nalele științifice ale Universității „Al. I. Cuza" din Iași
omul XXV, s. I a, f. 1

FINITE AUTOMATA WITH A VARIABLE STRUCTURE IN VT-ALGEBRAS

BY

V. FELEA

In this paper the notions of VT-algebra on the theory T, VT-free lgebra and finite automaton with a variable structure in VT-algebras re introduced. The behavior of the finite automaton with a variable structure in VT-algebras is characterized.

1. **VT-algebras.** Let S be the category of sets. It is known that in S the empty set $\varnothing$ is an initial object and the set $I=\{1\}$ is a terminal bject for S. The set $\{1,2,\dots,n\}$ will be denoted by $[n]$, for each integer $n=0,1,2,\dots$. Thus $[0]=\varnothing$ and $[1]=I$. Let S_0 be the subcategory of S, for which $[n]$ are the objects and the morphisms between objects are those from S. For every $1 \leqslant i \leqslant n$ we denote by $i_n: I \to [n]$ the morphisms in S, such that $1 i_n = i$.

Definition 1.1. [4]. *Let $\mathcal{A}$ be a category. The family of morphisms $\{\psi_i: A_i \to A,\ i=1,2,\dots,n\}$ in $\mathcal{A}$ is called a coproduct diagram if for every object B and for each family of morphisms $\{\varphi_i: B \to A_i,\ i=1,2,\dots,n\}$ there exists a unique morphism $\varphi: B \to A$, such that $\psi_i\,\varphi = \varphi_i$, for each* $i=1,2,\dots,n$. It follows that for each $n=1,2,\dots$ the family $\{i_1, i_2,\dots,i_n\}$ is a coproduct diagram in S and in S_0.

Definition 1.2. [4]. *A theory T is a subcategory with the following properties:*

(1) *the objects of T are $[n]$, for $n=0,1,2,\dots$,*
(2) *S_0 is a subcategory of T,*
(3) *for each $n=1,2,\dots$ the family $\{i_1, i_2,\dots,i_n\}$ is a coproduct diagram in T.*

Given the morphisms $\Phi_i: I \to [p]$, $i=1,2,\dots,n$ in T, the unique morphism $\Phi: [n] \to [p]$, such that $\Phi_i = i_n\,\Phi$, $i=1,2,\dots,n$, i.e. the morphism Φ determined by $\{\Phi_i,\ i=1,2,\dots,n\}$ and the coproduct diagram $\{i_1, i_2,\dots,i_n\}$, is denoted by $<\Phi_1, \Phi_2,\dots,\Phi_n>$. Let $T([n],[p])$ be the set of all morphisms $[n] \to [p]$ and $T_p = T(I,[p])$.

Definition 1.3. *Let T and T' be two theories. A morphism between T and T' is a functor between the categories T and T', such that:*

(1) *$[n]F = [n]$, for each object $[n]$ in S_0;*

---

[1]) For other types of algebras, see [1, 2].

(2) $fF = f$, *for each morphism* f *in* S_0;

(3) $<\Phi_1, \Phi_2, ..., \Phi_n> F = <\Phi_1 F, ..., \Phi_n F>$, *for each family of mor-phisms* $\{\Phi_1, ..., \Phi_n\}$ *in* T.

Definition 1.4. *Let* T *be a theory. A VT-algebra* A *on the theory* T *is a system* $<A, \alpha_i, \beta_i>$ *consisting of:*

(1) *an indexed family of finite sets* $A = (A_i)$, $i = 0, 1, 2, ...$;

(2) *the morphisms in* S $\alpha_i : A_{i+1} \to A_i$, $\beta_i : A_i \to A_{i+1}$, *for each* $i = 0, 1, 2, ...$;

(3) *for each morphism* $\Phi : [n] \to [p]$ *in* T, *there exists the family* *morphisms in* S $\{(\Phi\psi_i^{p,n}) : A_i^n \to A_{i+1}^n, i = 0, 1, 2, ...\}$, *such that:*

(3.1) *if* Φ *is in* S_0, $\Phi : [n] \to [p]$ *and* $(x_1^i, ..., x_p^i) \in A_i^p$, *then* $(x_1^i, ...,$ $(\Phi\psi_i^{p,n}) = (y_1^{i+1}, ..., y_n^{i+1})$, *for each* $i = 0, 1, 2, ...$, *where* $y_j^{i+1} = x_{1\Phi_j}^i \beta_i$, $\Phi_j = j$, $j = 1, 2, ..., n$;

(3.2) *If* $\Phi : [n] \to [p]$ *and* $\theta : [p] \to [k]$ *are the morphisms in* *then* $((\Phi\theta)\psi_i^{k,n}) = (\theta\psi_i^{k,p})\tilde{\alpha}_i(\Phi\psi_i^{p,n})$, $i = 0, 1, 2, ...$, *where* $\tilde{\alpha}_i$ *is defined by* $(z_1, z_2, ..., z_p)\tilde{\alpha}_i = (z_1\alpha_i, ..., z_p\alpha_i)$.

If $i = 0$, *we consider* θ *as the morphism in* S_0;

(3.3) *If* $\varphi : [n] \to [p]$ *in* $T, \varphi = <\varphi_1, \varphi_2, ..., \varphi_n>$, *then* $(<\varphi_1, ..., \varphi_n>$ $\psi_i^{p,n}) = <(\varphi_1\psi_i^{p,1}), ..., (\varphi_n\psi_i^{p,1})> = ((\varphi_1\psi_i^{p,1}), ..., (\varphi_n\psi_i^{p,1}))$, $i = 0, 1, 2, ...,$ *where* $\alpha(z_1, ..., z_n) = (\alpha z_1, ..., \alpha z_n)$, z_j *are the morphisms in* S, $j = 1, 2, ..., n$, $\alpha \in A$

Remark 1.1. A T-algebra A [3] is a VT-algebra considering $A_i = A$, $i = 0, 1, 2, ..., \alpha_i = \beta_i = 1_A$.

Remark 1.2, *If* $n = 1$ and $p = 0$, then $\Phi : [1] \to \emptyset$ and $(\;)(\Phi\psi_i^{0,1})$ $\in A_{i+1}$.

Remark 1.3. The morphisms $(\Phi\psi_i^{p,n})$ in (3) can be defined only for each morphism $\Phi : I \to [p]$ in T.

Definition 1.5. *Let* $A = (A_i, \alpha_i^A, \beta_i^A, i = 0, 1, 2, ...)$, $B = (B_i, \alpha_i^B, \beta_i^B,$ $i = 0, 1, 2, ...)$ *be two VT-algebras. A morphism between* A *and* B *is a* *family* $\xi = (\xi_i)$, $i = 0, 1, 2, ...$, *where* ξ_i *are the morphisms in* S, $\xi_i : A_i \to B_i$ *such that for each morphism* $\Phi : [n] \to [p]$ *in* T *the following diagram is* *commutative:*

$$
(1)\qquad
\begin{tikzcd}
A_i^p \arrow[r, "(\Phi\Psi_{A,i}^{p,n})"] \arrow[d, "\tilde{\xi}_i"'] & A_{i+1}^n \arrow[d, "\tilde{\xi}_{i+1}"] \\
B_i^p \arrow[r, "\Phi\Psi_{B,i}^{p,n}"'] & B_{i+1}^n
\end{tikzcd}
$$

for $i = 0, 1, 2, ...$, where $(x_1, ..., x_p)\tilde{\xi}_i = (x_1\xi_i, ..., x_p\xi_i)$

Thus, VT-algebras form a category denoted by C^{VT}. From the Remark 1.3 it follows that we can define the commutativity of the diagram (1) only for the morphisms $\Phi : I \to [p]$ in T.

Theorem 1.1. *Let* $\xi = (\xi_i)$, $i = 0, 1, 2, ...$ *be a morphism between VT-algebras* A *and* B. *For each* $i = 0, 1, 2, ...$ *the following diagram is commutative:*

$$
(2)\qquad
\begin{tikzcd}
A_i \arrow[r, "\beta_i^A"] \arrow[d, "\xi_i"'] & A_{i+1} \arrow[d, "\xi_{i+1}"] \\
B_i \arrow[r, "\beta_i^B"'] & B_{i+1}
\end{tikzcd}
$$

Proof. We have $(\Phi\psi_{A,i}^{p,n})\tilde{\xi}_{i+1} = \tilde{\xi}_i(\Phi\psi_{B,i}^{p,n})$, for each $\Phi : [n] \to [p]$ in T and $i = 0, 1, 2, ...$ Considering $\Phi = 1_{[n]} : [n] \to [n]$, it follows that $(x_1, ..., x_n)$ $(\Phi\psi_{A,i}^{n,n})\tilde{\xi}_{i+1} = (x_{1\Phi_1}\beta_i^A, ..., x_{1\Phi_n}\beta_i^A)\tilde{\xi}_{i+1} = (x_1, ..., x_n)\tilde{\beta}_i^A\tilde{\xi}_{i+1} = (x_1, ..., x_n)$ $\tilde{\xi}_i(\Phi\psi_{B,i}^{n,n}) = (x_1\xi_i, ..., x_n\xi_i)(\Phi\psi_{B,i}^{n,n}) = (x_1\xi_i\beta_i^B, ..., x_n\xi_i\beta_i^B) = (x_1, ..., x_n)\tilde{\xi}_i\tilde{\beta}_i^B$. But $(x_1, ..., x_n)(\Phi\psi_{A,i}^{n,n})\tilde{\xi}_{i+1} = (x_1, ..., x_n)\tilde{\xi}_i(\Phi\psi_{B,i}^{n,n})$, then $\tilde{\beta}_i^A\tilde{\xi}_{i+1} = \tilde{\xi}_i\tilde{\beta}_i^B$, therefore $\beta_i^A\xi_{i+1} = \xi_i\beta_i^B$.

Theorem 1.2. *Let* ξ *be a morphism between VT-algebras* A *and* B. *A sufficient condition for the commutativity of the diagram:*

$$
(3)\qquad
\begin{tikzcd}
A_{i+1} \arrow[r, "\alpha_i^A"] \arrow[d, "\xi_{i+1}"'] & A_i \arrow[d, "\xi_i"] \\
B_{i+1} \arrow[r, "\alpha_i^B"'] & B_i
\end{tikzcd}
$$

is:

(4) *there exist morphisms* $\theta : [p] \to [k]$, $\Phi : [n] \to [p]$ *in* T, *such that* $(\Phi\psi_{A,i}^{k,p})$ *and* $(\Phi\psi_{B,i}^{p,n})$ *are injective.*

Proof. From the commutativity of the diagram (1) for the morphism $\Phi\theta$, it results $((\Phi\theta)\psi_{A,i}^{k,n})\tilde{\xi}_{i+1} = \tilde{\xi}_i((\Phi\theta)\psi_{B,i}^{k,n})$. On the other hand, we have $((\Phi\theta)\psi_{A,i}^{k,n})\tilde{\xi}_{i+1} = (\theta\psi_{A,i}^{k,p})\tilde{\alpha}_i(\Phi\psi_{A,i}^{p,n})\tilde{\xi}_{i+1} = (\theta\psi_{A,i}^{k,p})\tilde{\alpha}_i^A\tilde{\xi}_i(\Phi\psi_{B,i}^{p,n})$.

Similarly $\tilde{\xi}_i((\Phi\theta)\Phi_{B,i}^{k,n}) = \tilde{\xi}_i(\theta\psi_{B,i}^{k,p})\tilde{\alpha}_i^B(\Phi\psi_{B,i}^{p,n}) = (\theta\psi_{A,i}^{k,p})\tilde{\xi}_{i+1}\tilde{\alpha}_i^B(\Phi\psi_{B,i}^{p,n})$. Consequently,

$$
(5)\qquad (\theta\psi_{A,i}^{k,p})\tilde{\alpha}_i^A\tilde{\xi}_i(\Phi\psi_{B,i}^{p,n}) = (\theta\psi_{A,i}^{k,p})\tilde{\xi}_{i+1}\tilde{\alpha}_i^B(\Phi\psi_{B,i}^{p,n}).
$$

Since $(\theta\psi_{A,i}^{k,p})$, $\tilde{\alpha}_i^A\tilde{\xi}_{i+1}$, $(\Phi\psi_{B,i}^{p,n})$, $\tilde{\xi}_{i+1}\tilde{\alpha}_i^B$ are morphisms in S, (5) implies $\tilde{\alpha}_i^A\tilde{\xi}_i = \tilde{\xi}_{i+1}\tilde{\alpha}_i^B$, if further $(\theta\psi_{A,i}^{k,p})$ is surjective and $(\Phi\psi_{B,i}^{p,n})$ is injective.

Remark 1.4. A sufficient condition for the commutativity of the diagram (3) is:

(6) There exist k, $\theta : I \to [k]$, $\Phi : I \to I$, Φ, θ in T, such that $(\theta\psi_{A,i}^{k,1})$ is surjective and $(\Phi\psi_{B,i}^{1,1})$ is injective. This follows from (3.3) and (if $(\Phi_j\psi)$ is injective, $1 \leqslant j \leqslant n$, then $\Phi\psi$ is injective, where $\Phi = <\Phi_1, \Phi_2, ..., \Phi_n>$).

2. VT-free algebras. Let T be a theory and $k \in N$. Consider $C_h = T_k$, $h = 1, 2, ...$ $C_0 = T_k \cap \mathrm{morf}_{S_0}(I, [k])$; α_0 is a morphism, which coincides on C_0 with 1_{C_0}, $\alpha_h = \beta_h = 1_{C_h}$, $h = 1, 2, ..., \gamma\beta_0 = \gamma, \forall\gamma \in C_0$. We organize

$C^k = (C_h)$, $h = 0, 1, 2, \dots$ as a VT-algebra in the following way: gi $\Phi: [n] \to [p]$ in T and $(x_1, x_2, \dots, x_p) \in C_h^p$, we have $x_j \in T_k$, $1 \leqslant j \leqslant$ $x_j: I \to [k]$; there exists an unique morphism $Z: [p] \to [k]$ in T, such

(7) $$x_i = i_p Z, \quad i = 1, 2, \dots, p.$$

We have $\Phi Z: [n] \to [k]$. If we consider $\gamma_j = j_n(\Phi Z)$, $j = 1, 2, \dots, n$, $\gamma_j: I \to [k]$, $\gamma_j \in T_k = C_{i+1}$. We define:

(8) $$(x_1, \dots, x_p)(\Phi\psi^{p,n}_{C^k,i}) = (\gamma_1, \gamma_2, \dots, \gamma_n) \text{ for each } p > 0,$$
$$(\ )(\Phi\psi^{0,1}_{C^k,i}) = \Phi O_k, \text{ where } O_k: \varnothing \to [k] \text{ is a}$$

unique morphism in T.

Let us show that (3.1), (3.2) and (3.3) are satisfied.

If $\Phi \in S_0$, $\Phi: [n] \to [p]$, then $j_n \Phi \in S_0$, $j = 1, 2, \dots, n$. But Φ $= <\Phi_1, \Phi_2, \dots, \Phi_n>$, $j_n \Phi = \Phi_j$, $j = 1, 2, \dots, n$; it follows $I(j_n\Phi) = I$ $j = 1, 2, \dots, n$, $\Phi_j \in S_0$. If $1\Phi_j = i$, $i \in [p]$, then $\Phi_j = i_p$, $\gamma_j = j_n(\Phi Z)$ $= (j_n\Phi)Z = i_pZ = x_i = x_{1\Phi_j}$, $j = 1, 2, \dots, n$, therefore (3.1). Let $\Phi: [n] \to$ and $\theta: [p] \to [h]$ be the morphisms in T. Let us show that:

(9) $$((\Phi\theta)\psi^{h,n}_{C^k,i}) = (\theta\psi^{h,p}_{C^k,i})(\Phi\psi^{p,n}_{C^k,i}).$$

If $(x_1, \dots, x_h) \in T_k^h$, there exists a unique morphism Z in T, $[h] \to [k]$, such that $x_i = i_h Z$, $i = 1, 2, \dots, h$.

By definition we have $(x_1, \dots, x_h)((\Phi\theta)(\psi^{h,n}_{C^k,i}) = (\xi_1, \xi_2, \dots, \xi_n)$, ξ_j $= j_n(\Phi\theta Z)$, $j = \overline{1, n}$; $(x_1, \dots, x_h)\ (\theta\psi^{h,p}_{C^k,i}) = (\eta_1, \eta_2, \dots, \eta_p)$, $\eta_l = l_p'($ $l = 1, 2, \dots, p$ și $\theta Z = <\eta_1, \eta_2, \dots, \eta_p>. (x_1, \dots, x_h)(\theta\psi^{h,p}_{C^k,i})(\Phi\psi^{p,n}_{C^k,i}) = (\eta_1, \eta_2,$ $\dots, \eta_p)(\Phi\psi^{p,n}_{C^k,i}) = (\delta_1, \delta_2, \dots, \delta_n)$, $\delta_j = j_n(\Phi(\theta Z)) = j_n(\Phi\theta Z) = \xi_j$, $j = \overline{1}$

Consequently (9) holds. Let $\Phi: [n] \to [p]$, $\Phi = <\Phi_1, \Phi_2, \dots, \Phi$ be in T. We have $(x_1, \dots, x_p)(\Phi\psi^{p,n}_{C^k,i}) = (\gamma_1, \dots, \gamma_n)$, $\gamma_j = j_n(\Phi Z)$, $j = 1,$ $Z = <x_1, \dots, x_p>$, $(x_1, \dots, x_p)(\Phi_h\psi^{p,1}_{C^k,i}) = \delta_h$, $\delta_h = h_1(\Phi_h Z)$, for $h = 1, 2, \dots$ where h_1 is a morphism in S_0, $h_1: I \to I$, i.e. $h_1 = 1_I$, so that $\delta_h = \Phi_h Z$ $(h_n\Phi)Z = h_n(\Phi Z) = \gamma_h$, $(\Phi\psi^{p,n}_{C^k,i}) = ((\Phi_1\psi^{p,1}_{C^k,i}), \dots, (\Phi_n\psi^{p,1}_{C^k,i}))$. We shall den this VT-algebra by C^k.

Remark 2.1. Let $\gamma \in C_{h+1}$. Then $(1_k, 2_k, \dots, k_k)(\gamma\psi^{k,1}_{C^k,h}) = \gamma$, for $h = 0, 1, 2, \dots$

Proof. If $k > 0$, then $(1_k, 2_k, \dots, k_k)(\gamma\psi^{k,1}_{C^k,h}) = \gamma < 1_k, 2_k, \dots, k_k >$ $= \gamma \,.\, 1_{[k]} = \gamma$, where $1_k, \dots, k_k$ are the morphisms in $S_0 \cap T_k$. For $k =$ we have $(\ )(\gamma\psi^{0,1}_{C^k,i}) = \gamma O_0 = \gamma$, because $O_0: \varnothing \to \varnothing$, $O_0 = 1_\varnothing$. Since th exists a $1-1$ corespondence between $[k]$ and S_k, where $S_k = \text{morf}_{S_0}(I, [k]$ $\cap T_k$, we shall identify i_k by i, for each $i = 1, 2, \dots, k$.

Theorem 2.1. *Let $A = (A_i, \alpha_i, \beta_i,\ i = 0, 1, 2, \dots)$ be a VT-al and $f_0: [k] \to A_0$ a morphism in S_0. Consider $f_{i+1} = f_i\beta_i$, $i = 0, 1, 2$ The family of morphisms $f = (f_i)$, $i = 0, 1, 2, \dots$ admits a unique exten $g = (g_i)$, $i = 0, 1, 2, \dots$ to a morphism between VT-algebras C^k and A.*

Proof. Let $\Phi \in C_0^k$. Then Φ is a morphism in $S_0: I \to [k]$. L

us define $\Phi g_0 = (1\Phi)_k f_0 = (1\Phi) f_0$. If $x_1, \dots, x_p \in C_i^k$ and $\Phi: I \to [p]$, then $\Phi < x_1, \dots, x_p > \in C_{i+1}^k$. We consider

(10) $$(\Phi < x_1, \dots, x_p >) g_{i+1} = (x_1 g_i, x_2 g_i, \dots, x_p g_i)(\Phi\psi^{p,1}_{A,i}).$$

Taking in (10) $p = k$, $x_j = j_p$, $j = 1, 2, \dots, p$, we obtain Φg_{i+1} for each $\Phi \in C_{i+1}$

(11) $$\Phi g_{i+1} = (1_k g_i, \dots, k_k g_i)(\Phi\psi^{k,1}_{A,i}).$$

We observe that the definition of g_{i+1} only for the elements from C_{i+1}^k is not sufficient, because the relation of morphism between VT-algebras C^k and A contains the composition of two morphisms in T. Let us show the commutativity of the diagram (1). Let $\Phi: [n] \to [p]$ a morphism in T and $x_1, \dots, x_p \in C_i^k$, $x_j: I \to [k]$, $j = 1, p$. We have $(x_1, \dots, x_p$ $(\Phi\psi^{p,n}_{C^k,i})\,\tilde{g}_{i+1} = (\gamma_1, \dots, \gamma_n)\tilde{g}_{i+1} = (\gamma_1 g_{i+1}, \dots, \gamma_n g_{i+1})$, where $\gamma_j = j_n\Phi < x_1,$ $x_2, \dots, x_p > = \Phi_j < x_1, \dots, x_p >$, $j = 1, 2, \dots, n$, $\gamma_j \in T_k$, $\Phi = <\Phi_1, \Phi_2, \dots, \Phi_n>$;

$$\gamma_j g_{i+1} = (x_1 g_i, \dots, x_p g_i)(\Phi_j \psi^{p,1}_{A,i});$$

$$(x_1, \dots, x_p)(\Phi\psi^{p,n}_{C^k,i})\,\tilde{g}_{i+1} = ((x_1 g_i, \dots, x_p g_i)(\Phi_1\psi^{p,1}_{A,i}), \dots, (x_1 g_i, \dots, x_p g_i)(\Phi_n\psi^{p,1}_{A,i})) =$$
$$= (x_1 g_i, \dots, x_p g_i)((\Phi_1\psi^{p,1}_{A,i}), \dots, (\Phi_n\psi^{p,1}_{A,i})) = (x_1 g_i, \dots, x_p g_i)(\Phi\psi^{p,n}_{A,i}) =$$
$$= (x_1, \dots, x_p)\,\tilde{g}_i(\Phi\psi^{p,n}_{A,i}),$$

therefore $(\Phi\psi^{p,n}_{C^k,i})\,\tilde{g}_{i+1} = \tilde{g}_i(\Phi\psi^{p,n}_{A,i})$. Now we show that $g = (g_i)$, $i = 0, 1, 2, \dots$ is a unique extension of $f = (f_i)$, $i = 0, 1, 2, \dots$ By definition, we have $\Phi g_0 = (1\Phi)_k f_0 = (1\Phi) f_0 = \Phi f_0$, because $\Phi \in C_0^k = T_k \cap \text{Morf}_{S_0}(I, [k]) =$ $= \{1_k, 2_k, \dots, k_k\}$. Assume that $j_k g_i = j_k f_i$, $j = 1, 2, \dots, n$, $i \geqslant 0$. From (11) for each $\Phi \in C_{i+1}^k \cap \text{Morf}_{S_0}(I, [k])$ we have $\Phi g_{i+1} = (1_k g_i, \dots, k_k g_i)$ $(\Phi\psi^{k,1}_{A,i})$. If $1\Phi = j_k$, $1 \leqslant j \leqslant k$, then $(1_k g_i, \dots, k_k g_i)(\Phi\psi^{k,1}_{A,i}) = (j_k g_i)\beta_i =$ $= (j_k f_i)\beta_i = j_k f_{i+1} = (1\Phi) f_{i+1} = (1\Phi)_k f_{i+1} = \Phi f_{i+1}$. It remains to verify that the extension g is unique. Let $t = (t_i)$, $i = 1, 0, 2, \dots$ be a morphism between VT-algebras C^k and A, which is a extension of $f = (f_i)$, $i =$ $= 0, 1, 2, \dots$ We have $(x_1, \dots, x_p)(\Phi\psi^{p,n}_{C^k,i})\,\tilde{t}_{i+1} = (x_1, \dots, x_p)\,\tilde{t}_i\,(\Phi\psi^{p,n}_{A,i})$, for $\Phi: [n] \to [p]$, $x_j \in C_i^k$, $1 \leqslant j \leqslant p$, $i = 0, 1, 2, \dots$ It results:

(12) $$(\Phi_j < x_1, \dots, x_p >) t_{i+1} = (x_1 t_i, \dots, x_p t_i)(\Phi_j \psi^{p,1}_{A,i}), \ j = 1, 2, \dots, n,$$
$$\Phi = <\Phi_1, \dots, \Phi_n>.$$

Moreover, we have for each $\Phi: I \to [k]$ in T

(13) $$\Phi t_{i+1} = (1_k t_i, \dots, k_k t_i)(\Phi\psi^{k,1}_{A,i}) \text{ for } k > 0 \text{ and}$$
$$\Phi t_{i+1} = (\ )(\Phi\psi^{0,1}_{A,i}) \text{ for } k = 0.$$

If $\Phi \in C_0^k$, $\Phi t_0 = \Phi f_0 = \Phi g_0$. Assume that $\Phi t_i = \Phi g_i$ for each $\Phi \in C_i^k$, $i \geqslant 0$. From (12) and (13) it follows:

$$(\Phi < x_1, \dots, x_p >) t_{i+1} = (\Phi < x_1, \dots, x_p >) g_{i+1}, \quad \Phi: I \to [p],$$
$$\Phi t_{i+1} = \Phi g_{i+1}, \quad \Phi: I \to [k].$$

Theorem 2.2. *VT-algebra C^0 is the initial object in the category C^{VT}.*

Proof. Consider $k = 0$ in the theorem 2.1. There exists a unique morphism in T, $f_0 : \varnothing \to A_0$; by the theorem 2.1 for each VT-alge[...] $A = (A_i, \alpha_i, \beta_i,\ i = 0, 1, 2, \ldots)$ there exists a unique morphism $\xi_A = ($[...] $i = 0, 1, 2, \ldots$ between C° and A.

Remark 2.4. If $\Phi \in T_0$, then $\Phi\xi_{i+1} = (\quad)(\Phi\psi^{0,1}_{A,i})$, $i = 0, 1, 2$[...]

Remark 2.5. $C^0_0 = \varnothing$, because $C^0_0 = T_0 \cap \mathrm{Morf}_S(I, \varnothing)$ and th[...] is no morphism $I \to \varnothing$ in S_0.

3. Finite automate with a variable structure in VT-algebras. In [...] following consider VT-algebras with the property:

(14) there exists $\Phi : I \to I$ in T, such that for each $i = 0, 1, 2, \ldots$ Φ[...] is injective.

Let T be a theory. An finite automaton with a variable structure [...] T (shortly VT-aoutomaton) is (A, t), where A is a VT-algebra, $t = ($[...] t_i is a subset of A_i, $i = 0, 1, 2, \ldots$ and there exists m, such that $|A_i| \leqslant$[...] $i = 0, 1, 2, \ldots$ Let (A, t_1), (B, t_2) be two automata. A morphism ξ betw[...] (A, t_1) and (B, t_2) is a morphism ξ between VT-algebras A and B, su[...] that $t_1 = t_2\xi^{-1}$. Let (A, t) be an VT-automaton. The behavior of A[...] defined as $\beta((A, t)) = t\xi_A^{-1}$, where ξ_A is the unique morphism $\xi_A : C^\circ$[...] from the theorem 2.2. Let A be a VT-algebra and ρ a congruence re[...] tion on A. The relation ρ is finite if there exists $m \in N$, such th[...] for each $i = 0, 1, 2, \ldots$ $|A_i/\rho_i| \leqslant m$. (A/ρ is the set of congruence class[...] of A mod ρ). Let A be a VT-algebra and $t = (t_i)$, $t_i \subseteq A_i$, $i = 1, 2$[...] We say that t is admissible in A, if there exists ρ a finite congruen[...] on A, such that t is closed for ρ, that is for each $i = 0, 1, 2, \ldots$ $a\rho$[...] and $a \in t_i$ implies $a' \in t_i$. It results that the class of all the families [...] sets from A, which are admissible in A, is closed relative to fin[...] Boolean operations.

Theorem 3.1. *The class of all the behaviors of VT-automata coinci[...] with the class of the admissible families in C°.*

Proof. Let A be an VT-automaton. From (14) and Remark 2[...] it follows (6). Then the morphism $\xi_A : C^\circ \to A$ defines a congruence [...] on C°: $a, a' \in C^0_i$, $a\rho_i a'$ if $a\xi_{A,i} = a'\xi_{A,i}$, $i = 0, 1, 2, \ldots$ Since A is finite, [...] follows that ρ is finite. The behavior of A is closed relative to ρ: if $a\rho_i$[...] and $a \in t_i\xi^{-1}_{A,i}$, then $a\xi_{A,i} = a'\xi_{A,i}$, $\exists b_i \in t_i$, $b_i = a\xi_{A,i} = a'\xi_{A,i}$, therefo[...] $a' \in b_i\xi^{-1}_{A,i} \subseteq t_i\xi^{-1}_{A,i}$, $i = 0, 1, 2, \ldots$, that is $\beta((A, t))$ is admissible in C[...] Conversely, let t_0 be a admissible family in C°. Then we have a co[...] gruence ρ on C°, such that t_0 is closed relative to ρ. There exists th[...] canonical epimorphism $\Psi : C^\circ \to C^\circ/\rho$. Let us consider $A = C^\circ/\rho$ and $t_i =$[...] $= t_{0,i}/\rho_i$, $t = (t_i)$, $i = 0, 1, 2, \ldots$ Since there exists a unique morphi[...] between C° and A, namely ξ_A, it results that $\xi_A = \chi$. Let us show th[...] $t_0 = t\xi_A^{-1}$, that means for each $i = 0, 1, 2, \ldots$

$$t_{0,i}\,\xi_{A,i} \subseteq t_i \text{ and} \tag{15}$$

$$\text{for every } a_i \in C^0_i,\ a_i\xi_{A,i} \in t_i \text{ implies } a_i \in t_{0,i}. \tag{16}$$

Indeed, if $a_i \in t_{0,i}$, then $a_i\xi_{A,i} = a_i\chi_i = Z_i$, where Z_i is the class whic[...] contains a_i; hence $Z_i \in t_i$; if $a_i \in C^0_i$ and $a_i\xi_{A,i} \in t_i$, then $a_i\chi_i \in t_i$[...] $a_i\chi_i = Z_i$, $Z_i \in t_i$, it follows that $a_i \in t_{0,i}$.

REFERENCES

1. A r b i b M. A., G i v e'o n Y. — *Algebra Automata I: Parallel Programming as a Prolegomena to the Categorical Approach*, Information and Control, vol. 12, nr. 4, 1968, 331—345.
2. C o h n P. M. — *Universal Algebra*, Harper, New York, 1965.
3. E i l e n b e r g S., W r i g h t J. — *Automata in General Algebras*, Information and Control, vol. 11, nr. 4, 1967, 452—470.
4. G i v e'o n Y., A r b i b M. A. — *The Categorical Framework for Dynamic Analysis*, Information and Control, vol. 12, nr. 4, 1968, 346—370.

Received 29.III.1978

Faculty of Mathematics
University of Iaşi
R. S. Romania