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ON A DISCRETE MAX-MIN PROBLEM

BY

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1. In this note, we are concerned with the following problem: *Given E a finite set and a function $f: E \times \mathcal{P}(E) \to R$, determine*

$$\max_{A \subseteq E} \min_{x \in A} f(x, A).$$

2. This problem is one of the most interesting in discrete optimization as it results from the applications **3** and **4**. When f is an arbitrary function, the problem considered here is very hard, but, in a special case we solve it with an algorithm suggested by F i n c k and S a c h s [2]. Finally, we shall deduce a possible application of this algorithm to obtain a lower bound for the stability number of a graph.

3. Many combinatorial optimization problems can be formulated in the following manner:

Let E be a finite set and $\mathcal{F}$ a system of subsets of E with property that any subset of a member of $\mathcal{F}$ belongs to $\mathcal{F}$ as well. Let $w: E \to R_+$ a weight function on E. If F is a subset of E, its weight is $w(F) = \sum_{e \in F} w(e)$. Then, the combinatorial optimization problem can be stated as:

$$\text{Determine} \max_{F \in \mathcal{F}} w(F).$$

A subset $C \subset E$ with property $C \notin \mathcal{F}$ but $C - \{x\} \in \mathcal{F}$ for every $x \in C$, is called a circuit of $\mathcal{F}$. If $A \subseteq E$ and $x \in A$, we denote $c(x, A)$ the number of circuits C of $\mathcal{F}$ such that $x \in C$ and $C \subseteq A$. Clearly, $F \in \mathcal{F}$ iff $c(x, F) = 0$ for every x in F. Hence the combinatorial optimization problem can be interpreted as an example of a problem **1**:

$$\text{Determine} \max_{A \subseteq E} \min_{x \in A} w(A)/(1 + Mc(x, A)),$$

where M is a very large (with respect to $w(E)$) real number.

4. Let $G = (X, U)$ be a finite undirected graph without loops and multiple edges. A set S of pairwise nonadjacent vertices is called a stable set of G. The maximum cardinality of a stable set of G is called the stability number of G and is denoted $\alpha(G)$. Obviously, we have $\alpha(G) = \max_{A \subseteq X} \alpha([A]_G)$, where $[A]_G$ denotes the subgraph induced by A in G. B e r g e [1] has determined that $\alpha(G) \geq [\,|X|/(1 + \Delta(G))]^*$, where $|X|$ denotes the cardinality of X, $\Delta(G)$ denotes the maximum degree

of a vertex in G and $[a]^*$ is the smallest integer greater or equal the real number a. Since for a maximum stable set S of G, the Berge lower bound for the stability number of $[S]_G$ is equal with $|S|$, we c write

$$\alpha(G) = \max_{A\subseteq X} [|A|/(1+\Delta([A]_G))]^*.$$

Hence $\alpha(G) = \max_{A\subseteq X}[|A|/(1+\max_{x\in A} d_{[A]_G}(x))]^*$, where $d_{[A]_G}(x)$ is the deg of vertex x in the induced subgraph $[A]_G$. Therefore

$$\alpha(G)=\max_{A\subseteq X}\min_{x\in A}[|A|/(1+d_{[A]_G}(x))]^*.$$

and the determination of the stability number of a graph is a particu case of problem 1.

With a similar argument we can deduce that the density numb of G (the maximum cardinality of a clique in G) is:

$$\omega(G)=\max_{A\subseteq X}\min_{x\in A}[|A|/(|A|-d_{[A]_G}(x))]^*.$$

5. The examples **3** and **4** show that the problem **1** is in general very hard problem. However, when f is nondecreasing in the second va riable, we give a time polynomial bounded algorithm for solving **1**. W note that this approach has been suggested by that used by Finc and Sachs [2] to determine the upper bound due to Wilf and Sze keres [3] for the chromatic number of a graph.

6. Let us consider the following algorithm:

I. $k:=1,\ E_1=E$;

II. for $k:=1,|E|/x_k^* := \min_{x\in E_k} f(x,E_k),\ E_{k+1}:=E_k-\{x_k^*\}/$;

III. $f_{\text{opt}} := \max_{k=1,|E|} f(x_k^*,E_k)$.

7. Theorem. *If for every* $F_1\subseteq F_2$ *and* $x\in F_1$: $f(x,F_1)\leqslant f(x,F_2)$ *then* $\max_{A\subseteq E}\min_{x\in A} f(x,A)=\max_{k=1,|E|} f(x_k^*,E_k)$.

Proof:

We have $\max_{k=1,|E|} f(x_k^*,E_k)=\max_{k=1,|E|}\min_{x\in E_k} f(x,E_k)\leqslant\max_{A\subseteq E}\min_{x\in A} f(x,A)$.

Conversely, let $A_0\subset E$ be such that $\max_{A\subseteq E}\min_{x\in A} f(x,A)=\min_{x\in A_0} f(x,A_0)$.

$i_0=\min\{i/x_i^*\in A_0\}$, then we have $A_0\subseteq E_{i_0}$. Hence $\min_{x\in A_0} f(x,A_0)\leqslant f(x_{i_0}^*,A_0)$

Because f is nondecreasing, we have $f(x^*_{i_0},A_0)\leqslant f(x^*_{i_0},E_{i_0})\leqslant\max_{k=1,|E|} f(x_k^*,E_k)$

Thus, by the choice of A_0, we have $\max_{A\subseteq E}\min_{x\in A} f(x,A)\leqslant\max_{k=1,|E|} f(x_k^*,E_k)$

8. If G is a graph and $\chi(G)$ is the chromatic number of G, Wi and Szekeres have proved in [3] that $\chi(G)\leqslant 1+\max_{G'\subseteq G}\min_{x\in G'} d$

and "the minimal degree sequences method" used by Finck and Sachs for computing the above right side is just the algorithm **6**.

9. We note that if f is not nondecreasing in the second variable, we can use the algorithm **6** for finding a lower bound for $\max_{A\subseteq E}\min_{x\in A} f(x,A)$.

For example, let us consider the graph $G=(X,U)$ where $X=\{1,2,\dots,7\}$ and $U=\{(1,2),(1,3),(1,4),(1,6),(2,3),(4,5),(6,7)\}$. We have $\alpha(G)=\max_{A\subseteq X}\min_{x\in A} f(x,A)$, where $f(x,A)=[|A|/(1+d_{[A]_G}(x))]^*$.

The algorithm 6 gives:

$(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*)=(1,3,4,6,7,5,2)$,

$(f(x_1^*,X_1), f(x_2^*,X_2),\dots,f(x_7^*,X_7))=(2,3,3,2,3,2,1)$.

Hence the lower bound for $\alpha(G)$ is $\max f(x_i^*,X_i)=3$, which is just $\alpha(G)$.

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