dary conditions) on the equation and boundary data as in [11]. We omit the details.

In concluding this paper we note that the inequality established in Theorem 2 can be used to study the boundedness, uniqueness and continuous dependence of the solutions of a partial integrodifferential equation of the form

$$(36)\qquad \frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} = f(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} F\left[x, y, s, t, u(s, t), \frac{\partial^2 u(s, t)}{\partial s\, \partial t}, \frac{\partial^4 u(s, t)}{\partial s^2\, \partial t^2}\right] ds\, dt,$$

under some suitable conditions on the functions involved in (36).

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Received 15.I.1978

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Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXV, s. I a, 1979, f. 1

SEMI-LINEAR EVOLUTION EQUATIONS WITH MULTIVALUED RIGHT HAND SIDE IN BANACH SPACES

BY

NICOLAE H. PAVEL and IOAN I. VRABIE

1. Introduction In the present paper we establish via compacity arguments some general existence results, extending those of Pavel [15], Pazy [17] and Schiaffino [19]. In particular, we obtain an existence theorem for the perturbed heat equation, as well as another one for the closed loop systems that appear in some control problems as those studied by Barbu in [4].

We would like to point out that our main result, Theorem 2.1., may be related to a result of Attouch [1] and to a theorem from Castaing and Valadier [8, Theorem VI.13.]. There are also several papers in some connection with the present one, which from we mention only Cellina [9] and Crandall and Nohel [11]. See also Crandall [10].

We begin with the basic notations and definitions used in the sequel.

In all what follows, X is a real Banach space with the norm denoted by $\|\cdot\|$, while X^* is its dual. If $x \in X$ and $r > 0$, denote by $S_X(x, r)$ the open ball with center x and radius r and by $B_X(x, r)$ the closed ball with center x and radius r. When no confusion can arise, we shall write $S(x, r)$ and $B(x, r)$ instead of $S_X(x, r)$ and $B_X(x, r)$.

If $x, y \in X$ and $A \subset X$, denote by $d(x, y)$ the distance between x and y and by $d(x, A)$ the distance between x and A. If F_1 and F_2 are two nonempty, bounded sets in X, denote by $\rho(F_1, F_2)$ the Hausdorff distance between F_1 and F_2, i.e.

$$(1.1)\qquad \rho(F_1, F_2) = \inf\{r > 0\,;\, F_1 \subset \bigcup_{x\in F_2} S(x, r),\ \ F_2 \subset \bigcup_{x\in F_1} S(y, r)\}.$$

It follows from (1.1), that $\rho(F_1, F_2) = 0$ if and only if F_1 and F_2 have the same closure. An excellent survey of the basic properties of the Hausdorff distance can be found in [8, Chapter III].

A set $G \subset X$ is called *semi locally closed* if

1) G can be represented as follows;

$$G = \bigcup_{\varepsilon \in]0, 1[} D_\varepsilon\,;$$

(2) *for each* $x \in G$, *there exists* $r > 0$ *such that* $B(x, r) \cap D_\varepsilon$ *is closed in* X *for all* $\varepsilon \in]0, 1[$;

(3) *for each* $r > 0$, $\varepsilon_0 \in]0, 1[$ *and* $x \in D_{\varepsilon_0}$, *there exists* $\delta > 0$, *such that the mapping* $\varepsilon \to B(x, r) \cap D_\varepsilon$ *is continuous in the Hausdorff metric on* $]\varepsilon_0 - \delta, \varepsilon_0 + \delta[$.

Let $\varphi : X \to]-\infty, +\infty]$ be a convex and lower semicontinuous function and $x \in X$. The function $\varphi^* : X^* \to]-\infty, +\infty]$ defined by:

$$\varphi^*(x^*) = \sup \{ <x, x^*> - \varphi(x) ;\ x \in X\}$$

is called the *conjugate* of φ. It is also well known, that φ^* is always convex and lower semicontinuous on X^* and if in addition X is reflexive, then $x^* \in \partial\varphi(x)$ if and only if $x \in \partial\varphi^*(x^*)$.

Consider $L : X \times U \to]-\infty, +\infty]$, a convex and lower semicontinuous function, where U is another Banach space, while X is as above. We shall use the notation:

$$L(x, u) = [\partial_x L(x, u),\ \partial_u L(x, u)],$$

where $\partial_x L(x, u) \subset X$ and $\partial_u L(x, u) \subset U$. It should be noted that generaly, $\partial_x L(x, u)$ and $\partial_u L(x, u)$ are not the subdifferentials of L with respect to x and u, excepting some particular cases, as for example the case in which L is everywhere finite and Gâteaux differentialeble at (x, u), when $\partial_x L(x, u) = \nabla_x L(x, u)$ and $\partial_u L(x, u) = \nabla L(x, u)$. The reader is referred to [3, Chapter 2] for further properties of convex functions defined on infinite dimensional spaces.

Let $A : X \to X$ be the infinitesimal generator of a C_0 semigroup of linear operators $S(t)$, with $S(t)$ compact for all $t > 0$. It is well known that $S(t)$ is compact if and only if the mapping $t \to S(t)$ is continuous in the uniform operator topology on $]0, +\infty[$ and the resolvent $R(\lambda; A) = (\lambda I - A)^{-1}$ is compact for some $\lambda > 0$. (See for instance P a z y [16]). For the definition and the basic properties of the linear C_0 semigroups, we refer the reader to [5, Chapter I, §3].

Consider the following problem:

$$\dot{x}(t) \in Ax(t) + F(t, x(t)), \quad a \leq t < T, \quad x(a) = x_0, \tag{1.3}$$

where A and F are as above and $x_0 \in G$.

A *mild solution* of (1.3) is a continuous function $x : [a, T[\to G$ such that there exists a strongly measurable selection f of $F(\cdot, x(\cdot))$ on $[a, T[$, i.e. $f(s) \in F(s, x(s))$ almost everywhere on $[a, T[$, f is strongly measurable from $[a, T[$ to X and the functions x, f verify together

$$x(t) = S(t-a)\,x_0 + \int_a^t S(t-s) f(s) ds, \quad a \leq t < T. \tag{1.4}$$

The equation (1.4) is called the *mild form* of (1.3). It is obvious that in the particular case of single-valued mapping F, this definition yields to the classical definition of the *mild solution* due to B r o w d e r [6].

Our main results are concerned with the problem of local existence of mild solutions for (1.3).

In the next Section, we shall formulate our theorems and we shall give some technical results used in the proofs.

2. Statement of the main theorems. We begin this Section with the basic assumptions we need throughout the paper.

(A_1) *X is a real reflexive Banach space.*

(A_2) *$A : X \to X$ generates a C_0 semigroup of linear operators $S(t)$, with $S(t)$ compact for all $t > 0$.*

(A_3) *$G = \bigcup_{\varepsilon \in]0,1[} D_\varepsilon$ is a semi locally closed set in X.*

(A_4) *The mapping $F : [a, b[\times G \to 2^X$ is nonempty, convex and weakly compact valued. Moreover, F is demiclosed and locally bounded.*

(A_5) *For each $x \in G$, for each $t \in [a, b[$ and for each $\varepsilon \in]0,1[$ such that $x \in D_\varepsilon$, the following condition:*

$$\lim_{h \to 0_+} \frac{1}{h} d\left(S(h)\,x + \int_t^{t+h} S(t+h-s) f(t, x)\, ds ;\ D_{\varepsilon + SMh}\right) = 0 \tag{2.1}$$

holds; here $S = \sup \{\|S(t)\| ;\ t \in [0, b-a]\}$, M is a fixed upper bound of F in a neighborhood of (t, x) and $f(t, x)$ is arbitrary in $F(t, x)$.

It is obvious that (A_5) is automatically satisfied if G is, for instance, an open set in X and F verifies (A_4). Moreover, if G is a locally closed set in X and $F : [a, b[\times G \to X$ is single-valued, then (A_5) becomes the condition used in [15], by taking $D_\varepsilon = D$ for all $\varepsilon \in]0, 1[$.

Let us remark also, that (2.1) is equivalent to:

$$\lim_{h \to 0_+} \frac{1}{h} d(S(h)\,x + hf(t, x) ;\ D_{\varepsilon + SMh}) = 0. \tag{2.2}$$

We need also the following hypotheses:

(B_1) *X is a real Banach space and $Y \subset X$ is a continuous included subspace of X which is also a Banach space;*

(B_2) *$A : X \to X$ satisfies (A_2) and the restriction of $S(t)$ to Y is a C_0 semigroup of linear operators on Y (not necessarily compact).*

(B_3) *$F : [a, b[\times S_Y(0, r_0) \to Y$ is a single-valued and continuous function in both the norm of X and the norm of Y*[2]

Now, we are able to formulate the main results of this paper.

Theorem 2.1. *Suppose that (A_1), (A_2), (A_3), (A_4) and (A_5) are satisfied. Then, for each $x_0 \in G$, there exists $T \in]a, b[$ such that the problem (1.3) has at least a mild solution on $[a, T[$.*

Theorem 2.2. *Suppose that (B_1), (B_2) and (B_3) are satisfied.*

Then, for each $x_0 \in S_Y(0, r_0)$, there exists $T \in]a, b[$, such that the problem (1.3) has at least a mild solution on $[a, T[$.

Before proceeding to the proof of Theorem 2.1, we point out, what our results become in some important particular cases.

We note that in the case in which $F : [a, b[\times G \to X$ is single-valued, G being a locally closed set in X, our Theorem 2.1. improves the "if" part of the main thorem of P a v e l [15, Theorem 1.1].

If in addition, we suppose that G is open in X, Theorem 2.1. yields to an existence result which improves the main theorem due to P a z y [17, Theorem 2.1].

It should be noted that in these two last theorems, F is supposed to be only *demiclosed* and *locally bounded*, conditions which can easily be derived from the continuity assumptions on F used by P a z y [17] and by P a v e l [15].

We would like also to point out that our second result, Theorem 2.2) is more general than Pazy's theorem quoted above, as we assume the compactness of $S(t)$ only in X but not in Y and thus, this theorem is more convenient in some practical situations.

We note that the idea of assuming the compactness of $S(t)$ is due to P a z y [17] (See also A t t o u c h [1]), while the method of proof consists in the construction of a sequence of approximations of the solution, by using the techniques of C a r t a n [7, Chapter II, Theorem 1.3.1], M a r t i n [13, 14] and those of W e b b [21], under the form developed in [15] by P a v e l.

Finally, we shall state for easy references, two technical results that we need in the sequel.

Lemma 2.6. *Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of nonempty, closed and bounded sets in X and let $\{x_n\}_{n=0}^{\infty}$ be a sequence of elements of X strongly convergent to x. If $\{A_n\}_{n=0}^{\infty}$ converges in the Hausdorff metric to A and $d(x\,;A)\leqslant\bar{\varepsilon}_0$, then there exists $n(\bar{\varepsilon}_0)\in N$, such that for all $n\geqslant n(\bar{\varepsilon}_0)$, the following inequality:*

$$d(x_n\,;A_n)\leqslant 2\bar{\varepsilon}_0$$

holds.

and the following result due to K a t o [12]:

Lemma 2.7. *Let X be a reflexive real Banach space and let $\{u_n\}_{n=1}^{\infty}$ be a sequence of $L^p(a,T\,;X)$, $p>1$, such that $u_n(t)$ is bounded for almost all $t\in[a,T]$. Let $Y(t)$ be the set of all weak cluster points of $u_n(t)$. If u_n converges weakly in $L^p(a,T\,;X)$ to u, then: $u(t)\in\overline{\operatorname{conv}}\,Y(t)$ almost everywhere on $[a,T]$.*

The next two sections are concerned with the proof of Theorem 2.1.

3. Proof of theorem 2.1. First part. We divided the first part of the proof of Theorem 2.1 in several steps.

First step. Construction of the approximations of the solution. Let $x_0\in G$. Then by (A_3), it follows that there exists $\varepsilon_0\in\,]0,1[$ such that $x_0\in D_{\varepsilon_0}$. As D_{ε_0} is locally closed, there exists $r>0$ such that $B(x_0,r)\cap D_{\varepsilon_0}$ is closed in X. From the fact that F is locally bounded, it follows that one can choose r small enough, and $\tilde{t}>a$ in order that the restriction of F to $[a,\tilde{t}]\times B(x_0\,,r)\cap D_\varepsilon$ is bounded by $M-1>0$ for all $\varepsilon\in\,]0,1[$. More precisely one has: $\|F(t,x)\|\leqslant M-1$ for all $t\in[a,\tilde{t}]$ and all $x\in B(x_0\,,r)\cap D_\varepsilon$ where $\varepsilon\in\,]0,1[$ and $\|F(t,x)\|$ is given by (1.2). Choose $\tilde{T}\in\,]a,\tilde{t}]$ such that:

$$\varepsilon_0+MT_0\leqslant 1 \tag{3.1}$$

and

$$\max_{0\leqslant t\leqslant T_0}\|S(t)\,x_0-x_0\|+T_0M\leqslant r, \tag{3.2}$$

where $T_0=\tilde{T}-a$. Let us remark that $\tilde{T}$ can be chosen small enough, such that in addition, for each $\varepsilon\in[\varepsilon\,,\ \varepsilon_0+T_0]$, $B(x_0\,,r)\cap D_\varepsilon$ is closed and the mapping $\varepsilon\to B(x_0\,,r)\cap D_\varepsilon$ is continuous in the Hausdorff metric on $[\varepsilon_0\,,\ \varepsilon_0+T_0]$. One may suppose without loss of generality that:

$$\sup_{0\leqslant t\leqslant T_0}\|S(t)\|\leqslant 1 \tag{3.3}$$

Fixe a natural number n and define the approximation x_n as follows. Put $t_0^n=a$, $x_0^n=x_0$ and suppose that we have constructed x_n on $[a,t_i^n]$, where $x_n(t_j^n)=x_j^n$ with $x_j^n\in D_{\varepsilon_0+b_j^n}$ for all $j=0,1,\ldots,i$. Here we have used the notation: $b_j^n=M\sum_{k=0}^{j-1}d_k^n$, $b_0^n=0$. If $t_i^n<\tilde{T}$, then choose the largest number $d_i^n\in\,]0,1/n]$ such that:

$$d\Big(S(d_i^n)\,x_i^n+\int_{t_i^n}^{t_i^n+d_i^n}S(t_i^n+d_i^n-s)\,f(t_i^n,x_i^n)\,ds\,;\ D_{\varepsilon_0+b_{i+1}^n}\Big)\leqslant d_i^n/2n. \tag{3.4}$$

and:

$$t_i^n+d_i^n=t_{i+1}^n\leqslant T. \tag{3.5}$$

In (3.4) $f(t_i^n,x_i^n)$ is an arbitrary element of $F(t_i^n,x_i^n)$ that we fixe in what follows, while b_{i+1}^n is defined as above. The existence of such d_i^n is guaranteed by (A_5). Now, one has to define x_{i+1}^n. From (3.4) it follows that there exists $x_{i+1}^n\in D_{\varepsilon_0+b_{i+1}^n}$ such that:

$$\Big\|S(d_i^n)\,x_i^n+\int_{t_i^n}^{t_i^n+d_i^n}S(t_i^n+d_i^n-s)\,f(t_i^n,x_i^n)\,ds-x_{i+1}^n\Big\|\leqslant d_i^n/n. \tag{3.6}$$

Denote by p_i^n:

$$p_i^n=\frac{1}{d_i^n}\Big(x_{i+1}^n-S(d_i^n)\,x_i^n-\int_{t_i^n}^{t_i^n+d_i^n}S(t_i^n+d_i^n-s)\,f(t_i^n,x_i^n)\,ds\Big). \tag{3.7}$$

Then, from (3.7) one obtains:

$$x_{i+1}^n=S(t_{i+1}^n-t_i^n)\,x_i^n+\int_{t_i^n}^{t_i^n+d_i^n}S(t_{i+1}^n-s)\,f(t_i^n,x_i^n)\,ds+(t_{i+1}^n-t_i^n)\,p_i^n, \tag{3.8}$$

where $\| p_i^n \| \leqslant 1/n$ and t_{i+1}^n is defined as before (i.e. $t_{i+1}^n = t_i^n + d_i^n$). Now, let us define for each $t \in [t_i^n, t_{i+1}^n]$,

$$x_n(t) = S(t - t_i^n)\, x_i^n + \int_{t_i^n}^{t} S(t-s) f(t_i^n, x_i^n)\, ds + (t - t_i^n)\, p_i^n. \tag{3.9}$$

Let us remark that $x_n(t_i^n) = x_i^n$ and thus x_n is continuous on its domain. By a simple calculation that we left to the reader we get from (3.9).

$$x_n(t) = S(t-a)\, x_0 + \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} S(t-s) f(t_j^n, x_j^n)\, ds + \int_{t_i^n}^{t} S(t-s) f(t_i^n, x_i^n) ds + \sum_{j=0}^{i-1} (t_{j+1}^n - t_j^n)\, S(t - t_{j+1}^n)\, p_j^n + (t - t_i^n)\, p_i^n, \text{ for all } t \in [t_i^n, t_{i+1}^n]. \tag{3.10}$$

Define now the step functions: $a_n(s) = t_i^n$ for $s \in [t_i^n, t_{i+1}^n[$, $a_n(\widetilde{T}) = \widetilde{T}$. Then, it follows that: $x_n(a_n(s)) = x_n(t_i^n) = x_i^n$ for all $s \in [t_i^n, t_{i+1}^n[$. Let us define the function g_n by:

$$g_n(t) = \sum_{j=0}^{i-1} (t_{j+1}^n - t_j^n)\, S(t - t_{j+1}^n)\, p_j^n + (t - t_i^n)\, p_i^n. \tag{3.11}$$

Using (3.3), we obtain from (3.11): $\| g_n(t) \| \leqslant (t-a)/n$ for all $t \in [a, \widetilde{T}]$.

One can easily remark that (3.10) may be rewritten (using (3.11)) as follows:

$$x_n(t) = S(t-a)\, x_0 + \int_a^t S(t-s) f_n(s)\, ds + g_n(s) \tag{3.12}$$

where $f_n(s) = f(a_n(s), x_n(a_n(s))) = f(t_i^n, x_i^n)$ for all $s \in [t_i^n, t_{i+1}^n[$.

We shall prove below that by the construction above, all the approximations of the solution are defined on the whole interval $[a, \widetilde{T}]$. Then, we suppose realised this fact.

Thus, we have constructed the approximations of the solution of (1.3), given by (3.12).

It should be noted that the values of x_n are not generaly in G; one can only assert that $x_n(t_i^n)$ are in G (by construction).

Second step. The convergence of the sequences $\{x_i^n\}_{i=0}^{\infty}$. We have to prove that by the construction above, for each fixed $n \in N$, one has $\lim_{i\to\infty} x_i^n = x_*^n$ and that $x_*^n \in D_{\varepsilon_0 + b_*^n}$. Here, b_*^n denotes the limit of the increasing sequence $\{b_i^n\}_{i=0}^{\infty}$.

Let us observe that for each fixed $n \in N$, the sequence $\{x_i^n\}_{i=0}^{\infty}$ is a Cauchy sequence and therefore, the limit x_*^n exists.

Indeed, let $j > i$. Using (3.10), we obtain:

$$\| x_i^n - x_j^n \| \leqslant \| S(t_j^n - t_i^n)\, x_0 - x_0 \| + \sum_{m=0}^{i-1} \| (S(t_j^n - t_i^n) - I) f(t_m^n, x_m^n) \| \, (t_{m+1}^n - t_m^n) + \sum_{m=0}^{i-1} \| S(t_j^n - t_i^n)\, p_m^n - p_m^n \| \, (t_{m+1}^n - t_m^n) + M(t_j^n - t_i^n). \tag{3.13}$$

Let $\varepsilon > 0$ be arbitrary and consider $k(\varepsilon) \in N$ large enough in order that:

$$t_j^n - t_i^n < \varepsilon'/6M \, ; \ \| S(t_j^n - t_i^n)\, x_0 - x_0 \| \leqslant \varepsilon'/6, \text{ for all } j \geqslant i \geqslant k(\varepsilon) \tag{3.14}$$

where $\varepsilon' = \varepsilon/2$.

Choose $\overline{k}(\varepsilon) \geqslant k(\varepsilon)$ such that the two relations:

$$\| S(t_j^n - t_i^n)\, p_m^n - p_m^n \| \leqslant \varepsilon'/3\widetilde{T}, \ m = 0, 1, \ldots, k(\varepsilon) - 1, \ j \geqslant i \geqslant \overline{k}(\varepsilon), \tag{3.15}$$

$$\| (S(t_j^n - t_i^n) - I) f(t_m^n, x_m^n) \| \leqslant \varepsilon'/3\widetilde{T}, \ m = 0, 1, \ldots, k(\varepsilon) - 1, \quad j \geqslant i \geqslant \overline{k}(\varepsilon) \tag{3.16}$$

hold. Then, it follows that:

$$\sum_{m=0}^{i-1} \| S(t_j^n - t_i^n)\, p_m^n - p_m^n \| \, (t_{m+1}^n - t_m^n) \leqslant \varepsilon' t_{k(\varepsilon)}^n / 3\widetilde{T} + 2(t_i^n - t_{k(\varepsilon)}^n)/n \leqslant \varepsilon'/3 + \varepsilon'/3nM \leqslant \varepsilon'/2. \tag{3.17}$$

if n is large enough.

One also obtains:

$$\sum_{m=0}^{i-1} \| (S(t_j^n - t_i^n) - I) f(t_m^n, x_m^n) \| \, (t_{m+1}^n - t_m^n) \leqslant \varepsilon' t_{k(\varepsilon)}^n / 3\bar{T} + 2(M-1)(t_i^n - t_{k(\varepsilon)}^n) \leqslant 2\varepsilon'/3. \tag{3.18}$$

Using (3.17) and (3.18), we can easily conclude:

$$\| x_i^n - x_j^n \| \leqslant \varepsilon'/6 + \varepsilon'/6 + \varepsilon'/2 + 2\varepsilon'/3 \leqslant 3\varepsilon'/2 < 2\varepsilon' = \varepsilon \text{ for all}$$

$j \geqslant i \geqslant \overline{k}(\varepsilon)$. Therefore, $\{x_i^n\}_{i=0}^{\infty}$ is strongly convergent to an element x_*^n.

One can also prove (by induction) that $x_n(t) \in B(x_0, r)$. Then, from the fact that $x_i^n \in B(x_0, r) \cap D_{\varepsilon_0 + b_i^n}$ and $\varepsilon \to B(x_0, r) \cap D_\varepsilon$ is continuous in the Hausdorff metric, it follows that x_*^n belongs to $B(x_0, r) \cap D_{\varepsilon_0 + b_*^n}$, as claimed.

Third step. The convergence of the sequences $\{t_i^n\}_{i=0}^{\infty}$ *to* $\widetilde{T}$. We shall prove that for each $n \in N$, $a + \sum_{i=0}^{\infty} d_i^n = \widetilde{T}$, i.e., that by the construction above, one may cover the whole interval $[a, \widetilde{T}]$.

The idea of the proof is essentialy due to C a r t a n [7, Chapter II, Theorem 1.31] and M a r t i n [13, 14]. However, in our context, we have to use the techniques of P a z y [17], involving the compactness of the semigroup $S(t)$ for all $t > 0$. Suppose by contradiction that for a fixed $n \in N$ one has: $\sum_{i=0}^{\infty} d_i^n = d < T_0$.

Let x_*^n be the strong limit of the sequence $\{x_i^n\}_{i=0}^{\infty}$. Then, by the preceeding step, it follows that $x_*^n \in D_{\varepsilon_0+Md}$. As the sequence $\{f(t_i^n, x_i^n)\}_{i=0}^{\infty}$ is bounded and X is reflexive, it follows that $\{f(t_i^n, x_i^n)\}_{i=0}^{\infty}$ is weakly compact in X. Let us consider now the set W of all weak cluster points of $\{f(t_i^n, x_i^n)\}_{i=0}^{\infty}$. From the remark above, one can easily assert that W is nonempty and in addition $W \subset F(a + d, x_*^n)$. Let us choose $f(a + d, x_*^n)$ in W. Then, there exists a subsequence of $\{f(t_i^n, x_i^n)\}_{i=0}^{\infty}$ denoted again by $\{f(t_i^n, x_i^n)\}_{i=0}^{\infty}$ weakly convergent to $f(a + d, x_*^n)$. From the condition (A_n) it follows that we can choose h^* small enough in $]0, 1/n]$ such that:

$$(3.19)\quad d\Big(S(h^*)\, x_*^n + \int_{a+d}^{a+d+h^*} S(a + d + h^* - s) f(a + d, x_*^n) ds\, ;\ D_{\varepsilon_0+Md+Mh^*}\Big) \leqslant h^*/4n$$

and:

$$(3.20)\qquad a + d + h^* < \tilde{T}.$$

Let us consider also the sequences:

$$v_i^n = S(h^*)\, x_i^n + \int_{t_i^n}^{t_i^n+h^*} S(t_i^n + h^* - s) f(t_i^n, x_i^n)\, ds, \text{ and:}$$

$$v_i^n(\varepsilon) = S(h^*)\, x_i^n + \int_{t_i^n}^{t^n+h^*-\varepsilon} S(t_i^n + h^* - s) f(t_i, x_i)\, ds, \text{ where } 0 < \varepsilon < h^*.$$

Then, using standard arguments involving the properties of the semigroup $S(t)$, one deduces:

$$(3.21)\qquad v_i^n(\varepsilon) = S(h^*)\, x_i^n + S(\varepsilon) \int_{t_i^n}^{t_i^n+h^*-\varepsilon} S(t_i^n + h^* - \varepsilon - s) f(t_i^n, x_i^n)\, ds.$$

From (3.21), as the sequence $\{f(t_i^n, x_i^n)\}_{i=0}^{\infty}$ is bounded in X and $S(t)$ is compact for all $t > 0$, we conclude that $\{v_i^n(\varepsilon)\}_{i=0}^{\infty}$ is precompact in X. On the other hand one has:

$$(3.22)\qquad \| v_i^n - v_i^n(\varepsilon) \| \leqslant (M - 1)\, \varepsilon.$$

Now, from (3.22) and the remark above, one easily deduces that $\{v_i^n\}_{i=0}^{\infty}$ is precompact in X and thus, there exists a subsequence of $\{v_i^n\}$,

denoted again by $\{v_i^n\}_{i=0}^{\infty}$, strongly convergent to an element v_*^n. Let us remark that:

$$(3.23)\qquad v_*^n = S(h^*)\, x_*^n + \int_{a+d}^{a+d+h^*} S(a + d + h^* - s) f(a + d, x_*^n)\, ds.$$

This simple fact follows from the condition that $f(t_i^n, x_i^n) \to f(a + d, x_*^n)$ and from the uniqueness of the weak limit.

We remark further that the continuity of the mapping $\varepsilon \to B(x_0, r) \cap \bigcap_{i} D_\varepsilon$ in the Hausdorff metric, implies:

$$(3.24)\quad d\Big(S(h^*)\, x_i^n + \int_{t_i^n}^{t_i^n+h^*} S(t_i^n + h^* - s) f(t_i^n, x_i^n) ds\, ;\ D_{\varepsilon_0+b_i^n+Mh^*}\Big) \leqslant h^*/2n$$

for infinitely many, sufficiently large $i \in N$. We recall that here $b_i^n = M \sum_{k=0}^{i-1} d_k^n$. Let us observe that we are in the hypotheses of Lemma 2.6 by taking $A_i = D_{\varepsilon_0+b_i^n+Mh^*} \cap B(x_0, r)$, $A = D_{\varepsilon_0+Md+Mh^*} \cap B(x_0, r)$, $x_i = v_i^n$, $x = v_*^n$ and $\varepsilon_0 = h^*/4n$.

But (3.24) contradicts the maximality of d_i^n for sufficiently large $i \in N$, as $\lim_{i\to\infty} d_i^n = 0$.

4. Proof of theorem 2.1. Second part. In this Section we shall prove that the sequence of approximations of the solution given by (3.12), has a strongly convergent subsequence in the norm topology of $C(a, T; X)$ $0 < T < \tilde{T}$.

To this end, we shall prove that $\{x_n\}_{n=0}^{\infty}$ (where x_n is given by (3.12), is relatively compact in $C(a, T; X)$, by applying the infinite dimensional version of A r z e l a and A s c o l i's Theorem.

To complete the proof of Theorem 2.1 we shall also prove that the limit of a such subsequence of $\{x_n\}_{n=0}^{\infty}$ is a mild solution for the problem (1.3).

We begin with the proof of the relative compactness of $\{x_n\}_{n=0}^{\infty}$.

First of all, let us observe that $\{x_n\}_{n=0}^{\infty}$ is uniformely bounded on $[a, T]$. Indeed, by a simple calculation involving (3.12) we get:

$$(4.1)\qquad \| x_n(t) \| \leqslant \| x_0 \| + T(M - 1) + (T - a)/n.$$

Now, let us remark that $\{x_n\}_{n=0}^{\infty}$ is equicontinuous on $[a, T]$. As $g_n \to 0$ in the uniform convergence topology and $S(t - a)\, x_0$ does not depend on n, it suffices to prove that:

$$y_n(t) = \int_a^t S(t - s) f_n(s)\, ds$$

is equicontinuous on $[a, T]$. We recall that $f_n(s) = f(a_n(s), x_n(a_n(s))) = f(t_i^n, x_i^n)$ for all $s \in [t_i^n, t_{i+1}^n[$.

After a simple calculation we get:

$$\|y_n(t) - y_n(s)\| \leq (M-1)\,|t-s| + (M-1)\int_a^s \|S(t-\tau) - S(s-\tau)\|\,d\tau, \tag{4.2}$$

for all $a \leq s \leq t \leq T$. But from (4.2), as $S(t)$ is continuous in the uniform operator topology for $t > 0$, it follows that $\{y_n\}_{n=0}^{\infty}$ is equicontinuous on $[a, T]$.

We must prove that for each $t \in [a, T]$ the set $\{y_n(t)\}_{n=0}^{\infty}$ is precompact in X. For $t = a$ this is clearly the case inasmuch as $y_n(a) = 0$ for all $n \in N$. Then, we must only consider $t \in]a, T]$. Let $\varepsilon \in]0, t-a[$ and define:

$$y_n^{\varepsilon}(t) = \int_a^{t-\varepsilon} S(t-s) f_n(s)\,ds = S(\varepsilon)\int_a^{t-\varepsilon} S(t-\varepsilon-s) f_n(s)\,ds. \tag{4.3}$$

As $S(t)$ is compact on X for all $t > 0$ and $\{f_n\}_{n=0}^{\infty}$ is bounded, it follows from (4.3) that $\{y_n(t)\}_{n=0}^{\infty}$ is precompact in X. On the other hand, one has:

$$\|y_n(t) - y_n^{\varepsilon}(t)\| \leq (M-1)\,\varepsilon. \tag{4.4}$$

Now, one can easily remark that (4.4) and the precompactness of $\{y_n^{\varepsilon}(t)\}_{n=0}^{\infty}$ imply the precompactness of $\{y_n(t)\}_{n=0}^{\infty}$. Thus, we have proved that $\{y_n\}_{n=0}^{\infty}$ satisfies the hypotheses of Arzelà and Ascoli's Theorem and therefore $\{y_n\}_{n=0}^{\infty}$ is relatively compact in $C(a, T; X)$. This last assertion implies that $\{x_n\}_{n=0}^{\infty}$ is relatively compact in $C(a, T; X)$ and consequently, one can choose a subsequence of $\{x_n\}_{n=0}^{\infty}$, denoted again by $\{x_n\}_{n=0}^{\infty}$, strongly convergent in $C(a, T; X)$ to some continuous function x. We shall prove that $x: [a, T] \to G$ and also that x is a mild solution of the problem (1.3) on $[a, T]$.

For prove the first assertion, let us consider an arbitrary $s \in [a, T]$ and also the set of all $i_n \in N$ such that $s \in [t_{i_n}^n, t_{i_n+1}^n[$.

Then, $s - t_{i_n}^n < t_{i_n+1}^n - t_{i_n}^n \leq 1/n$, $a_n(s) = t_{i_n}^n$, $x_n(a_n(s)) = x_{i_n}^n$. Therefore:

$$|a_n(s) - s| = |t_{i_n}^n - s| \leq 1/n,$$

$$\|x_n(a_n(s)) - x(s)\| = \|x_n(t_{i_n}^n) - x(s)\| \leq \|x_n(t_{i_n}^n) - x_n(s)\| + \|x_n(s) - x(s)\|. \tag{4.5}$$

From (3.12) we get succesively:

$$\|x_n(t_{i_n}^n) - x_n(s)\| \leq \|y_n(t_{i_n}^n) - y_n(s)\| + \|S(t_{i_n}^n - a)x_0 - S(s-a)x_0\| + T/n, \tag{4.6}$$

$$\|x_n(a_n(s)) - x(s)\| \leq \|x(s) - x_{i_n}^n\| + \|x_n(s) - x(s)\| + \|y_n(t_{i_n}^n) - y_n(s)\| + \|S(t_{i_n}^n - a)x_0 - S(s-a)x_0\| + 2T/n, \tag{4.7}$$

where $x_{i_n}^n = x_n(t_{i_n}^n)$ and y_n is defined as before, i.e., $y_n(t) = \int_a^t S(t-s) f_n(s)\,ds$.

Then, from (4.5), (4.6) and (4.7), as $x_i^n \in B(x_0, r) \cap D_{\varepsilon_0 + M(t_i^n - a)}$ which is closed in X, we get: $x(t) \in B(x_0, r) \cap D_{\varepsilon_0 + M(t-a)} \subset G$. We have used here the continuity of the mapping $\varepsilon \to B(x_0, r) \cap D_{\varepsilon}$ in the Hausdorff metric.

Therefore, x is valued in the domain of F. To complete the proof, we have only to prove that x is a mild solution for the problem (1.3) on $[a, T]$. To this end we shall apply Lemma 2.7.

Consider the sequence $\{f_n\}_{n=0}^{\infty}$, where $f_n(s) = f(a_n(s), x_n(a_n(s))) = f(t_i^n, x_i^n)$ for all $s \in [t_i^n, t_{i+1}^n[$. As $f_n(s) \in F(a_n(s), x_n(a_n(s)))$ for all $s \in [a, T]$, it follows that $\{f_n(s)\}_{n=0}^{\infty}$ is uniformly bounded on $[a, T]$ and thus relatively weakly compact in $L^2(a, T; X)$. On the other hand, the set $W(s)$ of all cluster points (in the weak topology of X) of $\{f_n(s)\}_{n=0}^{\infty}$ is nonempty and included in $F(s, x(s))$ since F is demiclosed.

Let us consider a subsequence of $\{f_n\}_{n=0}^{\infty}$, denoted again by $\{f_n\}_{n=0}^{\infty}$, which converges in the weak topology of $L^2(a, \widetilde{T}; X)$ to some function f. Then, as F is convex valued and $W(s) \subset F(s, x(s))$, it follows from Lemma 2.7 that f is a strongly measurable selection of $F(\cdot, x(\cdot))$.

Now, we shall prove that the sequence $\{y_n\}_{n=0}^{\infty}$, defined as above is strongly convergent in $C(a, T; X)$ to $y(t) = \int_a^t S(t-s) f(s)\,ds$, where f is the weak limit of $\{f_n\}_{n=0}^{\infty}$ in $L^2(a, T; X)$. Let $x^* \in X^*$ be an arbitrary element and let us consider:

$$\langle y_n(t), x^* \rangle = \int_a^t \langle S(t-s) f_n(s), x^* \rangle\,ds. \tag{4.8}$$

Here $\langle \ldots \rangle$ is the usual pairing between X and X^*, i.e. $\langle x, x^* \rangle = x^*(x)$.

From (4.8) one easily obtains:

$$\langle y_n(t), x^* \rangle = \int_a^t \langle f_n(s), S^*(t-s)x^* \rangle\,ds, \tag{4.9}$$

where $S^*(t-s)$ is the adjoint of $S(t-s)$.

Let us remark that for each fixed $t \in [a, T]$, the mapping $s \to S^*(t-s)x^*$ belongs to $L^2(a, t; X^*)$ and thus, taking the limit in (4.9), it follows:

$$\langle y(t), x^* \rangle = \int_a^t \langle S(t-s) f(s), x^* \rangle\,ds. \tag{4.10}$$

Let us remark that from the uniqueness of the weak limit an[d] from the strong convergence of the sequence $\{y_n\}_{n=0}^{\infty}$ in $C(a, T; X)$, w[e] get from (4.10):

(4.11) $$y(t) = \int_a^t S(t-s) f(s)\,ds.$$

Now, taking into account that in (3.12) $\{g_n\}_{n=0}^{\infty}$ is strongly conve[r]gent to 0 in $C(a, T; X)$, the relation (4.11) implies:

(4.12) $$x(t) = S(t-a)\,x_0 + \int_a^t S(t-s) f(s)\,ds.$$

Thus, we have proved that x is a mild solution of the proble[m] (1.3) on $[a, T]$, and this completes the proof of Theorem 2.1.

We conclude this Section with several remarks and comment[s].

Remark 4.1. First of all let us remark that our main result — The[o]rem 2.1 — remains valid if we suppose that X is the dual of a — nonre[]flexive Banach space, but in addition F is locally weakly compact, i.e. for each $(t_0, x_0) \in [a, b[\times G$, there exist $\delta_1 > 0$, $\delta_2 > 0$ and a weakl[y] compact set $K \subset X$ such that $F(t, x) \subset K$ for all $t \in [a, b[$ with $|t - t_0| \leqslant \delta_1$ and for all $x \in G$ with $\|x - x_0\| \leqslant \delta_2$. Indeed, in this cas[e] one can obtain an appropriate version of Lemma 2.7 and thus the pro[of] is very similar to the present one.

5. Proof of Theorem 2.2. We begin this Section with the nota[]tions we shall use.

X, Y are two real Banach spaces with $Y \subset X$ such that the in[]clusion above is continuous, i.e. there exists $c > 0$ such that $\|x\|_X \leqslant c\|x\|_Y$.

Here $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms in X and respectively in Y.

Denote by $\|S(t)\|_{L(Y)}$ the norm of the linear continuous operat[or] $S(t): Y \to Y$ in the uniform operator topology and by $\|S(t)\|_{L(X)}$ th[e] corresponding norm in the uniform operator topology on X.

Now, we proceed to the proof of Theorem 2.2. We point out th[at] several parts of the proof will be only outlined as these parts are ve[ry] similar to the corresponding ones from the preceeding proof.

Let $x_0 \in S_Y(0, r_0)$. As F is continuous in the norm topology of Y it follows that F is locally bounded in Y. Thus, there exist $r > 0$, $\tilde{T} >$ [] and $M > 0$ such that:

(5.1) $$\|F(t, y)\|_Y \leqslant M, \text{ for all } (t, y) \in [a, \tilde{T}] \times B_Y(x_0, r).$$

Using the continuity of the semigroup $S(t)$ on Y we can ass[ume] that there exists $K > 0$, such that:

(5.2.) $$\|S(t)\|_{L(Y)} \leqslant K, \text{ for all } t \in [a, \tilde{T}].$$

We choose T small enough such that in addition:

(5.3) $$\|S(t)\,x_0 - x_0\|_Y + T_0 M K \leqslant r, \text{ where } T_0 = \tilde{T} - a.$$

Now, we shall construct a sequence of approximations of the solution using the same techniques as in the proof of Theorem 2.1.

Let n be an arbitrary natural number and define $t_0^n = a$, $x_0^n = x_0$ and assume that we have constructed x_n on $[a, t_i^n]$ with $x_n(t_i^n) = x_i^n$, $x_i^n \in B_Y(x_0, r)$ and $t_i^n \leqslant \tilde{T}$. Choose the largest number $d_i^n \in]0, 1/n]$ such that:

(5.4) $$\|F(t, y) - F(t_i^n, x_i^n)\|_X \leqslant 1/n \text{ and } t_i^n + d_i^n \leqslant \tilde{T},$$

for all $(t, y) \in [a, \tilde{T}] \times S_Y(0, r_0)$ for which $|t - t_i^n| + \|y - x_i^n\|_Y \leqslant d_i^n$.

Set $t_{i+1}^n = t_i^n + d_i^n$ and:

(5.5) $$x_{i+1}^n = S(t_{i+1}^n - t_i^n)\,x_i^n + \int_{t_i^n}^{t_{i+1}^n} S(t_{i+1}^n - s)\,F(t_i^n, x_i^n)\,ds.$$

We shall prove later that $x_{i+1}^n \in B_Y(x_0, r)$.

One has to prove that there exists a natural number $i_0 = i_0(n)$ such that $t_{i_0(n)}^n = \tilde{T}$. To this end, let us observe firstly that $\{x_i^n\}_{i=0}^{\infty}$ is strongly convergent in X. Indeed, it is easy to verify that x_{i+1}^n defined by (5.5) can be rewritten under the form:

(5.6) $$x_{i+1}^n = S(t_{i+1}^n - a)\,x_0 + \sum_{j=0}^{i} \int_{t_j^n}^{t_{j+1}^n} S(t_{j+1}^n - s)\,F(t_j^n, x_j^n)\,ds.$$

From (5.6) it follows that:

(5.7) $$\|x_0 - x_{i+1}^n\|_Y \leqslant \|S(t-a)\,x_0 - x_0\|_Y + K T_0 M \leqslant r$$

and thus $x_{i+1}^n \in B_Y(x_0, r)$.

Now let $j > i$. Using (5.1), (5.2) and (5.6) one obtains:

(5.8) $$\|x_j^n - x_i^n\|_Y \leqslant C_1\|S(t_j^n - t_i^n)\,x_0 - x_0\|_X + C_1 \sum_{m=0}^{i-1} \|(S(t_j^n - t_i^n) - I)\,F(t_m^n, x_m^n)\|_X\,(t_{m+1}^n - t_m^n) + C_1 cM(t_j^n - t_i^n).$$

In (5.8) we have used the estimations:

(5.9) $$\|F(t_i, x_i)\|_X \leqslant c\|F(t_i, x_i)\|_Y \leqslant cM,$$

(5.10) $$\|S(t)\|_{L(X)} \leqslant C_1,$$

for all $t \in [0, T_0]$, where C_1 is a positive number.

We note that there exists such a number as $S(t)$ is a C_0 semigroup on X.

From (5.8), (5.9) and (5.10), using the same arguments as in the proof of Theorem 2.1, one may easily deduce that $\{x_i^n\}_{i=0}^{\infty}$ is a Cauchy sequence in X and thus it is convergent to some element denoted x_*^n. Suppose by contradiction that for all $i = 0, 1, 2, \ldots, t_i^n < \tilde{T}$. Th… it follows that $d_i^n = 1/n$ only for a finite number of i. Therefore, … may assume that $0 < d_i^n < 1/n$ for infinitely many i.

By (5.4) and by the maximality of $d_i^n \in \,]0, 1/n]$, it follows … for infinitely many $i \in N$ there exist $\tilde{t}_i^n, \tilde{x}_i^n$ such that:

$$(5.11) \quad d_i^n < | \tilde{t}_i^n - t_i^n | + \| \tilde{x}_i^n - x_i^n \|_Y \leqslant d_i^n + 1/i, \ \| F(\tilde{t}_i^n, \tilde{x}_i^n) - F(t_i^n, x_i^n) \|_X \geqslant 1/n.$$

But (5.11) implies that $\lim\limits_{i \to \infty} x = \lim\limits_{i \to \infty} x_i^n = x_*^n$ in X, and from … fact that $\lim\limits_{i \to \infty} t_i^n = t_*^n$ we get a contradiction. Indeed, the remark ab… and the second inequality in (5.11) are in contradiction. Thus we … proved that after a finite number of steps, one may cover the wh… interval $[a, \tilde{T}]$. Define now the function $x_n : [a, \tilde{T}] \to Y$ as follo…

$$(5.12) \qquad x_n(t) = S(t - t_i^n) x_i^n + \int_{t_i^n}^{t} S(t-s) F(t_i^n, x_i^n)\, ds,$$

for all $t \in [t_i^n, t_{i+1}^n[$ $i = 0, 1, 2, \ldots, i_0 - 1$, where i_0 is defined above.

Clearly, x_n is continuous on $[a, \tilde{T}]$ and (5.12) can be rewri… under the form:

$$(5.13) \quad x_n(t) = S(t-a) x_0 + \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} S(t-s) F(t_j^n, x_j^n)\, ds + \int_{t_i^n}^{t} S(t-s) F(t_i^n, \ldots$$

for all $t \in [t_i^n, t_{i+1}^n[$, $i = 0, 1, 2, \ldots, i_0 - 1$.

Combining (5.3) with (5.13), one obtains $x_n(t) \in B_Y(x_0, r)$, … tion which implies:

$$(5.14) \qquad \| x_n(t) - x_0 \|_X \leqslant cr$$

for all $t \in [a, \tilde{T}]$ and $n = 1, 2, \ldots$

Define now the step functions $a_n : [a, \tilde{T}] \to [a, \tilde{T}]$, by $a_n(s) \ldots$ for all $s \in [t_i^n, t_{i+1}^n[$, $a_n(\tilde{T}) = \tilde{T}$. Then, it follows that $x_n(a_n \ldots$ $= x_n(t_i^n) = x_i^n$ for all $s \in [t_i^n, t_{i+1}^n[$ and therefore x_n can be rewr… under the more convenient form:

$$(5.15) \quad x_n(t) = S(t-a) x_0 + \int_a^t S(t-s) F(a_n(s), x_n(a_n(s)))\, ds.$$

From (5.15) we get easily:

$$(5.16) \quad \begin{aligned} \| x_n(t) - x_n(s) \|_X \leqslant \| S(t-a) x_0 - S(s-a) x_0 \|_X + cC_1 M \, | t-s | + \\ + cM \int_a^s \| S(t-\tau) - S(s-\tau) \|_{L(X)}\, d\tau, \end{aligned}$$

for all $a \leqslant s \leqslant t \leqslant \tilde{T}$.

As $t \to S(t)$ is continuous on $]0, +\infty[$ in the uniform operator topology, from (5.16) we deduce that $\{x_n\}_{n=1}^{\infty}$ is equicontinuous on $[a, \tilde{T}]$. Let us introduce:

$$(5.17) \qquad x_n^{\varepsilon}(t) = S(t-a) x_0 + \int_a^{t-\varepsilon} S(t-s) F(a_n(s), x_n(a_n(s)))\, ds,$$

for $\varepsilon > 0$ arbitrary small.

Using the same arguments as in the proof of Theorem 2.1, we conclude that $\{x_n^{\varepsilon}(t)\}_{n=1}^{\infty}$ is precompact in X and consequently $\{x_n(t)\}_{n=1}^{\infty}$ is precompact in X for each $t \in [a, T]$, where $a < T < \tilde{T}$.

Thus by Arzelà and Ascoli's Theorem, we assume that $\{x_n\}_{n=1}^{\infty}$ has a subsequence, denoted again by $\{x_n\}_{n=1}^{\infty}$, strongly convergent in the topology $C(a, T; X)$ to some continuous function x.

One can easily prove that x is a mild solution of the problem (1.3) on $[a, T[$. The arguments used here being more simpler that those used for the preceeding proof, we left to the reader to prove this last assertion, thereby completing the proof of Theorem 2.2.

In the next Section we shall give some results on the continuation of the solution on a maximal interval, as well as we shall treat several examples and applications.

6. Continuation of the solutions, examples, applications and further remarks. We begin this Section with the following results concerning the continuation of the solutions in the two considered cases.

Theorem 6.1. *Suppose that all the hypotheses of Theorem 2.1 are satisfied, G is closed in X and in addition $F : [a, +\infty[\times G \to 2^X$ maps bounded sets in bounded sets. Then, for each $x_0 \in G$, there exists a mild solution of the problem* (1.3) *defined on a maximal interval of existence $[a, T_{\max}[$, where either $T_{\max} = +\infty$ or, if $T_{\max} < +\infty$ then the solution x is unbounded on $[a, T_{\max}[$.*

Theorem 6.2. *Suppose that all the hypotheses of Theorem 2.2 are satisfied, Y is closed in X and in addition $F : [a, +\infty[\times Y \to Y$ maps bounded sets (in the norm of X) in bounded sets (in the norm of X). Then, for each $x_0 \in Y$, there exists a mild solution of the problem* (1.3) *defined on a maximal interval of existence $[a, T_{\max}[$, where either $T_{\max} = +\infty$ or of $T_{\max} < +\infty$ then the solution x is unbounded on $[a, T_{\max}[$.*

If, in addition, we suppose a more restrictive growth condition on F, we obtain:

Theorem 6.3. *Suppose that all the hypotheses of Theorem 2.1 are satisfied, G is closed in X and in addition $F: [a, +\infty[\times G \to 2^X$ verifies:*

$$\|F(t, x)\| \le k_1 \|x\| + k_2, \tag{6.1}$$

for all $(t, x) \in [a, +\infty[\times G$, where $k_1 > 0$ and $k_2 \in R$.

Then, for each $x_0 \in G$, there exists a mild solution of the problem (1.3) defined on $[a, +\infty[$.

Theorem 6.4. *Suppose that all the hypotheses of Theorem 2.2. are satisfied, Y is closed in X and, in addition $F: [a, +\infty[\times Y \to Y$ verifies:*

$$\|F(t, x)\|_Y \le k_1 \|x\|_Y + k_2 \tag{6.2}$$

for all $(t, x) \in [a, +\infty[\times Y$, where $k_1 > 0$ and $k_2 \in R$.

Then, for each $x_0 \in Y$, there exists a mild solution of the problem (1.3) defined on $[a, +\infty[$.

As all these results may be proved by using standard arguments, we left to the reader the proofs of the four theorems stated above.

We would like only to point out that all the results concerning the continuation of the solution do not remain valid if G or Y are not closed. It should be also noted that using the same remark, one may correct an error that appears in [15, Theorem 3.2] by taking the set D as a closed set in X and not as a locally closed set.

We shall indicate several situations in which our results are applicable. We bagin with the perturbed heat equation case.

A. Let us consider an open and bounded set Ω in R^n, with sufficient smooth boundary Γ. Let f be a continuous function definrd on $[a, b] \times \overline{\Omega} \times R$ and consider the following problem:

$$\begin{cases} \dfrac{\partial y(t, x)}{\partial t} = \Delta y(t, x) + f(t, x, y(t, x)), & \text{for } (t, x)\ [a, T[\times \Omega, \\ y(t, x)|_\Gamma = 0, \\ y(0, x) = y_0(x), \end{cases} \tag{6.3}$$

where $y_0 \in L^\infty(\Omega)$ and $\Delta: L^2(\Omega) \to L^2(\Omega)$ is the Laplace operator.

Let us remark that (6.3) may be rewritten under the following form:

$$\begin{cases} y'(t) = Ay(t) + \tilde{f}(t, y(t)) & \text{for } t \in [a, T] \\ y(0) = y_0 \end{cases} \tag{6.4}$$

where $A = \Delta$, the domain of A being $H_0^1(\Omega) \cap H^2(\Omega)$, while $\tilde{f}: [a, b[\times L^\infty(\Omega) \to L^\infty(\Omega)$ is the Nemytskii operator induced by f, i.e.

(6.5) $\tilde{f}(t, y)(x) = f(t, x, y(t, x))$ for all $(t, x) \in [a, b[\times \Omega$ and $y \in L^\infty(\Omega)$.

One may easily deduce the following result:

Theorem 6.5. *Suppose that $f: [a, b[\times \tilde{\Omega} \times R \to R$ is a continuous mapping. Then, for each $y_0 \in L^\infty(\Omega)$, there exists $T \in]a, b[$ such that the problem (6.3) has at least a mild solution on $[a, T[$.*

Proof. We shall apply Theorem 2.1. Take $X = L^2(\Omega)$ and let us observe that the Nemytskii operator $\tilde{f}$ defined by (6.5) is continuous from $[a, b[\times L^\infty(\Omega)$ to $L^\infty(\Omega)$ in the topology induced by the norm topology of $L^2(\Omega)$. In other words, for each $\{(t_n, x_n)\}_{n=0}^{\infty}$ from $[a, b[\times L^\infty(\Omega)$ with $(t_n, x_n) \to (t, x) \in [a, b[\times L^\infty(\Omega)$ in the norm topology of $L^2(\Omega)$, one has also $\tilde{f}(t_n, x_n) \to f(t, x)$ in the norm topology of $L^2(\Omega)$.

Let k be a positive number such that $y_0 \in S_{L^\infty(\Omega)}(0, k + 1)$. Now, let us define $G = S_{L^\infty(\Omega)}(0, k + 1)$ and $D_\varepsilon = B_{L^\infty(\Omega)}(0, k + \varepsilon)$, for all $0 < \varepsilon < 1$. It is obvious that $G = \bigcup_{\varepsilon \in]0, 1[} D_\varepsilon$ and also that D_ε is closed in $L^2(\Omega)$ for each $\varepsilon \in]0, 1[$. On the other hand, the mapping $\varepsilon \to D_\varepsilon$ is continuous in the Hausdorff metric induced by the norm of $L^2(\Omega)$. All these remarks, show us that G satisfies the properties (1), (2) and (3) from introduction and thus, G is semi locally closed.

Now, let us remark that the linear C_0 semigroup $S(t)$ generated by the Laplace operator on $L^2(\Omega)$ is compact and therefore, we have proved that the hypotheses (A_1), (A_2), (A_3) and (A_4) of Theorem 2.1 are satisfied. Then, it remains to verify (A_5). To this end, take $x \in G$, $t \in [a, b[$ and $\varepsilon \in]0, 1[$, such that $x \in D_\varepsilon$. We shall prove that:

$$d\left(S(h)x + \int_t^{t+h} S(t + h - s) f(t, x)\, ds;\ D_{\varepsilon+Mh}\right) = 0 \tag{6.6}$$

holds, for all $h > 0$.

As $D_\varepsilon = B_{L^\infty(\Omega)}(0, k + \varepsilon)$ and $D_{\varepsilon+Mh} = B_{L^\infty(\Omega)}(0, k + \varepsilon + Mh)$, (6.6) is equivalent to:

$$\left\| S(h)x + \int_t^{t+h} S(t + h - s) f(t, x)\, ds \right\|_{L^\infty(\Omega)} \le k + \varepsilon + Mh, \tag{6.7}$$

But, let us remark that (6.7) is a direct consequence of the well known maximum principle for parabolic equations. Indeed, for each $h > 0$ one has $\|S(h)x\|_{L^\infty(\Omega)} \le \|x\|_{L^\infty(\Omega)}$ and, on the other hand, $\left\| \int_t^{t+h} S(t + h - s) f(t, x)\, ds \right\|_{L^\infty(\Omega)} \le Mh$, since M is the upper bound of $\tilde{f}$ on $[a, T[\times G$. Thus, it follows that all the hypotheses of Theorem 2.1 are satisfied and therefore the proof of Theorem 6.5 is complete.

Remark 6.1. (i) We note that the semi locally closed set G considered in the proof of Theorem 6.5 is not locally closed and thus our extension to semi locally closed sets is not trivial.

(ii) It should be pointed out that Theorem 6.5 may be also deduced from our second main result, Theorem 2.2, by taking $X = L^2(\Omega)$ and $Y = L^\infty(\Omega)$.

(iii) It should be very interesting to know if the conclusion Theorem 6.5 remains valid in the case in wich the initial data y_0 taken in $L^2(\Omega)$ such that $f(t, \cdot\, y_0(\cdot))$ is also in $L^2(\Omega)$ for all $t \in [a,$

(iv) We would like to mention that there are several results o tained by assuming additionally growth conditions on f, results ve closely related to our Theorem 6.5. See for instance [1], [15]. In these results, a growth condition of the type (6.1) is assumed in ord to assure that the domain of the Nemytskii operator is the whole $L^2($

B. The next example is concerned with the study of the lo existence problem for the closed loop system arising in a convex probl of Bolza.

Consider the following control problem:

$$\text{Minimize} \int_0^T L(x(t), u(t))\, dt \tag{6.8}$$

for $x \in C(0, T; X)$, $u \in L^2(0, T; U)$, subject to the constraint:

$$x(t) = S(t)\, x_0 + \int_0^t S(t-s)\, B(s)\, u(s)\, ds, \tag{6.9}$$

where X and U are two real Hilbert spaces with norms $\|\cdot\|_X$, $\|$ and inner products $<.\,,.>_X$ and $<.\,,.>_U$.

In all what follows, $L: X \times U \to]-\infty, +\infty]$ is a convex and low semicontinuous function, while $H: X \times U \to [-\infty, +\infty]$ is the Hami tonian function associated to L, i.e.

$$H(x, p) = \sup\, \{<p, u>_U - L(x, u)\,;\ u \in U\}. \tag{6.10}$$

It is obvious that H is the conjugate of L with respect to U.

For each $t \in [0, T]$, $B(t)$ is a linear continuous operator from to X, whose adjoint is denoted by $B^*(t)$. We suppose also that the map ping $t \to B(t)$ is continuous on $[0, T]$ in the uniform operator topolog Consider the following sets:

$$\mathrm{Dom}_1 H = \{x \in X\,;\ H(x, p) > -\infty \text{ for all } p \in U\},$$

$$\mathrm{Dom}_2 H = \{p \in U\,;\ H(x, p) < +\infty \text{ for all } x \in X\}.$$

The set $\mathrm{Dom}\, H = \mathrm{Dom}_1 H \times \mathrm{Dom}_2 H$ is called the effective d main of H.

Denote by C_L the set of all attainable pairs for L, i.e., the of all pairs $(x_0, x_T) \in X \times X$ such that there exists $x \in C(0, T;$ and $u \in L^2(0, T; U)$ verifying (6.9) with $x(0) = x_0$, $x(T) = x_T$ and:

$$L(x, u) \in L^1(0, T; R). \tag{6.11}$$

Now, we are able to formulate the hypotheses we need in th sequel, hypotheses that are derived from those of B a r b u [4], wh studied this problem in a more general setting.

(A_6) $\mathrm{Dom}_1 H = X$ *and* $0 \in \mathrm{int}\, \mathrm{Dom}_2 H$.

(A_7) *There exists at least a pair of functions* (x, u) *in* $C(0, T; X) \times L^2(0, T; U)$ *satisfying* (6.9) *and* (6.11).

Under the hypotheses (A_6) and (A_7) it was proved by B a r b u [4] that a pair $(x, u) \in C(0, T; X) \times L^2(0, T; U)$ is optimal for (6.8), (6.9) if and only if there exists functions $p \in C(0, T; X)$ and $q \in L^2(0, T; U)$ such that (x, u) verifies (6.9) and in addition:

$$p(t) = S^*(T-t)\, p(T) - \int_t^T S^*(s-t) q(s)\, ds, \tag{6.12}$$

$$(B(t)\, p(t),\ q(t)) \in \partial L(x(t), u(t)), \tag{6.13}$$

almost everywhere on $[0, T]$.

Let us remark that from (6.13) it follows that u may be expressed as:

$$u(t) \in \partial_p H(x(t),\ B^*(t)\, p(t)) \tag{6.14}$$

almost everywhere on $[0, T]$.

This simple fact may be deduced from the convexity assumptions on L and from the properties of the conjugate of a convex function, stated in introduction.

Let us consider the following function:

$$\Phi(t, x_0) = \inf \left\{ \int_t^T L(x(s)), u(s))\, ds\,;\ x(t) = x_0 \right\}, \tag{6.15}$$

where (x, u) verify together (6.9) on $[t, T]$.

Then, one asserts that for each $t \in [0, T]$, $\Phi(t, .)$ is convex and lower semicontinuous on X. The proof of this property of Φ may by found in [3, Chapter 4, §4].

Thus, from (6.12), (6.13) and from the subdifferentiability of $\Phi(t, .)$ one deduces:

$$p(t) \in -\partial \Phi(t, x(t)). \tag{6.16}$$

Combining (6.14) and (6.16), we get:

$$u(t) \in \partial_p H(x(t),\ B^*(t)\, (-\partial \Phi(t, x(t)))), \tag{6.17}$$

relation that represents the *optimal feedback law* for (6.8), (6.9).

Therefore, it is easy to see that the optimal arc x must satisfy the *closed loop system*:

$$x(t) = S(t)\, x_0 + \int_0^t S(t-s) f(s)\, ds, \tag{6.18}$$

where f is any strongly measurable selection of the multivalued mapping $t \to B(t)\, (\partial_p H(x(t),\ B^*(t)\, (-\partial \Phi(t, x(t)))))$.

Obviously, (6.18) is the mild form of:

$$\begin{cases} \dot{x}(t) = Ax(t) + B(t)\,(\partial_p H(x(t), B^*(t)\,(-\,\partial\Phi(t, x(t))))) \\ x(0) = x_0 \end{cases} \tag{6.19}$$

where A is the infinitesimal generator of the linear C_0 semigroup $S(t)$.

From Theorem 2.1, we get:

Theorem 6.6. *Suppose that* (A_6), (A_7) *are satisfied and, in addition, the operator* A *in* (6.19) *verifies* (A_2). *Suppose further that* $\partial\Phi$ *is continuous on* $[a, b[\times X$. *Then, for each* $x_0 \in X$, *there exists* $T \in]a, b[$ *such that the problem* (6.19) *has at least a mild solution on* $[a, T[$.

Remark 6.2. This last example shows us that Theorem 2.1 is a good tool in establishing existence results for the closed loop systems that appears in the study of various control problems.

The fact is, that generaly, we cannot obtain the *optimal feedback law*, and therefore we have to look for a *suboptimal feedback law*, that we must choose in order to obtain additionally properties of the solution. On the other hand, let us observe that such a suboptimal feedback law has to be chosen in such a way that, first of all, the suboptimal closed loop system has at least a solution.

Thus, in this context, Theorem 2.1. shows us what kind of suboptimal feedback laws can be considered in order that the corresponding suboptimal closed loop system has solutions, when A generates a linear C_0 semigroup $S(t)$ of compact operators for $t > 0$.

Several examples of such control problems, in which one need to consider suboptimal feedback lows, may be found in [20].

We conclude this paper with a remark concerning the *generalized Hamiltonian systems* of the form:

$$\dot{x}(t) \in H_1(t, x(t), p(t)),\quad \dot{p}(t) \in H_2(t, x(t), p(t)),\quad x(0) = x_0,\ p(0) = p_0 \tag{6.20}$$

where $H_i\colon [0, T] \times X \times X \to 2^X$, $i = 1, 2$, are two demiclosed and locally bounded mappings, while $(x_0, p_0) \in X \times X$.

In the particular case in which $H_1(t, x, p) = \partial_p H(t, x, p)$ and $H_2(t, x, p) = -\,\partial_x H(t, x, p)$, where $\partial H(t, x, p) = [-\,\partial_x H(t, x, p),\ \partial_p H(t, x, p)]$ is the subdifferential of an everywhere finite Hamiltonian function H, (6.20) becomes the system studied by Rockafellar [18] in the finite dimensional case and also by Barbu [2] in the infinite dimensional case. It should be noted, that in this last paper we mentioned above, (x_0, p_0) is supposed to be in an appropriate neighborhood of a saddle point of H.

However, it is well known that even in this very particular case, (6.20) may not have solutions if X is any infinite dimensional space and (x_0, p_0) is arbitrary in $X \times X$.

On the other hand, using our Theorem 2.1, one may assert that the *associated perturbed system*:

$$\begin{cases} \dot{x}(t) \in \varepsilon Ax(t) + H_1(t, x(t), p(t)),\quad \dot{p}(t) \in \varepsilon Ap(t) + H_2(t, x(t), p(t)) \\ x(0) = x_0,\ p(0) = p_0 \end{cases} \tag{6.21}$$

where $\varepsilon > 0$ is sufficiently small and $A\colon X \to X$ satisfies (A_2), has at least a mild solution if in addition the mappings H_i, $i = 1, 2$, are convex valued.

As (6.20) appears in the mathematical modelling of several problems in quantum mechanics, it should be very interesting to know what are the most appropriate operators A that we are allowed to use in (6.21) and what is the physical significance of such operators.

Acknowledgements. The authors would like to thank Professor V. Barbu for having proposed this subject, for his helpful comments and also to C. Ursescu, F. Iacob and C. Zălinescu for their useful remarks on this work.

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Received 12.VI.1978

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