where $\varepsilon_0 = \max\{\varepsilon_1 K_V, \varepsilon_2 K_V\}$. By B6, there exists $T_2 \geqslant T_1$ such tha
for $t \geqslant T_2$,

$$\int_0^t \| k_V(t, s; \omega) \|_\infty \, ds \leqslant \frac{\varepsilon}{5 \varepsilon_1 p}, \quad \int_0^\infty \| k_F(t, s; \omega) \|_\infty \, ds < \frac{\varepsilon}{5 \varepsilon_2 p}.$$

Thus, for $t \geqslant T_2 \geqslant T_1$, and $x, y \in S_0$,

$$\| T_V(\omega)\, x(t; \omega) + T_F(\omega)\, x(t; \omega) \|_2 \leqslant \| f(t; \omega) \|_2 + \int_0^t \| k_V(t, s; \omega) \|_\infty$$

$$\| g_V(s, x(s; \omega)) \|_2 \, ds + \int_0^\infty \| k_F(t, s; \omega) \|_\infty \| g_F(s, x(s; \omega)) \|_2 \, ds \leqslant$$

$$\leqslant \frac{\varepsilon}{5} + \varepsilon_1 \left\{ \int_0^{T_1} \| k_V(t,s; \omega) \|_\infty \| x(s; \omega) \|_2 \, ds + \int_{T_1}^t \| k_V(t, s; \omega) \|_\infty \| x(s; \omega) \|_2 \, ds \right.$$

$$\left. + \varepsilon_2 \left\{ \int_0^{T_1} \| k_F(t, s; \omega) \|_\infty \| x(s; \omega) \|_2 \, ds + \int_{T_1}^\infty \| k_F(t, s; \omega) \|_\infty \| x(s; \omega) \|_2 \, ds \right\} \right.$$

$$\leqslant \frac{\varepsilon}{5} + \varepsilon_1 p \int_0^t \| k_V(t, s; \omega) \|_\infty \, ds + \frac{\varepsilon_1 \varepsilon}{5 \varepsilon_0} \int_0^t \| k_V(t, s; \omega) \|_\infty \, ds +$$

$$+ \varepsilon_2 p \int_0^t \| k_F(t, s; \omega) \|_\infty \, ds + \frac{\varepsilon_2 \varepsilon}{5 \varepsilon_0} \int_0^t \| k_F(t, s; \omega) \|_\infty \, ds \leqslant$$

$$\frac{\varepsilon}{5} + \varepsilon_1 p \frac{\varepsilon}{5 \varepsilon_1 p} + \frac{\varepsilon_1 \varepsilon}{5 \varepsilon_0} K_V + \varepsilon_2 p \frac{\varepsilon}{5 \varepsilon_2 p} \frac{\varepsilon_2 \varepsilon}{5 \varepsilon_0} K_F \leqslant \varepsilon,$$

since $\varepsilon_0 = \max\{\varepsilon_1 K_V, \varepsilon_2 K_F\}$. Thus $[T_V(\omega)\, x(t; \omega) + T_F(\omega)\, y(t; \omega)] \in$ when $x, y \in S_0$. By Theorem 2.1 it follows that there exists $x \in$ such that, for $\omega \in \Omega_0$. $T_V(\omega)\, x(t; \omega) + T_F(\omega)\, x(t; \omega) = x(t; \omega)$, so (4.1) has a (unique) random solution which tends to zero as $t \to$ a.s. This completes the proof.

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Department of Mathe
University of Natal, D

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LOCAL CONTROLLABILITY OF NONLINEAR EVOLUTION EQUATIONS IN BANACH SPACES

BY

OVIDIU CÂRJĂ

1. **Introduction.** The purpose of this note is to discuss the local controllability to the problem

$$x'(t) = Ax(t) + F(x(t)) + Bu(t) \tag{1.1}$$

for $0 \leqslant t \leqslant T$, with initial data given at $t = 0$, $x(0) = x_0$, where $x_0 \in X$, X is a Banach space, $[0, T]$ is a finite closed interval of real line, $x(.)$ is a function from $[0, T]$ into X. A is the possibly unbounded infinitesimal generator of a strongly continuous semigroup of class (C_0) on X, F is a nonlinear mapping from X into X. Y denotes a Banach space and $u(.)$ is a function from $[0, T]$ into Y. B is a linear continuous operator from Y into X. The norm on X or Y is denoted by $\|.\|$.

Our result is very much in the spirit of F a t t o r i n i's work [1]. We prove that every state that is sufficiently small can be reached from the origin in a time T, with a control $u(.)$ subject to the constraint

$$\| u(t) \| \leqslant q \quad \text{a.e. on } [0, T].$$

The basic technique used here is the linearization of the equation and the application of an implicit function theorem.

2. **The main result.** We shall study the mild solution of the equation (1.1) i.e. the solution of the equation

$$\left\{ \begin{array}{l} x(t) = T(t)x_0 + \int_0^t T(t-s)F(x(s))ds + \int_0^t T(t-s)Bu(s)ds, \\ 0 \leqslant t \leqslant T, \ x_0 \in X, \end{array} \right. \tag{2.1}$$

where $T(t)$, $t \geqslant 0$, is the semigroup generated by A. Before this study we shall discuss the existence and uniqueness of the solution of (2.1). We recall the following existence result.

Proposition 1. *Assume that A is the generator of a strongly continuous semigroup of class C_0, $T(t), t \geqslant 0$, B is a linear continuous operator from Y into X and F is locally Lipschitz continuous from X into X. Let $q > 0$ and $x_0 \in X$.*

Then there exists $t_1 > 0$ such that for every $u(.)$ satisfying

$$\|u(t)\| \leqslant q \quad \text{a.e. on } [0, t_1[, \tag{2.2}$$

equation (2.1) *has a unique continuous solution $x(\,.\,)$ from $[0, t_1[$ into* [...] *Moreover, if F is global Lipschitz continuous on X, the solution x* [...] *is defined on every interval $[0, T]$, $T > 0$.*

Remark 1. If we consider $F(t, x)$ defined on $0 \leqslant t < +\infty$, $x \in$ [...] continuous in t and x, and Lipschitz continuous in x, then Propos[...]tion 1 holds.

2\. If $0 < T < t_1$, there exists $r > 0$ such that for every $u(\,.\,)$ sati[...]fying (2.2), the solution $x(.)$ for (2.1) satisfies $\|x(t)\| \leqslant r$, $t \in [0, T]$.

Let $q > 0$ and $T > 0$ be such that there exists the solution [...] the equation

$$x(t) = \int_0^t T(t-s)F(x(s))ds + \int_0^t T(t-s)Bu(s)ds, \tag{2.3}$$

for $0 \leqslant t \leqslant T$ and $\|u(t)\| \leqslant q$ a.e. on $[0, T]$.

We now state the basic assumptions which will be used in th[...] paper.

I. *A and B satisfy the conditions in Proposition 1.*

II. *$F : X \to X$ is a continuously Fréchet differentiable function* [...] *X, $F(0) = 0$.*

III. *The pair $(A + dF(0), B)$ is completely controllable on $[0, T]$.*

The condition III must be understood in the following sense: If $x_0 \in X$ is given, there exists $u(\,.\,) \in L^\infty(0, T; Y)$ such that th[...] solution of the equation.

$$x(t) = \int_0^t T(t-s)dF(0)x(s)ds + \int_0^t T(t-s)\, Bu(s)\, ds$$

satisfies $x(T) = x_0$.

We denote by $L^\infty(0, T; Y)$ the space of all strongly measura[...] essentially bounded functions from $]0, T[$ to Y, endowed with its us[...] supremum norm, $\|\cdot\|_\infty$.

The main result of this paper is the following:

Theorem 1. *Assume that conditions I, II and III hold. Then th[...] exists $\varepsilon > 0$, such that for every $x_0 \in X$ satisfying $\|x_0\| < \varepsilon$ there exi[...] $u \in L^\infty(0, T; Y)$ with*

$$\|u\|_\infty \leqslant q, \tag{2.4}$$

such that the solution of (2.3) *satisfies $x(T) = x_0$.*

3. Proof of Theorem 1. Let $U = B(0, q) \subset L^\infty(0, T; Y)$ be the op[...] ball with radius q centered at the origin. We consider in the sequel th[...] nonlinear map

$$\Phi_T : U \to X, \text{ defined by } \Phi_T(u) = x(T), \tag{3.1}$$

where $x(\,.\,)$ is the solution of (2.3). Consider the equation

$$y(t) = \int_0^t T(t-s)dF(x(s))y(s)\,ds + \int_0^t T(t-s)Bv(s)\,ds, \tag{3.2}$$

$v \in L^\infty(0, T; Y)$, which admits a continuous solution on $[0, T]$, (see Remark 1).

Lemma 1. *The map Φ_T is continuously Fréchet differentiable from U into X and $d\Phi_T(u)\,v = y(T)$ where $y(\,.\,)$ is the solution of* (3.2).

Proof. We set $\delta(t) = \tilde{x}(t) - x(t) - y(t)$, on $[0, T]$, where $\tilde{x}(t)$ is the solution of

$$\tilde{x}(t) = \int_0^t T(t-s)F(\tilde{x}(s))ds + \int_0^t T(t-s)B(u(s) + v(s))\,ds, \tag{3.3}$$

and $v \in L^\infty(0, T; Y)$ such that $u + v \in U$. $x(\,.\,)$ is the solution of (2.3) and $y(\,.\,)$ is the solution of (3.2). We shall prove that $\|\delta(t)\|/\|v\|_\infty \to 0$, if $\|v\|_\infty \to 0$. It is easy to see that $\delta(t)$ is the solution of

$$\begin{aligned}\delta(t) = &\int_0^t T(t-s)dF(x(s))\delta(s)ds + \\ &+ \int_0^t T(t-s)(F(\tilde{x}(s)) - F(x(s)) - dF(x(s))(\tilde{x}(s) - x(s)))\,ds.\end{aligned} \tag{3.4}$$

In the next estimates C will denote a positive constant. It follows immediately that

$$\|\delta(t)\| \leqslant C\int_0^t \|\delta(s)\|\,ds + C\int_0^t \|F(\tilde{x}(s)) - F(x(s)) - dF(x(s))(\tilde{x}(s) - x(s))\|ds.$$

Applying Gronwall's inequality we obtain

$$\|\delta(t)\| \leqslant C\int_0^T \|F(\tilde{x}s)) - F(x(s)) - dF(x(s))\,(\tilde{x}(s) - x(s))\|\,ds.$$

Taking into account that F is locally Lipschitz continuous and using Remark 2 we obtain.

$$\|\tilde{x}(t) - x(t)\| \leqslant C\int_0^t \|\tilde{x}(s) - x(s)\|\,ds + C\|v\|_\infty.$$

Using once again Gronwall's inequality we obtain

$$\|\tilde{x}(t) - x(t)\| \leq C\|v\|_\infty, \quad t \in [0, T].$$

If $v \to 0$ then $\|\tilde{x}(t) - x(t)\| \to 0$ uniformly on $[0, T]$. Taking into account that $F(\tilde{x}(\cdot)) - F(x(\cdot)) - dF(x(\cdot))(\tilde{x}(\cdot) - x(\cdot))$ is continuous from $[0, T]$ into X we may infer than there exists $\tau \in [0, T]$ such that

$$\int_0^T \|F(\tilde{x}(t)) - F(x(t)) - dF(x(t))(\tilde{x}(t) - x(t))\|dt =$$

$$= T\|F(\tilde{x}(\tau)) - F(x(\tau)) - dF(x(\tau))(\tilde{x}(\tau) - x(\tau))\|.$$

If $\tilde{x}(\tau) - x(\tau) \neq 0$ we obtain

$$\frac{\|\delta(t)\|}{\|v\|_\infty} \leq \frac{C\|\delta(t)\|}{\|\tilde{x}(\tau) - x(\tau)\|} \leq \frac{CT\|F(\tilde{x}(\tau)) - F(x(\tau)) - dF(x(\tau))(\tilde{x}(\tau) - x(\tau))\|}{\|\tilde{x}(\tau) - x(\tau)\|}.$$

Since F is Fréchet differentiable we obtain

(3.5) $$\frac{\|\delta(t)\|}{\|v\|} \to 0, \text{ uniformly on } [0, T], \text{ when } \|v\|_\infty \to 0.$$

If $\tilde{x}(\tau) - x(\tau) = 0$, we have $\delta(t) = 0$ on $[0, T]$ and (3.5) holds. We complete the proof of Lemma 1 by verifying that $d\Phi_T : U \to \mathcal{L}(L^\infty(0, T; Y), X)$ is continuous.

Let $\tilde{u}, u \in L^\infty(0, T; Y)$ be fixed and let $\tilde{x}(\cdot), x(\cdot)$ be the corresponding solution of (2.3), and let $\tilde{y}(\cdot), y(\cdot)$ be the solution of the equations

(3.6) $$\tilde{y}(t) = \int_0^t T(t-s)dF(\tilde{x}(s))\tilde{y}(s)ds + \int_0^t T(t-s)Bv(s)ds,$$

(3.7) $$y(t) = \int_0^t T(t-s)dF(x(s))y(s)ds + \int_0^t T(t-s)Bv(s)ds,$$

$$0 \leq t \leq T, \quad v \in L^\infty(0, T; Y).$$

Operating in the same way as in the beginning of the proof, we obtain

(3.8) $$\|\tilde{x}(t) - x(t)\| \leq C\|\tilde{u} - u\|_\infty.$$

On the other hand

$$\tilde{y}(t) - y(t) = \int_0^t T(t-s)(dF(\tilde{x}(s)) - dF(x(s)))\tilde{y}(s)ds +$$

$$+ \int_0^t T(t-s)dF(x(s))(\tilde{y}(s) - y(s))ds.$$

Applying once again Gronwall's inequality we obtain

$$\|\tilde{y}(t) - y(t)\| \leq C\int_0^t |||dF(\tilde{x}(s)) - dF(x(s))|||ds.$$

We denote by $|||\cdot|||$ the usual norm in $\mathcal{L}(X, X)$. If $\|u - \tilde{u}\|_\infty \to 0$, then $\|\tilde{x}(t) - x(t)\| \to 0$ uniformly in $0 \leq t \leq T$. Taking into account that $dF(\cdot)$ is continuous, we infer $\|\tilde{y}(t) - y(t)\| \to 0$ uniformly in $0 \leq t \leq T$, and uniformly with respect to $v \in L^\infty(0, T; Y)$.

We need also the following implicit function result:

Lemma 2. *Let E, F be two Banach spaces, $S = B(0, r) \subset E$ the open ball with radius r centered at the origin. Let f be a continuously Fréchet differentiable function from S into F such that $f(0) = 0$ and $df(0)$ is onto. Then $f(S)$ contains a neighborhood of zero in F.*

For the proof see [6] (Chapter 10) and [7].

Proof of Theorem 1. Nothing left to do except to combine Lemmas 1 and 2.

4. Aplications 1. The wave equation. Let H be a Hilbert space, A a self adjoint operator in H such that

$$(Au, u) \leq -\omega|u|^2, \quad u \in D(A) \text{ for some } \omega > 0,$$

where $(.,.)$ denotes the scalar product in H, and $|.|$ stands for the norm in H.

Let V be the domain of $(-A)^{\frac{1}{2}}$- the square root of $(-A)$- and let $F: V \to H$ be a continuously Fréchet differentiable function on V, $F(0) = 0$. It is well-known that A is the infinitesimal generator of a strongly continuous cosine family in H, and $\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ is the infinitesimal generator of a strongly continuous group in $V \times H$. The norm in V is defined by $\|u\| = |(-A)^{\frac{1}{2}}u|$, and the space $V \times H$ is endowed with its Hilbert product norm. Let $\mathcal{F}: V \times H \to V \times H$ be defined by

$$\mathcal{F}((x_1, x_2)) = \begin{pmatrix} 0 \\ F(x_1) \end{pmatrix},$$

and $B: H \to V \times H$ by $B(u) = \begin{pmatrix} 0 \\ u \end{pmatrix}$. Consider the equation

(4.1) $$\begin{cases} x''(t) = Ax(t) + F(x(t)) + u(t), \\ x(0) = x_0 \in V, \quad x'(0) = x_1 \in H, \quad 0 \leq t \leq T. \end{cases}$$

This problem can be reformulated as

(4.2) $$y'(t) = \mathcal{A}y(t) + \mathcal{F}(y(t)) + Bu(t)$$
$$0 \leq t \leq T, \quad y(0) = y_0, \text{ where } y(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}, \quad y_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

Let $C(t)$, $t \in R$ be the cosine family generated by A and $S(t)x = \int_0^t C(s)x ds$, $x \in V$, $t \in R$. Let $V(t)$ be the group generated by $\mathcal{A}$. It is easy to see that

(4.3) $$V(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}.$$

We apply Theorem 1 to obtain local controllability for (4.2), equivalently for (4.1). The mild solution of (4.2) is given by

(4.4) $$y(t) = V(t)y_0 + \int_0^t V(t-s)\mathcal{F}(y(s))ds + \int_0^t V(t-s)Bu(s)ds.$$

Taking into account (4.3) we obtain

(4.5) $$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)F(x(s))ds + \int_0^t S(t-s)u(s)\,ds.$$

The solution of (4.5) is the mild solution of (4.1). Let $q > 0$ be given. Let $[0, T]$ be an interval where there exists the solution of (4.5) for

$$|u(t)| \leqslant q \text{ a.e. on } [0, T].$$

Such intervals exists as a consequence of Proposition 1 applied to (4.4). If we define Φ_T as in (3.1), taking into account that $d\mathcal{F}((x_1, x_2) = \begin{pmatrix} 0 \\ dF(x_1) \end{pmatrix}$ and $F(0)=0$, we obtain $d\Phi_T(0)v = (z(T), z'(T))$ where $z(\,.\,)$ is the solution of

(4.6) $$z(t) = \int_0^t S(t-s)dF(0)z(s)ds + \int_0^t S(t-s)v(s)ds.$$

In order to apply Theorem 1 we must prove that $d\Phi_T(0)$ is onto. We need the following:

Lemma 3. *Let $z_0 \in V$, $z_1 \in H$ and $T > 0$. Then there exists $u \in L^\infty(0, T; H)$ such that $z(t) = \int_0^t S(t-s)u(s)ds$ satisfies $z(T) = z_0$, $z'(T) = z_1$.*

Postponing for a moment verification of this lemma and using the fact that $dF(0)$ is continuous we obtain that the control

(4.7) $$v(t) = u(t) - dF(0)z(t)$$

belongs to $L^\infty(0, T; H)$ and $z(t)$ satisfied (4.6) with $z(T) = z_0$, $z'(T) = z_1$.

The proof of Lemma 3 can be carried out essentially in the same way as that of Theorem 3.1 in [2], but Lemma 3 is not, strictly speaking, contained in this result. We sketch its proof here.

Using the invariance of the equation $z''(t) = Az(t) + u(t)$ with respect to time reversal, it is sufficiently to prove that, if $z_0 \in V$, $z_1 \in H$, then there exists $u(\,.\,) \in L^\infty(0, T; H)$ such that

(4.8) $$z(t) = C(t)z_0 + S(t)z_1 + \int_0^t S(t-s)u(s)ds$$

satisfies $z(T) = 0$, $z'(T) = 0$.

Consider $x_0, x_1 \in D(A)$ and $v(t) = \varphi(t)Ax_0 - \varphi''(t)x_0 + \psi(t)Ax_1 - \psi''(t)x_1$, where $\varphi(\,.\,)$, $\psi(\,.\,)$ are twice continuously differentiable scalar functions in $0 \leqslant t \leqslant T$, such that,

$$\varphi(0) = 0,\ \varphi(T) = -1,\ \varphi'(0) = 0,\ \varphi'(T) = 0,$$
$$\psi(0) = 0,\ \psi(T) = 0,\ \psi'(0) = 0,\ \psi'(T) = -1.$$

Then integration by parts shows that

(4.9) $$\int_0^T S(T-t)v(t)dt = x_0\,; \quad \int_0^T C(T-t)v(t)dt = x_1\,.$$

Clearly, $v(\,.\,)$ belongs to $L^\infty(0, T; H)$. Let $z_0, z_1 \in V$ and $u(t) = -\frac{2}{T}C(t)z_1 - \frac{2}{T}AS(t)z_0 + u_1(t)$ where $u_1(t)$ satisfies (4.9) with $x_0 = -\frac{1}{T}S(T)z_0$ and $x_1 = \frac{1}{T}S(T)z_1$.

We obtain

(4.10) $$\int_0^T S(T-t)u(t)dt + C(T)z_0 + S(T)z_1 = 0,$$

(4.11) $$\int_0^T C(T-t)u(t)dt + AS(T)z_0 + C(T)z_1 = 0.$$

If $z_0 \in V$, $z_1 \in H$, we define the invertible operator

$$N(T)x = \int_0^T C(t)^2 x\,dt,\ x \in V.$$

In this case consider $u_1(t) = u_2(t) + u_3(t)$, where

$$u_2(t) = \frac{1}{T}C(T-t)N(T)^{-1}S(T)z_1$$

and $u_3(t)$ satisfies (4.9) with $x_0 = -\frac{1}{T} S(T)z_0 - \int_0^T S(T-t)\, u_2(t)\, dt$ and $x_1 = 0$. We obtain once again (4.10) and (4.11). It is clear that $u(\,.\,) \in L^\infty(0, T; H)$. (see [2] for further details).

We obtain then the following

Corollary 1. There exists $\varepsilon > 0$, such that for every $x_0 \in V$, $x_1 \in H$ satisfying $\|x_0\| \leq \varepsilon$, $|x_1| < \varepsilon$, there exists $u(\,.\,) \in L^\infty(0, T; H)$, such that the solution of

$$x(t) = \int_0^t S(t-s)F(x(s))ds + \int_0^t S(t-s)u(s)ds, \quad 0 \leq t \leq T, \tag{4.12}$$

satisfies $x(T) = x_0$, $x'(T) = x_1$.

We can apply Corrollary 1 to the Klein–Gordon equation

$$\Box\, x(t) + m^2 x(t) + K|x(t)|^{p-1} x(t) = u(t),$$

$$\Box = -\frac{d^2}{dt^2} + \Delta, \ K > 0.$$

In this case Ω is a bounded domain in Euclidian space R^n, $n \geq 1$, with sufficiently smooth boundary Γ. $A = \Delta$, $H = L^2(\Omega)$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, $V = H_0^1(\Omega)$, $n \leq \frac{2p}{p-1}$, $F: V \to H$ is defined by $F(x)(t) = m^2 x(t) + K|x(t)|^{p-1} x(t)$. $H_0^1(\Omega)$ and $H^2(\Omega)$ are usually Sobolev spaces.

Remark. If we consider the control $u(\,.\,) \in L^2(0, T; Y)$, the results of §2 hold. Clearly, the constraint (2.2) becomes $\|u\|_2 \leq q$, where $\|\,.\,\|_2$ denotes the norm in $L^2(0, T; Y)$.

Let A be the operator in $L^2(0, L)$ defined by $Ay(x) = y''(x)$ with domain $D(A)$ consisting of all functions y, continuously differentiable in $[0, L]$, such that y' is absolutely continuous, $y'' \in L^2(0, L)$ and satisfying one of the following conditions

a. $y(0) = y(L) = 0$,

b. $y(0) = y'(L) = 0$,

c. $y'(0) = y'(L) = 0$.

In this case $Bu = \begin{pmatrix} 0 \\ bu \end{pmatrix}$, $b \in H$, $u \in L^2(0, T)$. (see [1], [5])

2. The group case. We consider A the infinitesimal generator of a strongly continuous group, $T(t)$, in X (or a semigroup which satisfies $T(t)X = X$). Let $B = I$ be the identity operator in X and F satisfies condition II. Then the pair (A, I) is completely controllable (see e.g. [3], [4]).

Using an argument similar to the one that led to (4.7), we obtain that the pair $(A + F(0), I)$ is completely controllable. Thus, in this case condition III is instantly verified.

We can apply the results obtained to the Schrödinger equation

$$x'(t) + i\Delta x(t) + F(x(t)) = u(t)$$

In this case $A = -i\Delta$, $X = (L^2(\Omega))^2$. An element $x \in X$ has the following form $x = (x_1, x_2) = (\mathcal{R}e\, x, \mathcal{I}m.\, x)$ F satisfies condition II.

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Faculty of Mathematics
University of Iași
R. S Romania