Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXV, s. I a, 1979, f 1

ON THE *NP*-COMPLETNESS OF A PROBLEM IN FINITE AUTOMATA THEORY

BY

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0. Introduction. Beginning with the papers of C o o k [2] and K a r p [4], one of the most important open problems in theoretical computer science is: does $P = NP$? (in other words, given a nondeterministic Turing machine M, which runs in polynomial time, does there exist a deterministic Turing machine M' with polynomial time complexity too and both M and M' recognize the same language?). The importance of the concept of "NP-complete problem" for the question $P = NP$, was emphazized by K a r p [4]. Starting with the vertex cover problem, which is NP-complete (see [1], [4]), we will show that the feedback edge set problem, [1], with some restrictions is NP-complete and a related problem of finite automata theory is NP-complete, too.

1. Preliminary facts. Let $G = (V, E)$ be an undirected graph. *A vertex cover* of G is a subset $S \subseteq V$ such that each edge of G is incident upon some vertex in S. The *vertex cover problem* is: Given an undirected graph $G = (V, E)$ and a natural number k, does G have a vertex cover S with k vertices $(|S| = k)$?

Let $G = (V, E)$ be a directed graph. A *feedback edge set* is a subset $F \subseteq E$ such that every circuit of G contains an edge from F. The *feedback edge set problem* is: Given a directed graph $G = (V, E)$ and a natural number k, does G have a feedback edge set F with $|F| = k$?

Theorem 1. *The vertex cover and feedback edge set problems are NP-complete.*

Proof see [1], p. 387.

2. New *NP*-complete problems. We first define a restriction of the feedback edge set problem.

Definition 1. *Let $G = (V, E)$ be a directed graph and let v_0 be a vertex from V. An initial feedback edge set (i.f.e.s) with respect to v_0 is a subset $F \subseteq E$ such that every circuit of G passing through v_0 with at least 4 edges contains an edge from F.*

The i.f.e.s problem is: Given a directed graph $G = (V, E)$, a vertex $v_0 \in V$ and a natural number k, when does G have an i.f.e.s with respect to v_0 and $|F| = k$.

Theorem 2. *The i.f.e.s problem is NP-complete.*

Proof. The NP-completness of the i.f.e.s problem follows from the fact that it belongs to NP, (which is obvious), and from the polynomially reducibility of the vertex cover problem to this one.

Let $G=(V,E)$ be an undirected graph and k a natural number. To prove the theorem it remains to construct in deterministic polynomial time a directed graph $G'=(V',E')$ such that, G' has an i.f.e.s F' with respect to some v_0, $|F'|=k$, if and only if G has a vertex cover with k members. Let $V=\{v_1, v_2, \ldots, v_n\}$; then we define $V'=\{v_0\} \cup \cup\{[v',0], [v',1], 1 \leqslant i \leqslant n\}$ and E' consists of:

i) $([v,0], [v,1])$ for each v from V;

ii) $([v_i,1], [v_j,0])$ for each undirected edge $(v_i,v_j) \in E$ and $1 \leqslant i < j \leqslant n$.

iii) $(v_0, [v_i,0])$, $([v_j,1], v_0)$ for each $([v_i,1], [v_j,0]) \in E'$ (constructed in ii)).

Now, let $S \subseteq V$ be a vertex cover of G, and $|S|=k$.

We claim that the set $F'=\{([v,0], [v,1])/v \in S\}$ is an i.f.e.s with respect to v_0. Indeed, let $v_0, u_1, u_2, \ldots, u_c, v_0$ be a circuit in G', with $l \geqslant 3$. It follows that this circuit contains at least one edge of the form $([v_i,1], [v_j,0])$, $i<j$ (because of the construction of E'). But $([v_i,1], [v_j,0])$ is an edge of G' if and only if (v_i,v_j) is an edge of G. Therefore either v_i or v_j belong to S. We note that, if $([v_i,1], [v_j,0])$ is an edge in circuit, then both $([v_i,0], [v_i,1])$ and $([v_j,0], [v_j,1])$ are in circuit. Hence, either $([v_i,0], [v_i,1])$ or $([v_j,0], [v_j,1])$ belong to F'.

Conversely, let F' be an i.f.e.s with respect to v_0 in G' and $|F'|=k$. The adges in F' can be of the form i), ii), or iii) from the definition of E'. We note that if we have in G' a circuit of length at least 4, then:

1) If it passes through $(v_0, [v,0])$, then it passes through $([v,0], [v,1])$, too;

2) If it passes through $([v,1], v_0)$, then it passes through $([v,0], [v,1])$, too;

3) If it passes through $([v_i,1], [v_j,0])$, then it passes through $([v_j,0], [v_j,1])$, too;

Replacing the edges of the form ii) and iii) in F' by the corresponding edges defined in 1), 2), 3), we obtain a set F'' which is an i.f.e.s with respect to v_0 and $|F''| \leqslant |F'|$. Therefore, without loss of generality, we suppose that F' contains only edges of the form $([v,0], [v,1])$. We claim that $S=\{v/v \in V, ([v,0], [v,1]) \in F''\}$ is a vertex cover of G. Indeed, let (v_i,v_j) be an edge of E $(i<j)$. Then we have in G' the circuit: $v_0, [v_i,0], [v_i,1], [v_i,0], [v_j,1], v_0$, which must have an edge of F'. Hence, either $([v_i,0], [v_j,1])$ or $([v_j,0], [v_j,1])$ are in F' and, of course, either v_i or v_j are in S; S is a vertex cover for G.

It is obvious that, the construction of G' is made in polynomial time depending on the dimension of G. This completes the proof.

Definition 2. *Let $G=(V,E)$ be a directed graph and let v_0 be a vertex of V. A l-extended i.f.e.s with respect to v_0 is a subset $F \subseteq E$, such that every circuit of G passing through v_0 with at least l edges contains an edge from F, $l \geqslant 4$.*

The l-extended i.f.e.s problem is: Given a directed graph $G=(V,E)$, a vertex v_0 in V and a natural number k, when does G have a l-extended i.f.e.s with respect to v_0 and $|F|=k$.

Theorem 3. *The l-extended i.f.e.s problem is NP-complete.*

Proof. We will show that the vertex cover problem is polynomially reducible to the l-extended i.f.e.s problem.

Let $G=(V,E)$ be an undirected graph, $V=\{v_1, v_2, \ldots, v_n\}$, and k a natural number. Let us consider the directed graph $G'=(V',E')$ where:

$$V'=\{v_0\} \cup \{[v_i,0], [v_i,1], \ldots, [v_i, l-3], 1 \leqslant i \leqslant n\},$$

E' consists of:

i) $([v,h], [v,h+1])$ for each $v \in V$, $0 \leqslant h < l-3$;

ii) $([v_i, l-3], [v_j,0])$ for each undirected edge $(v_i,v_j) \in E$ and $1 \leqslant i < j \leqslant n$;

iii) $(v_0, [v_i,0])$, $([v_j, l-3], v_0)$ for each $([v_i, l-3], [v_j,0]) \in E'$.

It is clear that G' is obtained from G in polynomial time depending on the dimension of G. It is easy to prove that G has a vertex cover with k vertices if and only if G' has a l-extended i.f.e.s with respect to v_0 with k edges (we will omit this proof which is similar to the proof of the Theorem 2).

Consider, [3] p. 67, the automaton $A=(I,S,f,s_0)$, where: I (the set of inputs) and S (the set of states) are finite sets, $f: S \times I \to S$ is the transition function of automaton and s_0 is the initial state $(s_0 \in S)$.

Denote the language accepted by the finite automaton A, with initial state s_0 and final state $S_1 \subseteq S$ by

$$L(A, s_0, S_1) = \{p/p \in I^*, f(s_0,p) \in S_1\}.$$

The language accepted by the finite automaton A, with the initial state set S_1 and final state set $S_2 (S_1, S_2 \subseteq S)$ is the set:

$$L(A, S_1, S_2) = \bigcup_{s_1 \in S_1} L(A, s_1, S_2).$$

Let us consider the following *problem* (denoted by $P(A,k)$):
Given a finite automaton $A=(I,S,f,s_0)$ and a natural number k, does there exist $I_1 \subseteq I$, $|I_1|=k$ and $S_1, S_2 \subseteq S$ such that:

$$I^n \cap L(A, s_0, \{s_0\}) \subseteq L(A, s_0, S_1) \, . \, I_1 \, . \, L(A, S_2, \{s_0\}), \ \forall n \geqslant 4?$$

In other words, does there exist a subset I_1 of I with k symbols, such that every word in $L(A, s_0, \{s_0\})$ with length at least 4 has a symbol of I_1?

The next theorem shows that this problem is "hard„: if there exist a deterministic algorithm for solving this problem in polynomial time, then $P=NP$.

Theorem 4. *The problem $P(A,k)$ is NP-complete.*

Proof. We will show that the i.f.e.s problem is polynomially transformable to the $P(A,k)$ problem. From Theorem 2, i.f.e.s problem is NP-complete, therefore $P(A,k)$ problem is NP-complete, too.

Let us consider a directed graph $G=(V,E)$, a vertex v_0 in V and a natural number k. We construct the following finite automaton:

$A=(I,S,f,s_0)$ where: i) $S=V \cup \{\bar{s}\}$, $\bar{s} \notin V$, ii) $I=V \times V$,

iii) $s_0=v_0$, iv) $f: S \times I \to S$ defined by:

$$\forall u \in V, \ \forall (v,w) \in V \times V, \ f(u,(v,w)) = \begin{cases} w \text{ if } u=v \text{ and } (v,w) \in E \\ \bar{s} \text{ otherwise} \end{cases}$$

$$\forall (v, w) \in V \times V,\ f(s, (v, w)) = \bar{s}$$

Let us prove that the problem $P(A, k)$ has an afirmative answer, if and only if, the graph $G = (V, E)$ has an i.f.e.s, F, with respect to v_0, $|F| = k$.

If. Let $F = \{(v_{i_1}, w_{i_1}), (v_{i_2}, w_{i_2}), \ldots, (v_{ik}, w_{ik})\} \subseteq E$ be an i.f.e.s with respect to v_0. Let us consider $I_1 = F \subseteq V \times V = I$ and $S_1 = \{v / v \in V,$ $\exists w \in V, (v, w) \in F\}$, $S_2 = \{w / w \in V, \exists v \in V, (v, w) \in F\}$. We claim that :

$$I^n \cap L(A, s_0, \{s_0\}) \subseteq L(A, s_0, S_1) \,.\, I_1 \,.\, L(A, S_2, \{s_0\}), \quad \forall n \geq 4.$$

Indeed, let $p \in L(A, s_0, \{s_0\})$, $p = \sigma_1 \sigma_2 \ldots \sigma_n$, $n \geq 4$. From $f(s_0, p) = s_0$ and the definition of A, it follows that there exists the circuit $(v_0, v_1,$ $v_2, v_3, \ldots, v_n = v_0)$ in G where $\sigma_r = (v_{r-1}, v_r)$ and $v_r = f(v_{r-1}, \sigma_r)$ $1 \leq r \leq n$. Now, F is an i.f.e.s with respect to v_0, hence, there exists one $j (0 \leq$ $\leq j \leq n - 1)$ such that $(v_j, v_{j+1}) \in F$ and, therefore, $\sigma_{j+1} \in I_1$. We note that $f(s_0, \sigma_1 \sigma_2 \ldots \sigma_j) = v_j$, $v_j \in S_1$ and $p_1 = \sigma_1 \ldots \sigma_j \in L(A, s_0, S_1)$; $v_{j+1} \in$ $\in S_2$, $f(v_{j+1}, \sigma_{j+2} \ldots \sigma_m) = s_0$ and $p_2 = \sigma_{j+2} \ldots \sigma_n \in L(A, S_2, \{s_0\})$, therefore $p = p_1 \sigma_{j+1} p_2$ and the above inclusion is proved.

Only if. Let us suppose then, there exist $I_1 \subseteq I$, $|I_1| = k$, S_1, $S_2 \subseteq S$ such that $\forall p \in L(A, s_0, \{s_0\})$, $|p| \geq 4$, $p = p_1 \sigma p_2$ with $p_1 \in$ $\in L(A, s_0, S_1)$, $\sigma \in I_1$ and $p_2 \in L(A, S_2, \{s_0\})$. Then, we have $f(s_0, p_1) =$ $= s_1 \in S_1$, $f(s_1, \sigma) = s_2 \in S_2$, $f(s_2, p) = s_0$. It follows that, if $s_1 = v \in V$, $s_2 = w \in V$ then $\sigma = (v, w) \in E$. Therefore I_1 is a subset of edges from E. We claim that $F = I_1$ is an i.f.e.s with respect to v_0. Indeed, let $(v_0, v_1, \ldots, v_m = v_0)$ be a circuit in G, $m \geq 4$. Then we have the word $(v_0, v_1)(v_1, v_2)(v_2, v_3) \ldots (v_{m-1}, v_m)$ in $L(A, s_0, \{s_0\})$ and therefore there exist $\sigma \in I_1$ and $j (1 \leq j \leq m)$, such that $(v_{j-1}, v_j) = \sigma \in I_1$. Hence we have the edge (v_{j-1}, v_j) of circuit in F and the proof is completed.

The analogous problem for vertices is studied in [5].

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Received 25.IV.1978

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXV, s. I a, f. 1

ON CERTAIN VECTOR SPACES ASSOCIATED TO THE ORIENTED FINITE GRAPHS

BY

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1. Introduction. In this paper we use the vector spaces $\mathrm{Im}\nabla$ and $\mathrm{Ker}\Delta$ associated to an oriented finite graph in [4], in order to define and characterize two relations (denoted by "$\mathcal{J}$" and "$\mathcal{K}$") introduced on the set of arcs of such a graph.

We will show that $a_\alpha \mathcal{J} a_\beta$ if and only if there exists a non null vector $X_{[\alpha,\beta]} \in \mathrm{Ker}\,\Delta$, so that $a_\alpha \in \mathcal{A}(X_{[\alpha,\beta]})$ or $a_\beta \in \mathcal{A}(X_{[\alpha,\beta]})$ and for every $X =$ $= \sum_{k=1}^{q} x_k a_k \in \mathrm{Ker}\Delta$ we have $|x_\alpha| = |x_\beta|$.

From the graph's theory point of view, this result shows that if $a_\alpha \mathcal{J} a_\beta$, then there exists at least an elementary cycle ω, so that $a_\alpha \in \mathcal{A}(\omega)$ or $a_\beta \in \mathcal{A}(\omega)$, (we have denoted by $\mathcal{A}(\omega)$ the set of arcs of ω).

Concerning the relation "$\mathcal{K}$", we show that $a_\alpha \mathcal{K} a_\beta$, if and only if there exists a not null vector $Y_{[\alpha,\beta]} \in \mathrm{Im}\nabla$, so that $a_\alpha \in \mathcal{A}(Y_{[\alpha,\beta]})$ or $a_\beta \in$ $\in \mathcal{A}(Y_{[\alpha,\beta]})$ and for every

$$Y = \sum_{k=1}^{q} y_k a_k \in \mathrm{Im}\nabla, \text{ we have } |y_\alpha| = |y_\beta|.$$

This results means that if $a_\alpha \mathcal{K} a_\beta$, then there exists at least an elementary section $\mathcal{S}$, so that $a_\alpha \in \mathcal{A}^*$ or $a_\beta \in \mathcal{A}^*$ (we have denoted by $\mathcal{A}^*$ the set of arcs of $\mathcal{S}$).

2. Preliminary considerations. Throughout this paper we have in view an *oriented finite graph* $G = \langle \mathcal{N}, \mathcal{A} \rangle$, defined in [1], where $\mathcal{N} =$ $= \{n_1, n_2, \ldots, n_p\}$ is the set of *nodes*, and $\mathcal{A} = \{a_1, a_2, \ldots, a_q\}$ the set of *arcs*. An arc $a \in \mathcal{A}$ will be noted by $a = \langle n, m \rangle$, $n, m \in \mathcal{N}$. We consider the functions $\Delta^+ : \mathcal{A} \to \mathcal{N}$ and $\Delta^- : \mathcal{A} \to \mathcal{N}$, where $\Delta^+(a) = n$, and $\Delta^-(a) = m$.

We call *matrix of incidence* (see [2]), the matrix $\Lambda = (\Lambda^i_k)_{\substack{i=1,2,\ldots,p \\ k=1,2,\ldots,q}}$ defined as follows :

$$\Lambda^i_k = \begin{cases} 1, \text{ if } n_i = \Delta^+(a_k) \neq \Delta^-(a_k), \\ -1, \text{ if } n_i = \Delta^-(a_k) \neq \Delta^+(a_k), \\ 0, \text{ otherwise.} \end{cases}$$