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ON SOME INTEGRODIFFERENTIAL INEQUALITIES SIMILAR TO WENDROFF'S INEQUALITY

BY

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1. Introduction. In the study of qualitative properties of solutions of differential and integral equations, it is well known that the classical Gronwall inequality [4] has been a very powerful and versatile technique. Unfortunately, the two independent variable generalization of this inequality given by Wendroff [1. p. 154] seems to have been overlooked by the researches in integral inequalities and their applications. Wendroff's inequality, which has its origin in the field of partial differential equations has recently evoked lively interest as may be seen from the papers of Snow [11, 12], Ghoshal and Masood [2, 3], and Pchpatte [6]— [9] which are motivated by certain applications in the theory of hyperbolic partial differential and integrodifferential equations. Our objective here is to present some new partial integrodifferential inequalities similar to Wendroff's inequality which can be used in some problems in the theory of hyperbolic partial integrodifferential equations.

2. Main results. In this section we state and prove our main results on integrodifferential inequalities in two independent variables which claim the following as their origin.

Lemma (Wendroff [1, p. 154]). *Let $\Phi(x, y)$ and $b(x, y)$ be nonnegative continuous functions defined for $x \geqslant 0$, $y \geqslant 0$, for which the inequality:*

$$\Phi(x, y) \leqslant a + \int_0^x \int_0^y b(s, t)\, \Phi(s, t)\, ds\, dt,$$

holds for all $x \geqslant 0$, $y \geqslant 0$, where a is a nonnegative constant. Then

$$\Phi(x, y) \leqslant a \exp\left(\int_0^x \int_0^y b(s, t)\, ds\, dt\right),$$

for all $x \geqslant 0$, $y \geqslant 0$.

In [2, 3], [11, 12], the authors have obtained some useful generalizations of this Lemma by using Riemann's functions. Recently, in a series of papers [6]— [9], the present author has established a number of new integral and integrodifferential inqualities of the Wendroff type which can be used in the theory of partial integrodifferential equations as handy tools. Our results given below are partly motivated by the integrodifferential inequalities recently established by this author in [5].

A useful two independent variable integrodifferential inequality of the Wendroff type is embodied in the following theorem.

Theorem 1. *Suppose* $\Phi(x, y)$, $\partial^2\Phi(x, y)/\partial x\,\partial y$, $\partial^4\Phi(x, y)/\partial x^2\partial y^2$, $a(x, y)$, $b(x, y)$ *and* $c(x, y)$ *are nonnegative continuous functions defined on a domain* D *and* $\Phi(x, y_0) = \Phi(x_0, y) = 0$. *Let* $P_0(x_0, y_0)$ *and* $P(x, y)$ *be two points in* D *such that* $(x - x_0)(y - y_0) > 0$ *and let* R *be the rectangular region whose opposite corners are the points* P_0 *and* P. *Let* $v(s, t; x, y)$ *be the solution of the characteristic initial value problem*

$$(1)\qquad L[v] = \frac{\partial^2 v}{\partial s\,\partial t} - [1 + b(s, t) + c(s, t) + b(s, t)\,c(s, t)]\,v = 0,$$

$$v(s, y) = v(x, t) = 1,$$

and let D^+ *be a connected subdomain of* D *which contains* P *and on which* $v > 0$ (see Fig. 1). *If* $R \subset D^+$ *and* $\Phi(x, y)$ *satisfies*

$$(2)\qquad \frac{\partial^4\Phi(x, y)}{\partial x^2\,\partial y^2} \leqslant a(x, y) + b(x, y)\left[\Phi(x, y) + \frac{\partial^2\Phi(x, y)}{\partial x\,\partial y} + \right.$$

$$\left. + \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\left(\Phi(s, t) + \frac{\partial^2\Phi(s, t)}{\partial s\,\partial t} + \frac{\partial^4\Phi(s, t)}{\partial s^2\,\partial t^2}\right) ds\,dt\right],$$

then $\Phi(x, y)$ *also satisfies*

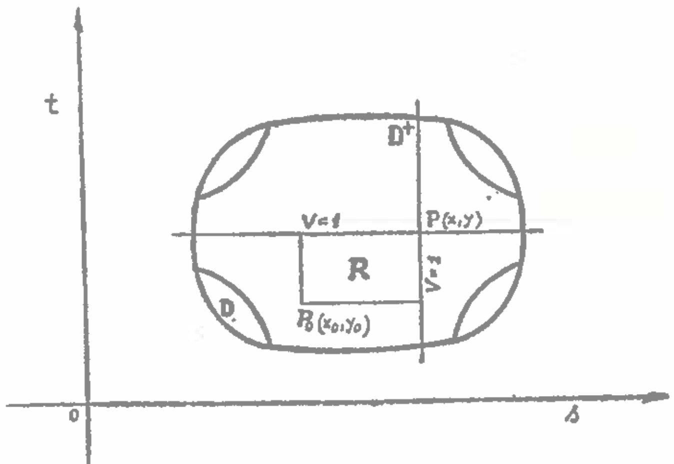

Fig. 1

$$(3)\qquad \frac{\partial^4\Phi(x, y)}{\partial x^2\,\partial y^2} \leqslant a(x, y) + b(x, y)\left[\int_{x_0}^{x}\int_{y_0}^{y} a(s, t)\,[1 + \right.$$

$$\left. c(s, t)]\,v(s, t; x, y)\,ds\,dt\right].$$

The proof of this theorem is obtained by reducing the integrodifferential inequality to a differential inequality and then integrating it by Riemann's method for hyperbolic partial differential equations [10]. The function $v(s, t; x, y)$ involved in theorem is a Riemann function relative to the point $P(x, y)$ for the selfadjoint operator L. There is such a function and a domain D^+ on which $v > 0$ since $v = 1$ on the vertical and horizontal lines through P and since v is continuous.

Proof. Define

$$(4)\qquad u(x, y) = \Phi(x, y) + \frac{\partial^2\Phi(x, y)}{\partial x\,\partial y} + \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\left(\Phi(s, t) + \right.$$

$$\left. + \frac{\partial^2\Phi(s, t)}{\partial s\,\partial t} + \frac{\partial^4\Phi(s, t)}{\partial s^2\,\partial t^2}\right) ds\,dt, \quad u(x, y_0) = u(x_0, y) = 0,$$

then

$$(5)\qquad \frac{\partial^2 u(x, y)}{\partial x\,\partial y} = \frac{\partial^2\Phi(x, y)}{\partial x\,\partial y} + \frac{\partial^4\Phi(x, y)}{\partial x^2\,\partial y^2}$$

$$+ c(x, y)\left(\Phi(x, y) + \frac{\partial^2\Phi(x, y)}{\partial x\,\partial y} + \frac{\partial^4\Phi(y, x)}{\partial x^2\,\partial y^2}\right).$$

Using the facts that $\partial^2\Phi/\partial x\,\partial y \leqslant u(x, y)$ and $\Phi(x, y) + \partial^2\Phi(x, y)/\partial x\partial y \leqslant u(x, y)$ from (4) and $\partial^4\Phi(x, y)/\partial x^2\,\partial y^2 \leqslant a(x, y) + b(x, y)\,u(x, y)$ from (2) in (5) we see that the inequality

$$(6)\qquad L[u] = \frac{\partial^2 u(x, y)}{\partial x\,\partial y} - [1 + b(x, y) + c(x, y) +$$

$$+ b(x, y)\,c(x, y)]\,u(x, y) \leqslant a(x, y)\,[1 + c(x, y)],$$

is satisfied.

The operator L is self-adjoint and hyperbolic. For any twice continuously differentiable u and v the operator L satisfies the identity

$$(7)\qquad vL[u] - uL[v] = -\frac{\partial}{\partial x}\left(u\,\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(v\,\frac{\partial u}{\partial x}\right).$$

Let P_0 and P be any points as in the theorem and label the directed sides and corners of the rectangle R as shown in Fig. 2.

Using s and t as independent variables, we integrate the identity (7) over R and use Green's theorem to obtain

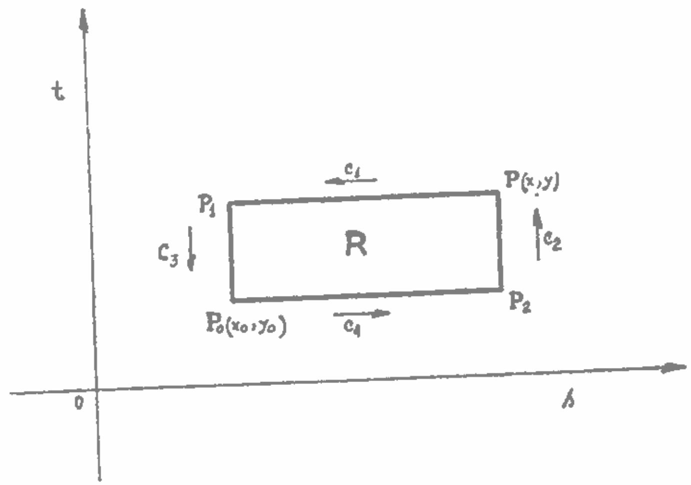

Fig. 2

$$\iint_R (vL[u] - uL[v])\,ds\,dt = -\int_{c_1+c_2+c_3+c_4} v\frac{\partial u}{\partial s}\,ds + u\frac{\partial v}{\partial t}\,dt.$$

This holds for any functions in C^2.

For the particular function u defined earlier we have $u = 0$ on c and $u = \partial u/\partial s = 0$ on c_4; so the right hand side reduces to

$$-\int_{c_1} v\frac{\partial u}{\partial s}\,ds - \int_{c_2} u\frac{\partial v}{\partial t}\,dt. \tag{8}$$

Now suppose v satisfies

$$L[v] = \frac{\partial^2 v}{\partial s\,\partial t} - [1 + b(x,y) + c(x,y) + b(x,y)\,c(x,y)]\,v = 0, \tag{9}$$

$$v = 1 \text{ on } c_1, \tag{10}$$

$$\frac{\partial v}{\partial t} = 0 \text{ on } c_3. \tag{11}$$

Then (10) and (11) imply that

$$v = 1 \text{ on } c_2. \tag{12}$$

Since $v \geqslant 0$ on R and $u(p_1) = 0$, by using (6) identity (8) becomes

$$u(x,y) \leqslant \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)\,[1 + c(s,t)]\,v(s,t;x,y)\,ds\,dt.$$

Now substituting this value of $u(x,y)$ in (2) we obtain the desi[red] bound in (3).

Another interesting and useful integrodifferential inequality of the Wendroff type may be stated as follows.

Theorem 2. *Suppose* $\Phi(x,y)$, $\partial^2\Phi(x,y)/\partial x\,\partial y$, $\partial^4\Phi(x,y)/\partial x^2\partial y^2$, $(\Phi \in C^4)$ *and* $a(x,y)$ *are nonnegative functions defined on a domain* D; $\Phi(x,y_0) = \Phi(x_0,y) = 0$, *and* $b(x,y) \geqslant 1$, $c(x,y) \geqslant 1$ *are continuous functions defined on a domain* D. *Let* $P_0(x_0,y_0)$ *and* $P(x,y)$ *be two points in* D, *such that* $(x - x_0)(y - y_0) > 0$ *and let* R *be the rectangular region whose opposite corners are the points* P_0 *and* P. *Let* $w(s,t;x,y)$ *be the solution of the characteristic initial value problem*

$$M[W] = \frac{\partial^2 W}{\partial s\,\partial t} - b(s,t)\,[1 + 2c(s,t)]\,W = 0, \quad W(s,y) = W(t,x) = 1, \tag{13}$$

and let D^+ *be a connected subdomain of* D *which contains* P *and on which* $W > 0$ (see Fig. 1). *If* $R \subset D^+$ *and* $\Phi(x,y)$ *satisfies*

$$\frac{\partial^4\Phi(x,y)}{\partial x^2\,\partial y^2} \leqslant a(x,y) + b(x,y)\left[\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\left(\Phi(s,t) + \frac{\partial^2\Phi(s,t)}{\partial s\,\partial t} + \frac{\partial^4\Phi(s,t)}{\partial s^2\,\partial t^2}\right)ds\,dt\right], \tag{14}$$

then $\Phi(x,y)$ *also satisfies*

$$\frac{\partial^4\Phi(x,y)}{\partial x^2\,\partial y^2} \leqslant a(x,y) + b(x,y)\left[\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,(a(s,t) + b(s,t)\,Q(s,t))\,ds\,dt\right], \tag{15}$$

where

$$Q(x,y) = \int_{x_0}^{x}\int_{y_0}^{y}\left[a(s,t)\,[1 + c(s,t)] + 2b(s,t)\,c(s,t)\cdot\int_{x_0}^{s}\int_{y_0}^{t} a(m,n)\,[2 + c(m,n)]\,W(m,n;s,t)\,dm\,dn\right]ds\,dt. \tag{16}$$

Proof. Define

$$u(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\left(\Phi(s,t) + \frac{\partial^2\Phi(s,t)}{\partial s\,\partial t} + \frac{\partial^4\Phi(s,t)}{\partial s^2\,\partial t^2}\right)ds\,dt, \tag{17}$$

$$u(x,y_0) = u(x_0,y) = 0,$$

then (14) can be restated as

$$\frac{\partial^4\Phi(x,y)}{\partial x^2\,\partial y^2} \leqslant a(x,y) + b(x,y)\,u(x,y). \tag{18}$$

Differentiating (17), first with respect to x and then with respect t y we have

$$\frac{\partial^2 u(x, y)}{\partial x\, \partial y} = c(x, y)\left(\Phi(x, y) + \frac{\partial^2 \Phi(x, y)}{\partial x\, \partial y} + \frac{\partial^4 \Phi(x, y)}{\partial x^2\, \partial y^2}\right). \tag{19}$$

Integrating both sides of (18) with respect to y first from y_0 to y an then with respect to x from x_0 to x we have

$$\frac{\partial^2 \Phi(x, y)}{\partial x\, \partial y} \leqslant \int_{x_0}^{x}\int_{y_0}^{y} [a(s, t) + b(s, t)\, u(s, t)]\, ds\, dt. \tag{20}$$

Again integrating both sides of (20) with respect to y first from y_0 t y and then with respect to x from x_0 to x we have:

$$\Phi(x, y) \leqslant \int_{x_0}^{x}\int_{y_0}^{y}\left(\int_{x_0}^{s}\int_{y_0}^{t} [a(m, n) + b(m, n)\, u(m, n)]\, dm\, dn\right) ds\, dt. \tag{21}$$

Now, using (18), (20), (21), and $b(x, y) \geqslant 1$ in (19) we obtain

$$\frac{\partial^2 u(x, y)}{\partial x\, \partial y} \leqslant c(x, y)\left[a(x, y) + b(x, y)\left\{u(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} [a(s, t) + \right.\right.$$

$$\left.\left. + b(s, t)\, u(s, t)]\, ds\, dt + \int_{x_0}^{x}\int_{y_0}^{y}\left(\int_{x_0}^{s}\int_{y_0}^{t} [a(m, n) + b(m, n) u(m, n)] dm dn\right) ds\, dt\right\}\right] \tag{22}$$

Define

$$r(x, y) = u(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} [a(s, t) + b(s, t)\, u(s, t)]\, ds\, dt +$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y}\left(\int_{x_0}^{s}\int_{y_0}^{t} [a(m, n) + b(m, n) u(m, n)] dm\, dn\right) ds\, dt,\ r(x, y_0) = r(x_0, y) = \tag{23}$$

then we have

$$\frac{\partial^2 r(x, y)}{\partial x\, \partial y} = \frac{\partial^2 u(x, y)}{\partial x\, \partial y} + a(x, y) + b(x, y)\, u(x, y) + \tag{24}$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y} [a(m, n) + b(m, n)\, u(m, n)]\, dm\, dn.$$

Using the facts that $\partial^2 u(x, y)/\partial x\, \partial y \leqslant c(x, y)\, [a(x, y) + b(x, y)\, r(x,$ from (22) and $u(x, y) \leqslant r(x, y)$ from (23) in (24) and since $b(x, y) \geqslant$ $c(x, y) \geqslant 1$ we see that the inequality

$$\frac{\partial^2 r(x, y)}{\partial x\, \partial y} \leqslant a(x, y)\, [1 + c(x, y)] + 2b(x, y)\, c(x, y)\, . \tag{25}$$

$$.\left[r(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} [a(m, n) + b(m, n)\, r(m, n)]\, dm\, dn\right],$$

is satisfied. If we put

$$h(x, y) = r(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} [a(m, n) + b(m, n)\, r(m, n)]\, dm\, dn, \tag{26}$$

$$h(x, y_0) = h(x_0, y) = 0,$$

then we have

$$\frac{\partial^2 h(x, y)}{\partial x\, \partial y} = \frac{\partial^2 r(x, y)}{\partial x\, \partial y} + a(x, y) + b(x, y)\, r(x, y). \tag{27}$$

Using $\partial^2 r(x, y)/\partial x\, \partial y \leqslant a(x, y)\, [1 + c(x, y)] + 2b(x, y)\, c(x, y)\, h(x, y)$ from (25) and $r(x, y) \leqslant h(x, y)$ from (26) in (27), we see that the inequality

$$M[h] = \frac{\partial^2 h(x, y)}{\partial x\, \partial y} - b(x, y)\, [1 + 2c(x, y)]\, h(x, y) \leqslant a(x, y)\, [2 + c(x, y)],$$

is satisfied.

Now by following the same steps as in the proof of Theorem 1 we obtain the estimate for $h(x, y)$ such that

$$h(x, y) \leqslant \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)\, [2 + c(s, t)]\, W(s, t\,;\, x, y)\, ds\, dt.$$

Substituting this bound on $h(x, y)$ in (25) and integrating both sides with respect to y first from y_0 to y and then with respect to x from x_0 to x we have

$$r(x, y) \leqslant \int_{x_0}^{x}\int_{y_0}^{y}\left[a(s, t)\, [1 + c(s, t)] + 2b(s, t)\, c(s, t)\, .\right.$$

$$\left.\left(\int_{x_0}^{s}\int_{y_0}^{t} a(m, n)\, [2 + c(m, n)]\, W(m, n\,;\, s, t)\, dm\, dn\right)\right] ds\, dt = Q(x, y).$$

Substituting the bound on $r(x, y)$ in (22) and integrating both sides with respect to y first from y_0 to y and then with respect to x from x_0 to x we have

$$u(x, y) \leqslant \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, [a(s, t) + b(s, t)\, Q(s, t)]\, ds\, dt.$$

Now substituting this bound on $u(x, y)$ in (14) we obtain the desir bound in (15).

We note that as in [11], Theorems 1 and 2 in the special cas when $a(x, y) = 0$, and when $a(x, y)$ and $b(x, y)$ are constants can b used in some applications. However, if the inequalities in (2) and (1 are reversed, then so are not the inequalities in (3) and (15).

3. Some Applications. In this section we present some applic tions of the inequalities established in this paper. There are many po sible applications of the inequalities established in this paper, but tho presented here are sufficient to convey the importance of our resul

Example 1. As a first application, we obtain the bound on th nonlinear hyperbolic partial integrodifferential equation

$$\frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} = f(x, y) + F\Bigg[x, y, u(x, y), \frac{\partial^2 u(x, y)}{\partial x \partial y},$$

$$\int_{x_0}^{x}\int_{y_0}^{y} g\left(x, y, s, t, u(s, t), \frac{\partial^2 u(s, t)}{\partial s \partial t}, \frac{\partial^4 u(s, t)}{\partial s^2 \partial t^2}\right) ds\, dt\Bigg], \tag{28}$$

with the boundary conditions $u(x, y_0) = u(x_0, y) = 0$, where f, F, and are continuous of their argument and such that

$$|f(x, y)| \leqslant a(x, y), \tag{29}$$

$$\left|F\left[x, y, u, \frac{\partial^2 u}{\partial x \partial y}, v\right]\right| \leqslant b(x, y)\left[|u| + \left|\frac{\partial^2 u}{\partial x \partial y}\right| + |v|\right], \tag{30}$$

$$\left|g\left(x, y, s, t, u, \frac{\partial^2 u}{\partial s \partial t}, \frac{\partial^4 u}{\partial s^2 \partial t^2}\right)\right| \leqslant c(s, t)\Bigg[|u| + \left|\frac{\partial^2 u}{\partial s \partial t}\right| + \left|\frac{\partial^4 u}{\partial s^2 \partial t^2}\right|\Bigg], \tag{31}$$

where $a(x, y)$, $b(x, y)$ and $c(x, y)$ are nonegative continuous functio of their argument. If $\Phi(x, y)$ is any solution of the equation (28), th we have

$$\frac{\partial^4 \Phi(x, y)}{\partial x^2 \partial y^2} = f(x, y) + F\Bigg[x, y, \Phi(x, y), \frac{\partial^2 \Phi(x, y)}{\partial x \partial y},$$

$$\int_{x_0}^{x}\int_{y_0}^{y} g\left(x, y, s, t, \Phi(s, t), \frac{\partial^2 \Phi(s, t)}{\partial s \partial t}, \frac{\partial^4 \Phi(s, t)}{\partial s^2 \partial t^2}\right) ds\, dt\Bigg]. \tag{32}$$

Using (29), (30), and (31) in (32) and applying Theorem 1 we hav

$$\left|\frac{\partial^4 \Phi(x, y)}{\partial x^2 \partial y^2}\right| \leqslant a(x, y) + b(x, y)\Bigg[\int_{x_0}^{x}\int_{y_0}^{y} a(s, t)\,[1 + c(s, t)]\, v(s, t\,; x, y)\, ds\, dt$$

where $v(s, t\,; x, y)$ is the solution of the characteristic initial value problem (1). Further integrating both sides of the above inequality twice with respect to y first from y_0 to y and then with respect to x from x_0 to x we obtain the bound on the solution $\Phi(x, y)$ of (28).

Example 2. As a second application, we establish the uniqueness of solutions of the nonlinear hyperbolic integrodifferential equation (28) with the same boundary conditions. We assume that the functions g and F in (28) satisfy

$$\left|g\left(x, y, s, t, u, \frac{\partial^2 u}{\partial s \partial t}, \frac{\partial^4 u}{\partial s^2 \partial t^2}\right) - g\left(x, y, s, t, u, \frac{\partial^2 \overline{u}}{\partial s \partial t}, \frac{\partial^4 \overline{u}}{\partial s^2 dt^2}\right)\right|$$

$$\leqslant c(s, t)\Bigg[|u - u| + \left|\frac{\partial^2 u}{\partial s \partial t} - \frac{\partial^2 \overline{u}}{\partial s \partial t}\right| + \left|\frac{\partial^4 u}{\partial s^2 \partial t^2} - \frac{\partial^4 \overline{u}}{\partial s^2 \partial t^2}\right|\Bigg], \tag{33}$$

$$\left|F\left[x, y, u, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^4 u}{\partial x^2 \partial y^2}, r\right] - F\left[x, y, \overline{u}, \frac{\partial^2 \overline{u}}{\partial x \partial y}, \frac{\partial^4 \overline{u}}{\partial x^2 \partial y^2}, r\right]\right| \leqslant b(x, y)\Bigg[|u - \overline{u}| + \left|\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y}\right| + \left|\frac{\partial^4 u}{\partial x^2 \partial y^2} - \frac{\partial^4 \overline{u}}{\partial x^2 \partial y^2}\right| + |r - \overline{r}|\Bigg] \tag{34}$$

for any two solutions $\overline{u}(x, y)$ and $u(x, y)$ of the given equation (28), where $b(x, y)$ and $c(x, y)$ are nonnegative continuous functions of their argument. Since $u(x, y)$ and $\overline{u}(x, y)$ are any two solutions of (28), we have

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} - \frac{\partial^4 u}{\partial x^2 \partial y^2} = F\Bigg[x, y, u, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^4 u}{\partial x^2 \partial y^2},$$

$$\int_{x_0}^{x}\int_{y_0}^{y} g\left(x, y, s, t, u, \frac{\partial^2 u}{\partial s \partial t}, \frac{\partial^4 u}{\partial s^2 \partial t^2}\right) ds\, dt\Bigg] - F\Bigg[x, y, u, \frac{\partial^2 \overline{u}}{\partial x \partial y}, \tag{35}$$

$$\frac{\partial^4 \overline{u}}{\partial x^2 \partial y^2}, \int_{x_0}^{x}\int_{y_0}^{y} g\left(x, y, s, t, \overline{u}, \frac{\partial^2 \overline{u}}{\partial s \partial t}, \frac{\partial^4 \overline{u}}{\partial s^2 \partial t^2}\right) ds\, dt\Bigg].$$

Now, if $(x - x_0)(y - y_0) \geqslant 0$, then using (33) and (34) in (35) and applying Theorem 1 when $a(x, y) = 0$, we have $|\partial^4 u/\partial x^2 \partial y^2 - \partial^4 \overline{u}/\partial x^2 \partial y^2| \leqslant 0$ and consequently we have $|u - \overline{u}| \leqslant 0$. Therefore $u = \overline{u}$; i.e. there is almost one solution of the problem.

We note that the integrodifferential inequality established in Theorem 1 can be used to study the continuous dependence of the solutions of (28) (when the left side is $\partial^6 u(x, y)/\partial x^3 \partial y^3$, with the given boundary conditions) on the equation and boundary data as in [11]. We omit the details.

In concluding this paper we note that the inequality established in Theorem 2 can be used to study the boundedness, uniqueness and continuous dependence of the solutions of a partial integrodifferential equation of the form

$$(36)\qquad \frac{\partial^4 u(x,y)}{\partial x^2\,\partial y^2} = f(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} F\left[x, y, s, t, u(s,t), \frac{\partial^2 u(s,t)}{\partial s\,\partial t}, \frac{\partial^4 u(s,t)}{\partial s^2\,\partial t^2}\right] ds\,dt,$$

under some suitable conditions on the functions involved in (36).

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SEMI-LINEAR EVOLUTION EQUATIONS WITH MULTIVALUED RIGHT HAND SIDE IN BANACH SPACES

BY

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1. Introduction In the present paper we establish via compacity arguments some general existence results, extending those of Pavel [15], Pazy [17] and Schiaffino [19]. In particular, we obtain an existence theorem for the perturbed heat equation, as well as another one for the closed loop systems that appear in some control problems as those studied by Barbu in [4].

We would like to point out that our main result, Theorem 2.1., may be related to a result of Attouch [1] and to a theorem from Castaing and Valadier [8, Theorem VI.13.]. There are also several papers in some connection with the present one, which from we mention only Cellina [9] and Crandall and Nohel [11]. See also Crandall [10].

We begin with the basic notations and definitions used in the sequel.

In all what follows, X is a real Banach space with the norm denoted by $\|\cdot\|$, while X^* is its dual. If $x \in X$ and $r > 0$, denote by $S_X(x, r)$ the open ball with center x and radius r and by $B_X(x, r)$ the closed ball with center x and radius r. When no confusion can arise, we shall write $S(x, r)$ and $B(x, r)$ instead of $S_X(x, r)$ and $B_X(x, r)$.

If $x, y \in X$ and $A \subset X$, denote by $d(x, y)$ the distance between x and y and by $d(x, A)$ the distance between x and A. If F_1 and F_2 are two nonempty, bounded sets in X, denote by $\rho(F_1, F_2)$ the Hausdorff distance between F_1 and F_2, i.e.

$$(1.1)\qquad \rho(F_1, F_2) = \inf\{r > 0 ; F_1 \subset \bigcup_{x \in F_2} S(x, r),\ \ F_2 \subset \bigcup_{x \in F_1} S(y, r)\}.$$

It follows from (1.1), that $\rho(F_1, F_2) = 0$ if and only if F_1 and F_2 have the same closure. An excellent survey of the basic properties of the Hausdorff distance can be found in [8, Chapter III].

A set $G \subset X$ is called *semi locally closed* if

1) G can be represented as follows;

$$G = \bigcup_{\varepsilon \in]0,1[} D_\varepsilon ;$$

(2) *for each* $x \in G$, *there exists* $r > 0$ *such that* $B(x, r) \cap D_\varepsilon$ *is closed in* X *for all* $\varepsilon \in]0, 1[$;