty\}$ respectively into L_∞, are continuous,

B3. (B, D) is admissible with respect to the random operators

$$U_V(\omega)\, x(t;\omega) = \int_0^t k_V(t, s;\omega)\, g_V(s, x(s;\omega))\, ds, \quad t \in R^+,$$

$$T_F(\omega)\, x(t;\omega) = \int_0^\infty k_F(t, s;\omega)\, g_F(s, x(s;\omega))\, ds, \quad t \in R^+,$$

where $g_V, g_F : R^+ \times D \to B$. Let $k_V(\cdot)$, $k_F(\cdot)$ denote the norms of these operators respectively,

B4. For every $\varepsilon > 0$ and $t \in R^+$, $\| k_F(t + \tau, s;\omega) - k_F(t, s;\omega) \|_B$ tends to zero as $|\tau| \to 0$ almost surely.

B5. g_V and g_F are functions from $R^+ \times D \to B$ satisfying the following conditions :

i) for every $\varepsilon_1 > 0$ there exists a $\delta_1 \equiv \delta(\varepsilon_1) > 0$ such that, for all x, y in D with $\| x(t;\omega) \|_D \leqslant \delta_1$ and $\| y(t;\omega) \|_D \leqslant \delta_1$

$$\| g_V(t, x(t;\omega)) - g_V(t, y(t;\omega)) \|_B \leqslant \varepsilon_1 \| x(t;\omega) - y(t;\omega) \|_D,$$

ii) for every $\varepsilon_2 > 0$ there exists a $\delta_2 \equiv \delta(\varepsilon_2) > 0$ such that, for all x, y in D with $\| x(t;\omega) \|_D \leqslant \delta_2$ and $\| y(t;\omega) \|_D \leqslant \delta_2$,

$$\| g_F(t, x(t;\omega)) - g_F(t, y(t;\omega)) \|_B \leqslant \varepsilon_2 \| x(t;\omega) - y(t;\omega) \|_D.$$

iii) $g_V(t, 0) \equiv g_F(t, 0) = 0$ a.s.

Theorem 4.1. *Under the above assumptions, there exits δ_0 such that, if $\delta \in (0, \delta_0]$, then there is an $\eta > 0$ such that there is a unique random solution $x(t;\omega)$ of (4.1) for which $\|x(t;\omega)\|_D \leqslant \varepsilon$ a.s. provided that $\|f(t;\omega)\|_D \leqslant \eta$ a.s.*

Remark : the proof parallels that of Theorem 3.1, and is accordingly omitted. As was the case in that theorem, the solution $x(t;\omega)$ obtained here — by considering fully the random nature of $T_F(\cdot)$ and $T_V(\cdot)$ — is in fact one for which $x \in S(p)$, where $p(\omega) \leqslant \delta$ a.s. and $S(p) = \{x(t;\omega) \in D : \| x(t;\omega) \|_D \leqslant p(\omega)\}$.

Suppose next that $B = D = C$, and consider the assumptions :

B4*. For every $\varepsilon > 0$ and $t \in R^+$,

$$\int_0^\infty \| k_F(t+\tau, s;\omega) - k_F(t, s;\omega) \|_\infty \, ds$$

tends to zero as $|\tau| \to 0$.

B5*. i) For every $\varepsilon_1 > 0$ there exists a $\delta_1 \equiv \delta(\varepsilon_1) > 0$ such that for all x, y in C with $\| x(t;\omega) \|_C \leqslant \delta_1$ and $\| y(t;\omega) \|_C \leqslant \delta_2$,

$$\| g_V(t, x(t;\omega)) - g_V(t, y(t;\omega)) \|_2 \leqslant \varepsilon_1 \| x(t;\omega) - y(t;\omega) \|_2$$

ii) For every $\varepsilon_2 > 0$ there exists a $\delta \equiv \delta(\varepsilon_2) > 0$ such that for all x, y in C with $\| x(t;\omega) \|_C \leqslant \delta_2$ and $\| y(t;\omega) \|_C \leqslant \delta_2$,

$$\| g_F(t, x(t;\omega)) - g_F(t, y(t;\omega)) \|_2 \leqslant \varepsilon_2 \| x(t;\omega) - y(t;\omega) \| \delta_2,$$

iii) $$\sup_{0<t<\infty} \int_0^t \| k_V(t, s;\omega) \|_\infty \, ds \equiv K_V < \infty,$$

iv) $$\sup_{0<t<\infty} \int_0^\infty \| k_F(t, s;\omega) \|_\infty \, ds \equiv K_F < \infty.$$

Corollary 4.1. *Under assumptions $B1 - B3$, $B4^*$, $B5^*$ there exists δ_0 such that, if $\delta \in (0, \delta_0]$, then there is an $\eta > 0$ such that there is a unique random solution $x(t;\omega)$ of (4.1) for which $\| x(t;\omega) \|_C \leqslant \delta$, provided that $\| f(t;\omega) \|_C \leqslant \eta$.*

Proof : in Theorem 3.2 let

$$h_V(t, s, x(s;\omega);\omega) = k_V(t, s;\omega)\, g_V(s, x(s;\omega)),$$

$$h_F(t, s, x(s;\omega);\omega) = k_F(t, s;\omega)\, g_F(s, x(s;\omega)),$$

$$k_V(t, s) = \| k_V(t, s;\omega) \|_\infty, \quad k_F(t, s) = \| k_F(t, s;\omega) \|_\infty .$$

Then the assumptions of this theorem imply those of Theorem 3.2, and the desired result is immediate.

Corollary 4.2. *Under the assumptions $B1 - B3$, $B4^*$, $B5^*$, if in addition $f(t;\omega) \to 0$ a.s. as $t \to \infty$ and if, for each $T > 0$,*

B6 i) $$\lim_{t\to\infty} \int_0^t \| k_V(t, s;\omega) \|_\infty \, ds = 0 \text{ and}$$

ii) $$\lim_{t\to\infty} \int_0^T \| k_F(t, s;\omega) \|_\infty \, ds = 0.$$

then $x(t;\omega) \to 0$ a.s. as $t \to \infty$, where x is the random solution of (4.1).

Proof : Given $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists $p > 0$ such that B_L^* (i), (ii) are satisfied. Letting

$$S(p) = \{x(t;\omega) \in C : \| x(t;\omega) \|_C \leqslant p\},$$

set

$$S_0 = \{x \in S(p) : \lim_{t\to\infty} x(t;\omega) = 0 \text{ a.s.}\}.$$

Clearly S_0 is a closed, bounded and convex subset of $S(p)$. Furthermore, there exists $\Omega_0 \in A$ such that $B1 - B3$, $B4^*$, $B5^*$, $B6$ hold everywhere on Ω_0 and $f(t;\omega) \to 0$ as $t \to \infty$.

Let T_V, $T_F : \Omega_0 \times C \to C$ be defined by

$$T_V(\omega)\, x(t;\omega) = f(t;\omega) + \int_0^t k_V(t, s;\omega)\, g_V(s, x(s;\omega))\, ds,$$

$$T_F(\omega)\, x(t;\omega) = \int_0^\infty k_F(t, s;\omega)\, g_V(s, x(s;\omega))\, ds.$$

Proceeding as in the proof of Theorem 3.1, we can show that T_V, T_F are random continuous operators, that T_V is a uniform contraction random operator on S_0, and that T_F is a compact random operator. Let $x, y \in S_0$, and let $\varepsilon > 0$ be given. Then there exists $T_1 \in R^+$ such that, for $t \geqslant T_1$,

$$\| x(t;\omega) \|_C \leqslant \frac{\varepsilon}{5\varepsilon_0}, \; \| y(t,\omega) \|_C \leqslant \frac{\varepsilon}{5\varepsilon_0}, \; \| f(t;\omega) \|_C \leqslant \frac{\varepsilon}{5},$$

where $\varepsilon_0 = \max\{\varepsilon_1 K_V, \varepsilon_2 K_V\}$. By B6, there exists $T_2 \geqslant T_1$ such tha[t] for $t \geqslant T_2$,

$$\int_0^t \| k_V(t, s; \omega) \|_\infty ds \leqslant \frac{\varepsilon}{5\varepsilon_1 p}, \quad \int_0^\infty \| k_F(t, s; \omega) \|_\infty ds < \frac{\varepsilon}{5\varepsilon_2 p}.$$

Thus, for $t \geqslant T_2 \geqslant T_1$, and $x, y \in S_0$,

$$\| T_V(\omega)\, x(t; \omega) + T_F(\omega)\, x(t; \omega) \|_2 \leqslant \| f(t; \omega) \|_2 + \int_0^t \| k_V(t, s; \omega) \|_\infty$$

$$\| g_V(s, x(s; \omega)) \|_2 ds + \int^\infty \| k_F(t, s; \omega) \|_\infty \| g_F(s, x(s; \omega)) \|_2 ds \leqslant$$

$$\leqslant \frac{\varepsilon}{5} + \varepsilon_1 \left\{ \int_0^{T_1} \| k_V(t, s; \omega) \|_\infty \| x(s; \omega) \|_2 ds + \int_{T_1}^t \| k_V(t, s; \omega) \|_\infty \| x(s; \omega) \|_2 ds \right.$$

$$\left. + \varepsilon_2 \left\{ \int_0^{T_1} \| k_F(t, s; \omega) \|_\infty \| x(s; \omega) \|_2 ds + \int_{T_1}^\infty \| k_F(t, s; \omega) \|_\infty \| x(s; \omega) \|_2 ds \right\} \right.$$

$$\leqslant \frac{\varepsilon}{5} + \varepsilon_1 p \int_0^t \| k_V(t, s; \omega) \|_\infty ds + \frac{\varepsilon_1 \varepsilon}{5\varepsilon_0} \int_0^t \| k_V(t, s; \omega) \|_\infty ds +$$

$$+ \varepsilon_2 p \int_0^t \| k_F(t, s; \omega) \|_\infty ds + \frac{\varepsilon_2 \varepsilon}{5\varepsilon_0} \int_0^t \| k_F(t, s; \omega) \|_\infty ds \leqslant$$

$$\frac{\varepsilon}{5} + \varepsilon_1 p \frac{\varepsilon}{5\varepsilon_1 p} + \frac{\varepsilon_1 \varepsilon}{5\varepsilon_0} K_V + \varepsilon_2 p \frac{\varepsilon}{5\varepsilon_2 p} \frac{\varepsilon_2 \varepsilon}{5\varepsilon_0} K_F \leqslant \varepsilon.$$

since $\varepsilon_0 = \max\{\varepsilon_1 K_V, \varepsilon_2 K_F\}$. Thus $[T_V(\omega)\, x(t; \omega) + T_F(\omega)\, y(t; \omega)] \in$ when $x, y \in S_0$. By Theorem 2.1 it follows that there exists $x \in$ such that, for $\omega \in \Omega_0$, $T_V(\omega)\, x(t; \omega) + T_F(\omega)\, x(t; \omega) = x(t; \omega)$, so (4.1) has a (unique) random solution which tends to zero as $t \to$ a.s. This completes the proof.

REFERENCES

1. Bharucha-Reid A. T. — *Random Integral Equations* Academic Press, New York, 19[…]
2. Miller R. K. — *Nonlinear Volterra Integral Equations*. Benjamin, Menlo Park, C[…] fornia, 1971.
3. Prakasa Rao B. L. S. — *Stochastic integral equations of mixed type* II. J. Math[…] tical and Physical Sci. VII, 245—260, 1973.
4. Prakasa Rao, B. L. S. Rama Mohana Rao, M. — *Stochastic integral equati[ons of] mixed type*. Analele Ştiinţifice, XVIII, 351—359, 1972.
5. Tsokos C. P. Padgett W. J. — *Random Integral Equations with Applicati[ons to] Stochastic Systems*. Springer-Verlag, Berlin-Heidelberg-New York, 1971.

Department of Mathe[matics] University of Natal, D[…]

Received 10.VI.1978

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXV, s. I a, 1979, f. 1

LOCAL CONTROLLABILITY OF NONLINEAR EVOLUTION EQUATIONS IN BANACH SPACES

BY

OVIDIU CÂRJĂ

1. **Introduction.** The purpose of this note is to discuss the local controllability to the problem

$$x'(t) = Ax(t) + F(x(t)) + Bu(t) \tag{1.1}$$

for $0 \leqslant t \leqslant T$, with initial data given at $t = 0$, $x(0) = x_0$, where $x_0 \in X$, X is a Banach space, $[0, T]$ is a finite closed interval of real line, $x(.)$ is a function from $[0, T]$ into X. A is the possibly unbounded infinitesimal generator of a strongly continuous semigroup of class (C_0) on X, F is a nonlinear mapping from X into X. Y denotes a Banach space and $u(.)$ is a function from $[0, T]$ into Y. B is a linear continuous operator from Y into X. The norm on X or Y is denoted by $\|.\|$.

Our result is very much in the spirit of Fattorini's work [1]. We prove that every state that is sufficiently small can be reached from the origin in a time T, with a control $u(.)$ subject to the constraint

$$\| u(t) \| \leqslant q \quad \text{a.e. on } [0, T].$$

The basic technique used here is the linearization of the equation and the application of an implicit function theorem.

2. **The main result.** We shall study the mild solution of the equation (1.1) i.e. the solution of the equation

$$\begin{cases} x(t) = T(t)x_0 + \int_0^t T(t-s)F(x(s))ds + \int_0^t T(t-s)Bu(s)ds, \\ 0 \leqslant t \leqslant T, \ x_0 \in X, \end{cases} \tag{2.1}$$

where $T(t)$, $t \geqslant 0$, is the semigroup generated by A. Before this study we shall discuss the existence and uniqueness of the solution of (2.1). We recall the following existence result.

Proposition 1. *Assume that A is the generator of a strongly continuous semigroup of class C_0, $T(t), t \geqslant 0$, B is a linear continuous operator from Y into X and F is locally Lipschitz continuous from X into X. Let $q > 0$ and $x_0 \in X$.*