$$\forall (v, w) \in V \times V,\ f(s, (v, w)) = \bar{s}$$

Let us prove that the problem $P(A, k)$ has an afirmative answer, if and only if, the graph $G = (V, E)$ has an i.f.e.s, F, with respect to v_0, $|F| = k$.

If. Let $F = \{(v_{i_1}, w_{i_1}), (v_{i_2}, w_{i_2}), \ldots, (v_{i_k}, w_{i_k})\} \subseteq E$ be an i.f.e.s with respect to v_0. Let us consider $I_1 = F \subseteq V \times V = I$ and $S_1 = \{v/v \in V, \exists w \in V, (v, w) \in F\}$, $S_2 = \{w/w \in V, \exists v \in V, (v, w) \in F\}$. We claim that:

$$I^n \cap L(A, s_0, \{s_0\}) \subseteq L(A, s_0, S_1) \,.\, I_1 \,.\, L(A, S_2, \{s_0\}), \quad \forall n \geqslant 4.$$

Indeed, let $p \in L(A, s_0, \{s_0\})$, $p = \sigma_1 \sigma_2 \ldots \sigma_n$, $n \geqslant 4$. From $f(s_0, p) = s_0$ and the definition of A, it follows that there exists the circuit $(v_0, v_1, v_2, v_3, \ldots, v_n = v_0)$ in G where $\sigma_r = (v_{r-1}, v_r)$ and $v_r = f(v_{r-1}, \sigma_r)$ $1 \leqslant r \leqslant n$. Now, F is an i.f.e.s with respect to v_0, hence, there exists one $j (0 \leqslant \leqslant j \leqslant n-1)$ such that $(v_j, v_{j+1}) \in F$ and, therefore, $\sigma_{j+1} \in I_1$. We note that $f(s_0, \sigma_1 \sigma_2 \ldots \sigma_j) = v_j$, $v_j \in S_1$ and $p_1 = \sigma_1 \ldots \sigma_j \in L(A, s_0, S_1)$; $v_{j+1} \in \in S_2$, $f(v_{j+1}, \sigma_{j+2} \ldots \sigma_m) = s_0$ and $p_2 = \sigma_{j+2} \ldots \sigma_n \in L(A, S_2, \{s_0\})$, therefore $p = p_1 \sigma_{j+1} p_2$ and the above inclusion is proved.

Only if. Let us suppose then, there exist $I_1 \subseteq I$, $|I_1| = k$, S_1, $S_2 \subseteq S$ such that $\forall p \in L(A, s_0, \{s_0\})$, $|p| \geqslant 4$, $p = p_1 \sigma p_2$ with $p_1 \in \in L(A, s_0, S_1)$, $\sigma \in I_1$ and $p_2 \in L(A, S_2, \{s_0\})$. Then, we have $f(s_0, p_1) = = s_1 \in S_1$, $f(s_1, \sigma) = s_2 \in S_2$, $f(s_2, p) = s_0$. It follows that, if $s_1 = v \in V$, $s_2 = w \in V$ then $\sigma = (v, w) \in E$. Therefore I_1 is a subset of edges from E. We claim that $F = I_1$ is an i.f.e.s with respect to v_0. Indeed, let $(v_0, v_1, \ldots, v_m = v_0)$ be a circuit in G, $m \geqslant 4$. Then we have the word $(v_0, v_1)(v_1, v_2)(v_2, v_3) \ldots (v_{m-1}, v_m)$ in $L(A, s_0, \{s_0\})$ and therefore there exist $\sigma \in I_1$ and $j (1 \leqslant j \leqslant m)$, such that $(v_{j-1}, v_j) = \sigma \in I_1$. Hence we have the edge (v_{j-1}, v_j) of circuit in F and the proof is completed.

The analogous problem for vertices is studied in [5].

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Faculty of Mathematics
University of Iaşi
R. S. Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
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ON CERTAIN VECTOR SPACES ASSOCIATED TO THE ORIENTED FINITE GRAPHS

BY

DĂNUŢ MARCU

1. Introduction. In this paper we use the vector spaces $\mathrm{Im}\nabla$ and $\mathrm{Ker}\Delta$ associated to an oriented finite graph in [4], in order to define and characterize two relations (denoted by "$\mathcal{J}$" and "$\mathcal{K}$") introduced on the set of arcs of such a graph.

We will show that $a_\alpha \mathcal{J} a_\beta$ if and only if there exists a non null vector $X_{[\alpha,\beta]} \in \mathrm{Ker}\,\Delta$, so that $a_\alpha \in \mathcal{A}(X_{[\alpha,\beta]})$ or $a_\beta \in \mathcal{A}(X_{[\alpha,\beta]})$ and for every $X = = \sum_{k=1}^{q} x_k a_k \in \mathrm{Ker}\Delta$ we have $|x_\alpha| = |x_\beta|$.

From the graph's theory point of view, this result shows that if $a_\alpha \mathcal{J} a_\beta$, then there exists at least an elementary cycle ω, so that $a_\alpha \in \mathcal{A}(\omega)$ or $a_\beta \in \mathcal{A}(\omega)$, (we have denoted by $\mathcal{A}(\omega)$ the set of arcs of ω).

Concerning the relation "$\mathcal{K}$", we show that $a_\alpha \mathcal{K} a_\beta$, if and only if there exists a not null vector $Y_{[\alpha,\beta]} \in \mathrm{Im}\nabla$, so that $a_\alpha \in \mathcal{A}(Y_{[\alpha,\beta]})$ or $a_\beta \in \in \mathcal{A}(Y_{[\alpha,\beta]})$ and for every

$$Y = \sum_{k=1}^{q} y_k a_k \in \mathrm{Im}\nabla, \text{ we have } |y_\alpha| = |y_\beta|.$$

This results means that if $a_\alpha \mathcal{K} a_\beta$, then there exists at least an elementary section $\mathcal{S}$, so that $a_\alpha \in \mathcal{A}^*$ or $a_\beta \in \mathcal{A}^*$ (we have denoted by $\mathcal{A}^*$ the set of arcs of $\mathcal{S}$).

2. Preliminary considerations. Throughout this paper we have in view an *oriented finite graph* $G = <\mathcal{N}, \mathcal{A}>$, defined in [1], where $\mathcal{N} = = \{n_1, n_2, \ldots, n_p\}$ is the set of *nodes*, and $\mathcal{A} = \{a_1, a_2, \ldots, a_q\}$ the set of *arcs*. An arc $a \in \mathcal{A}$ will be noted by $a = <n, m>$, $n, m \in \mathcal{N}$. We consider the functions $\Delta^+ : \mathcal{A} \to \mathcal{N}$ and $\Delta^- : \mathcal{A} \to \mathcal{N}$, where $\Delta^+(a) = n$, and $\Delta^-(a) = m$.

We call *matrix of incidence* (see [2]), the matrix $\Lambda = (\Lambda_k^i)_{\substack{i=1,2,\ldots,p \\ k=1,2,\ldots,q}}$ defined as follows:

$$\Lambda_k^i = \begin{cases} 1, \text{ if } n_i = \Delta^+(a_k) \neq \Delta^-(a_k), \\ -1, \text{ if } n_i = \Delta^-(a_k) \neq \Delta^+(a_k), \\ 0, \text{ otherwise.} \end{cases}$$

We denote by (see [3]) $V[\mathcal{A}, R]$ the set of linear forms X with $\mathcal{A}$ as basis and R (the field of real numbers) as domain of coefficients, $X = \sum_{k=1}^{q} x_k a_k$, $x_k \in R$, where $\pm a_i$ is the vector with all $x_k = 0$ except for $x_i = \pm 1$ and $\theta_{\mathcal{A}}$ the vector with all $x_k = 0$. Similary we define the vector space $V[\mathcal{N}, R]$ as all vectors $Z = \sum_{i=1}^{p} z_i n_i$, $z_i \in R$, where $0_{\mathcal{N}}$ is the vector with all $z_i = 0$, and $\pm n_i$ is the vector with all $z_j = 0$ for $z_i = \pm 1$. $V[\mathcal{A}, R]$ is R^q and $V[\mathcal{N}, R]$ is R^p.

We will attach (see [4]) to the vector spaces $V[\mathcal{A}, R]$ and $V[\mathcal{N}, R]$ the mappings $\Delta : V[\mathcal{A}, R] \to V[\mathcal{N}, R]$, $\nabla : V[\mathcal{N}, R] \to V[\mathcal{A}, R]$ defined as follows:

$$\Delta(X) = \sum_{i=1}^{p} \left(\sum_{k=1}^{q} \Lambda_k^i x_k \right) n_i ; \quad \nabla(Z) = \sum_{k=1}^{q} \left(\sum_{i=1}^{p} \Lambda_k^i z_i \right) a_k, \text{ where}$$

$$X = \sum_{k=1}^{q} x_k a_k \in V[\mathcal{A}, R] \text{ and } Z = \sum_{i=1}^{p} z_i n_i \in V[\mathcal{N}, R].$$

From the definitions of Δ and ∇ it results that:

$$\Delta(a_k) = \sum_{i=1}^{p} \Lambda_k^i n_i, \; k = 1, 2, \dots, q \text{ and } \nabla(n_i) = \sum_{k=1}^{q} \Lambda_k^i a_k, \; i = 1, 2, \dots, p$$

implying that Δ and ∇ are two homomorphisms between $V[\mathcal{A}, R]$ and $V[\mathcal{N}, R]$ with Λ as linear transformation matrix.

We denote by $\operatorname{Im}\nabla$ the subspace $\nabla(V[\mathcal{N}, R]) = \{Y \in V[\mathcal{A}, R] \mid$ there exists $Z \in V[\mathcal{N}, R]$ so that $\nabla(Z) = Y\}$ and by $\operatorname{Ker}\Delta$ the subspace $\operatorname{Ker}\Delta = \{X \in V[\mathcal{A}, R] \mid \Delta(X) = \theta_{\mathcal{N}}\}$.

For two vectors $X = \sum_{k=1}^{q} x_k a_k$, $Y = \sum_{k=1}^{q} y_k a_k$ of $V[\mathcal{A}, R]$ we define the *inner product* $[X \mid Y] = \sum_{k=1}^{q} x_k y_k = \sum_{k=1}^{q} y_k x_k = [Y \mid X]$.

If $\mathfrak{X}$ is a finit dimensioned vector space with $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ as basis and R as domain of coefficients and $X = \sum_{i=1}^{n} x_i b_i \in \mathfrak{X}$, we denote by $\mathcal{B}(X)$ the set $\mathcal{B}(X) = \{b_i \in \mathcal{B} \mid x_i \neq 0\}$.

A vector $X \in \mathfrak{X}$ is *minimal*, related to a non null subspace $\mathcal{W}$ of $\mathfrak{X}$, if and only if X is not null, $X \in \mathcal{W}$, and for every not null $Y \in \mathfrak{X}$ with $\mathcal{B}(Y) \subset \mathcal{B}(X)$, results $Y \notin \mathcal{W}$. (see [4]).

A vector $X = \sum_{k=1}^{q} x_k a_k \in \operatorname{Ker}\Delta$ (or $\operatorname{Im}\nabla$) is *elementary*, if and only if it is minimal related to $\operatorname{Ker}\Delta$ (or $\operatorname{Im}\nabla$), and $x_k \in \{-1, 0, 1\}$ for all $k = 1, 2, \dots, q$.

3. Relation "$\mathcal{J}$". In this paragraph we introduce on the $\mathcal{A}$ set the relation "$\mathcal{J}$", giving a characterization theorem of it ("$\mathcal{J}$").

3.1. Definition. *Let $\mathcal{N}^* \subset \mathcal{N}$, $\mathcal{N}^* \neq \varnothing$ be. We call section, induced by $\mathcal{N}^*$ a set of arcs with sign $\mathcal{S} = \{\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \dots, \varepsilon_r a_{k_r}\}$, $(\varepsilon_j = \pm 1, j = 1, 2, \dots, r)$, so that for every a_{k_j}, $j = 1, 2, \dots, r$, one of the following relations is verified:*

a) $\Delta^+(a_{k_j}) \in \mathcal{N}^*$ and $\Delta^-(a_{k_j}) \in \mathcal{N} - \mathcal{N}^*$, if $\varepsilon_j = 1$,

b) $\Delta^-(a_{k_j}) \in \mathcal{N}^*$ and $\Delta^+(a_{k_j}) \in \mathcal{N} - \mathcal{N}^*$, if $\varepsilon_j = -1$,

the set $A^* = \{a_{k_1}, a_{k_2}, \dots, a_{k_r}\}$ being maximal (related to the inclusion of the set) *with this property.*

3.2. Definition. *Let $a_\alpha, a_\beta \in \mathcal{A}$ be $\alpha, \beta \in \{1, 2, \dots, q\}$, $\alpha \neq \beta$. We say that $a_\alpha \mathcal{J} a_\beta$ if and only if we have simultaneously:*

(3.1) *there doesn't exist any section with $\mathcal{A}^* = \{a_\alpha\}$ or $\mathcal{A}^* = \{a_\beta\}$,*

(3.2) *there exists a not null vector $Y^{[0]}_{[\alpha,\beta]} \in \operatorname{Im}\nabla$, so that*

$$\mathcal{A}(Y^{[0]}_{[\alpha,\beta]}) = \{a_\alpha, a_\beta\}.$$

3.1. Lemma. *If $a_\alpha \mathcal{J} a_\beta$, then the vector $Y^{[0]}_{[\alpha,\beta]}$ is minimal related to* $\operatorname{Im}\nabla$.

Proof. Suppose that $Y^{[0]}_{[\alpha,\beta]}$ is not minimal related to $\operatorname{Im}\nabla$. In this case there exists $\tilde{Y} \in \operatorname{Im}\nabla$, $\tilde{Y} \neq 0_{\mathcal{A}}$, so that $\mathcal{A}(\tilde{Y}) \subset \mathcal{A}(Y^{[0]}_{[\alpha,\beta]})$. Because $\mathcal{A}(Y^{[0]}_{[\alpha,\beta]}) = \{a_\alpha, a_\beta\}$, then $\mathcal{A}(\tilde{Y}) = \{a_\alpha\}$ or $\mathcal{A}(\tilde{Y}) = \{a_\beta\}$, and the vector $\tilde{Y}$ has one (and only one) of the forms:

$$\tilde{Y} = \tilde{y}_\alpha a_\alpha, \; \tilde{y}_\alpha \in R - \{0\}, \tag{3.3}$$

$$\tilde{Y} = \tilde{y}_\beta a_\beta, \; \tilde{y}_\beta \in R - \{0\}. \tag{3.4}$$

Evidently, in this case, $\tilde{Y}$ is minimal related to $\operatorname{Im}\nabla$ and, from the theorem 2.2 from [4], it exists a section $\mathcal{S} = \{\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \dots, \varepsilon_r a_{k_r}\}$ and $\tau \in R - \{0\}$ so that:

$$\tilde{Y} = \sigma \cdot \sum_{j=1}^{r} \varepsilon_j a_{k_j}. \tag{3.5}$$

Taking into account the relations (3.3) and (3.4), it results $r = 1$, and from (3.5) we obtain $\tilde{Y} = \sigma \cdot \varepsilon_1 a_{k_1}$. Hence $\mathcal{A}^* = \{a_{k_1}\}$. If $\tilde{Y} = \tilde{y}_\alpha a_\beta$, then $\sigma \cdot \varepsilon_1 = \tilde{y}_\alpha$, $a_{k_1} = a_\alpha$ and $\mathcal{A}^* = \{a_\alpha\}$ and when $\tilde{Y} = \tilde{y}_\beta a_\beta$, then $\sigma \cdot \varepsilon_1 = \tilde{y}_\beta$, $a_{k_1} = a_\beta$ and $\mathcal{A}^* = \{a_\beta\}$. So, supposing that $Y^{[0]}_{[\alpha,\beta]}$ is not minimal related to $\operatorname{Im}\nabla$, we obtain a section $\mathcal{S}$ with $\mathcal{A}^* = \{a_\alpha\}$ or $\mathcal{A}^* = \{a_\beta\}$, i.e. in contradiction with the hypothesis (see (3.1)). Hence the vector $Y^{[0]}_{[\alpha,\beta]}$ is minimal related to $\operatorname{Im}\nabla$. (Q.E.D.).

3.1. Corollary. *If $a_\alpha \mathcal{J} a_\beta$, then $a_\alpha \notin \operatorname{Im}\nabla$ and $a_\beta \notin \operatorname{Im}\nabla$.*

Proof. If $a_\alpha \in \operatorname{Im}\nabla$ then a_α is minimal related to $\operatorname{Im}\nabla$ and by a similar judgement for $\tilde{Y}$, we obtain a section with $\mathcal{A}^* = \{a_\alpha\}$, that implies a contradiction with (3.1). Hence $a_\alpha \notin \operatorname{Im}\nabla$.

Similarly $a_\beta \notin \operatorname{Im}\nabla$.

3.1. Theorem. *If $a_\alpha, a_\beta \in \mathcal{A}$, $\alpha, \beta \in \{1, 2, \dots, q\}$, $\alpha \neq \beta$, then $a_\alpha \mathcal{J} a_\beta$, if and only if there exists $X_{[\alpha,\beta]} \in \operatorname{Ker}\Delta$, $X_{[\alpha,\beta]} \neq 0_{\mathcal{A}}$, with $a_\alpha \in \mathcal{A}(X_{[\alpha,\beta]})$*

or $a_\beta \in \mathcal{A}(X_{[\alpha,\beta]})$, *and for every* $X = \sum_{k=1}^{q} x_k a_k \in \operatorname{Ker}\Delta$, *we have* $|x_\alpha| = |x_\beta|$.

Proof. Suppose that $a_\alpha \mathcal{J} a_\beta$. By the definition of $\mathcal{J}$, there exists the vector $Y^{[0]}_{[\alpha,\beta]} \in \operatorname{Im}\nabla$, $Y^{[0]}_{[\alpha,\beta]} \neq 0_{\mathcal{A}}$, with $\mathcal{A}(Y^{[0]}_{[\alpha,\beta]}) = \{a_\alpha, a_\beta\}$. Since $\mathcal{A}(Y^{[0]}_{[\alpha,\beta]}) = \{a_\alpha, a_\beta\}$, the vector $Y^{[0]}_{[\alpha,\beta]}$ has the form:

(3.6) $$Y^{[0]}_{[\alpha,\beta]} = y^{[\alpha]}_{[\alpha,\beta]} a_\alpha + y^{[\beta]}_{[\alpha,\beta]} a_\beta ; \quad y^{[\alpha]}_{[\alpha,\beta]},\ y^{[\beta]}_{[\alpha,\beta]} \in R - \{0\}.$$

Since $Y^{[0]}_{[\alpha,\beta]}$ is minimal related to $\operatorname{Im}\nabla$, (see lemma 3.1) and from the theorem 2.2 from [4] there exists a section $\mathcal{S} = \{\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \dots, \varepsilon_r a_{k_r}\}$ and $\sigma \in R - \{0\}$ so that $Y^{[0]}_{[\alpha,\beta]} = \sigma \cdot \sum_{j=1}^{r} \varepsilon_j a_{k_j}$.

Hence, in view of (3.6), we obtain:

(3.7) $$r = 2,\ \sigma\varepsilon_1 = y^{[\alpha]}_{[\alpha,\beta]},\ a_\alpha = a_{k_1},\ \sigma \cdot \varepsilon_2 = y^{[\beta]}_{[\beta,\alpha]},\ a_\beta = a_{k_2}.$$

Let us suppose now that for every $X = \sum_{k=1}^{q} x_k a_k \in \operatorname{Ker}\Delta$, $X \neq 0_{\mathcal{A}}$, we have $a_\alpha \notin \mathcal{A}(X)$ and $a_\beta \notin \mathcal{A}(X)$, which implies $x_\alpha = 0$ and $x_\beta = 0$.

Since $a_\alpha \notin \operatorname{Im}\nabla$, $a_\beta \notin \operatorname{Im}\Delta$ (see corollary 3.1), then, from the theorems 4.1 and 4.2 from [4] we have: $0 \neq [a_\alpha \mid X] = x_\alpha = 0$, and $0 \neq [a_\beta \mid X] = x_\beta = 0$; contradiction. So, there exists $X_{[\alpha,\beta]} \in \operatorname{Ker}\Delta$, $X_{[\alpha,\beta]} \neq 0_{\mathcal{A}}$, so that $a_\alpha \in \mathcal{A}(X_{[\alpha,\beta]})$ or $a_\beta \in \mathcal{A}(X_{[\alpha,\beta]})$.

Let now $X = \sum_{k=1}^{q} x_k a_k \in \operatorname{Ker}\Delta$, be arbitrary chosen. From the theorems 4.1 and 4.2 from [4] and from (3.7) we have: $0 = [X \mid Y^{[0]}_{[\alpha,\beta]}] = x_\alpha y^{[\alpha]}_{[\alpha,\beta]} + x_\beta y^{[\beta]}_{[\alpha,\beta]} = x_\alpha \sigma\varepsilon_1 + x_\beta \sigma\varepsilon_2 = \sigma(\varepsilon_1 x_\alpha + \varepsilon_2 x_\beta)$. Hence $\varepsilon_1 x_\alpha = -\varepsilon_2 x_\beta$, and because $\varepsilon_{i=1,2} = \pm 1$, we obtain $|x_\alpha| = |x_\beta|$. (Q.E.D.).

Suppose now that: $a_\alpha, a_\beta \in \mathcal{A}$, $a_\alpha \neq a_\beta$, there exists $X_{[\alpha,\beta]} \in \operatorname{Ker}\Delta$, $X_{[\alpha,\beta]} \neq 0_{\mathcal{A}}$ with $a_\alpha \in \mathcal{A}(X_{[\alpha,\beta]})$ or $a_\beta \in \mathcal{A}(X_{[\alpha,\beta]})$, and for every $X = \sum_{k=1}^{q} x_k a_k \in \operatorname{Ker}\Delta$, we have $|x_\alpha| = |x_\beta|$.

Considering $X_{[\alpha,\beta]} = \sum_{k=1}^{q} x_k^{[\alpha,\beta]} a_k$, we have $|x^{[\alpha,\beta]}_\alpha| = |x^{[\alpha,\beta]}_\beta|$, fact that implies:

(3.8) $$x^{[\alpha,\beta]}_\alpha = \varepsilon^{[\alpha,\beta]} \cdot x^{[\alpha,\beta]}_{[\beta]} ; \quad \varepsilon^{[\alpha,\beta]} = \pm 1,\ x^{[\alpha,\beta]}_\alpha \neq 0 ;\ x^{[\alpha,\beta]}_\beta \neq 0.$$

Let $X = \sum_{k=1}^{q} x_k a_k$ be arbitrary in $\operatorname{Ker}\Delta$. We define the vector $Y = a_\alpha - \varepsilon^{[\alpha,\beta]} a_\beta$.

(3.9) If $X = 0_{\mathcal{A}}$, then $[X \mid Y] = 0$.

If $X = X_{[\alpha,\beta]}$, then $[X \mid Y] = x^{[\alpha,\beta]}_\alpha - \varepsilon^{[\alpha,\beta]} \cdot x^{[\alpha,\beta]}_\beta$, and by (3.8) it results

(3.10) $$[X \mid Y] = 0.$$

If $x_\alpha = 0$, then (as $|x_\alpha| = |x_\beta|$ and $x_\beta = 0$)

(3.11) $$[X \mid Y] = 0.$$

Suppose that $x_\alpha \neq 0$; then $x_\beta \neq 0$ and (since $|x_\alpha| = |x_\beta|$ we can consider

(3.12) $$x_\alpha = \varepsilon x_\beta, \quad \varepsilon = \pm 1.$$

If $\varepsilon \neq \varepsilon^{[\alpha,\beta]}$ we can consider (without loss of generality) $\varepsilon = -1$ and $\varepsilon^{[\alpha,\beta]} = 1$.

In this case X and $X_{[\alpha,\beta]}$ have the forms:

$$X = \sum_{\substack{k=1 \\ k \neq \alpha,\beta}}^{q} x_k a_k + \tau a_\alpha - \tau a_\beta, \quad X_{[\alpha,\beta]} = \sum_{\substack{k=1 \\ k \neq \alpha,\beta}}^{q} x_k^{[\alpha,\beta]} \cdot a_k + \tau^{[\alpha,\beta]} a_\alpha + \tau^{[\alpha,\beta]} a_\beta,$$

where $0 \neq \tau = x_\alpha = -x_\beta$ and $0 \neq \tau^{[\alpha,\beta]} = x^{[\alpha,\beta]}_\alpha = x^{[\alpha,\beta]}_\beta$.

Since X and $X_{[\alpha,\beta]}$ belongs to $\operatorname{Ker}\Delta$, the vector $X^* = \tau^{[\alpha,\beta]} \cdot X + \tau X_{[\alpha,\beta]} = \sum_{\substack{k=1 \\ k \neq \alpha,\beta}}^{q} (\tau^{[\alpha,\beta]} \cdot x_k + \tau x_k^{[\alpha,\beta]}) a_k + 2\tau\tau^{[\alpha,\beta]} a_\alpha + 0 \cdot a_\beta$ belongs to $\operatorname{Ker}\Delta$, and we have $|2\tau\tau^{[\alpha,\beta]}| = 0$. But $\tau \cdot \tau^{[\alpha,\beta]} \neq 0$, and we obtain a contradiction. Hence $\varepsilon = \varepsilon^{[\alpha,\beta]}$.

So, if for $X = \sum_{k=1}^{q} x_k a_k$ arbitrary in $\operatorname{Ker}\Delta$ we have $x_\alpha \neq 0$, then, from (3.12) we obtain

(3.13) $$[X \mid Y] = x_\alpha - \varepsilon^{[\alpha,\beta]} x_\beta = x_\alpha - \varepsilon x_\beta = 0.$$

From (3.9), (3.10), (3.11) and (3.13) results $[X \mid Y] = 0$, for every $X \in \operatorname{Ker}\Delta$, and from the theorems 4.1 and 4.2 from [4], we have $Y \in \operatorname{Im}\nabla$.

(3.14) If $Y^{[0]}_{[\alpha,\beta]} = Y$, we obtain $\mathcal{A}(Y^{[0]}_{[\alpha,\beta]}) = \{a_\alpha, a_\beta\}$.

In order to show that $a_\alpha \mathcal{J} a_\beta$ we must also show that there doesn't exist any section with $\mathcal{A}^* = \{a_\alpha\}$ or $\mathcal{A}^* = \{a_\beta\}$. Obviously, if a section $\mathcal{S}$ with $\mathcal{A}^* = \{a_\alpha\}$ or $\mathcal{A}^* = \{a_\beta\}$, exists, then, from theorem 2.4 from [4] we have $a_\alpha \in \operatorname{Im}\nabla$ or $a_\beta \in \operatorname{Im}\nabla$, and from the theorems 4.1 and 4.2 from [4] it results $0 = [a_\alpha \mid X_{[\alpha,\beta]}] = x^{[\alpha,\beta]}_\alpha$ or $0 = [a_\beta \mid X_{[\alpha,\beta]}] = x^{[\alpha,\beta]}_\beta$; contradiction with (3.8).

So, in view of (3.14) we can say that $a_\alpha \mathcal{J} a_\beta$. (Q.E.D.).

3.1. *Remark*. In [4] we have shown that any vector $Y \in \operatorname{Ker}\Delta$, $Y \neq 0_{\mathcal{A}}$ has the form $Y = \sum_{i=1}^{m} \overline{Y}_i$, where $\overline{Y}_i = \alpha_i \overline{X}_i$ with $\alpha_i \in R - \{0\}$ and $\overline{X}_i$ elementary in $\operatorname{Ker}\Delta$, $\overline{X}_i$ corresponding to an elementary cycle $(\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \dots, \varepsilon_{r_i} a_{k_{r_i}})$, $\overline{X}_i = \sum_{j=1}^{r_i} \varepsilon_j a_{k_j}$. With this result, we can say $X_{[\alpha,\beta]} = \sum_{i=1}^{m} \mu_i \overline{X}_i^{[\alpha,\beta]}$, and every $\overline{X}_i^{[\alpha,\beta]}$ corresponds to an elementary cycle, wherefrom we obtain $\mathcal{A}(X_{[\alpha,\beta]}) = \bigcup_{i=1}^{m} \mathcal{A}(\overline{X}_i^{[\alpha,\beta]})$, i.e. $\mathcal{A}(X_{[\alpha,\beta]})$ is a union of arcs belonging to certain elementary cycles. According to theorem 3.1 if $a_\alpha \mathcal{J} a_\beta$, then exists at least an elementary cycle ω_i, so that $a_\alpha \in \mathcal{A}(\omega_i)$ or $a_\beta \in \mathcal{A}(\omega_i)$.

4. Relation "$\mathcal{K}$". In this paragraph we introduce on the set A the relation "$\mathcal{K}$" giving a characterization theorem of it ("$\mathcal{K}$").

4.1. Definition. *An arc $a_\alpha \in A$, $\alpha \in \{1, 2, \ldots, q\}$ is a loop if and only if $\Delta^+(a_\alpha) = \Delta^-(a_\alpha)$.*

4.2. Definition. *We define a cycle of r length as a sequence of nodes $(n_{i_0}, n_{i_1}, n_{i_2}, \ldots, n_{i_r} = n_{i_0})$ and a sequence of r arcs with sign $(\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \ldots, \varepsilon_r a_{k_r})$ where $\varepsilon_j = 1$ or -1, and for every $j = 1, 2, \ldots, r$ we have*

$$n_{i_{j-1}} = \begin{cases} \Delta^+(a_{k_j}), & \text{if } \varepsilon_j = 1, \\ \Delta^-(a_{k_j}), & \text{if } \varepsilon_j = -1, \end{cases} \quad \text{and } n_{i_j} = \begin{cases} \Delta^-(a_{k_j}), & \text{if } \varepsilon_j = 1, \\ \Delta^+(a_{k_j}), & \text{if } \varepsilon_j = -1. \end{cases}$$

We denote a cycle by $\omega[n_{i_0}]$.

4.3. Definition. *Let a_α, a_β, α, $\beta \in \{1, 2, \ldots, q\}$, $\alpha \neq \beta$, be so that $a_\alpha, a_\beta \in A$. $a_\alpha \mathcal{K} a_\beta$ if and only if we have simultaneously:*

(4.1) *the arcs a_α, and a_β are not loops,*

(4.2) *there exists $X^{[0]}_{[\alpha,\beta]} \in \operatorname{Ker}\Delta$, $X^{[0]}_{[\alpha,\beta]} \neq 0_{\mathcal{A}}$, so that*

$$A(X^{[0]}_{[\alpha,\beta]}) = \{a_\alpha, a_\beta\}.$$

Using similarly judgements as those made for "$\mathcal{J}$", we obtain the following results for "$\mathcal{K}$":

4.1. Lemma. *If $a_\alpha \mathcal{K} a_\beta$, then the vector $X^{[0]}_{[\alpha,\beta]}$ is minimal related to* $\operatorname{Ker}\Delta$.

4.1. Corollary. *If $a_\alpha \mathcal{K} a_\beta$, then $a_\alpha \notin \operatorname{Ker}\Delta$ and $a_\beta \notin \operatorname{Ker}\Delta$.*

4.1. Theorem. *If $a_\alpha, a_\beta \in A$, α, $\beta \in \{1, 2, \ldots, q\}$, $\alpha \neq \beta$, then, $a_\alpha \mathcal{K} a_\beta$, if and only if there exists $Y_{[\alpha,\beta]} \in \operatorname{Im}\nabla$, $Y_{[\alpha,\beta]} \neq 0_{\mathcal{A}}$ with $a_\alpha \in A(Y_{[\alpha,\beta]})$ or $a_\beta \in A(Y_{[\alpha,\beta]})$,* and for every $Y = \sum_{k=1}^{q} y_k a_k \in \operatorname{Im}\nabla$, we have $|y_\alpha| = |y_\beta|$.

4.1. Remark. In [4] we have shown that any $Y \in \operatorname{Im}\nabla$, $Y \neq 0_{\mathcal{A}}$ has the form $Y = \sum_{i=1}^{\tilde{m}} \overline{Y}_i$, where $\overline{Y}_i = \alpha_i \overline{X}_i$ with $\alpha_i \in R - \{0\}$ and $\overline{X}_i$ is elementary in $\operatorname{Im}\nabla$, $\overline{X}_i$ corresponding to an elementary section

$$\mathcal{S} = \{\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \ldots, \varepsilon_{r_i} a_{k_{r_i}}\},\ \overline{X}_i = \sum_{j=1}^{r_i} \varepsilon_j a_{k_j}.$$

With this result, we can say $Y_{[\alpha,\beta]} = \sum_{i=1}^{\tilde{m}} \mu_i \overline{X}_i^{[\alpha,\beta]}$, every $\overline{X}_i^{[\alpha,\beta]}$ corresponding to an elementary section, wherefrom we obtain $A(Y_{[\alpha,\beta]}) = \bigcup_{i=1}^{\tilde{m}} A(\overline{X}_i^{[\alpha,\beta]})$ i.e. $A(Y_{[\alpha,\beta]})$ is a union of arcs belonging to certain elementary sections.

As a consequence of theorem 4.1 if, $a_\alpha \mathcal{K} a_\beta$, then there exists at least an elementary section $\mathcal{S}_i$, so that $a_\alpha \in A_i^*$ or $a_\beta \in A_i^*$.

*

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Faculty of Mathematics
University of Bucharest
R. S. România