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A NOTE ON TIME-VARYING GRAMMARS

BY

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The terminology and the results used in this note are those of Salomaa (1973).

Definition 1. *A time-varying grammar of type i, $i = 0, 1, 2, 3$ [1] is a pair (G, φ) where $G = (V_N, V_T, S, P)$ is a type-i grammar (for $i = 3$ we take the right-linear grammars) and $\varphi : N \to \mathcal{P}(P)$.*

For (x, i), $(y, j) \in V_G^* \times N$, we write $(x, i) \Rightarrow (y, j)$ iff $j = i + 1$, $x = x_1 z x_2$, $y = x_1 w x_2$ and $z \to w$ is a rule in $\varphi(i)$ (V_G denotes the set $V_N \cup V_T$ and V^* is the free monoid generated by V under the operation of concatenation and the null element λ). Let $\overset{*}{\Rightarrow}$ be the reflexive transitive closure of $\Rightarrow$. The language generated by (G, φ) is

$$L(G, \varphi) = \{x \in V_T^* \mid (S, 1) \overset{*}{\Rightarrow} (x, j) \text{ for some } j\}.$$

It is known that any language can be generated by a (fixed for a given vocabulary) type-3 time-varying grammar with arbitrary mapping φ. Moreover, if φ is a periodic mapping, then the generative capacity of type-i, $i = 0, 1, 3$ grammars is not increased whereas the type-2 periodically time-varying grammars exactly generate the context-free matrix languages.

Following a suggestion in [1], (p. 154), we investigate an intermediate case, namely, the case when φ is a recursive mapping (that is a mapping for which there exists a Turing machine which computes $\varphi(n)$ for any $n \in N$).

Let $\mathcal{RV}_i$, $i = 0, 1, 2, 3$ be the family of languages generated by recursively time-varying grammars of type i. From Church's thesis, it follows that $\mathcal{RV}_0 \subset \mathcal{L}_0$. The inclusions $\mathcal{L}_i \subset \mathcal{RV}_i$ are trivial for any $i = 0, 1, 2, 3$. (The families $\mathcal{L}_i$, $i = 0, 1, 2, 3$ are the four families in the Chomsky hierarchy). Therefore $\mathcal{RV}_0 = \mathcal{L}_0$. Clearly $\mathcal{RV}_i \subset \mathcal{RV}_{i-1}$, $i = 1, 2, 3$. In the following theorem we shall prove that $\mathcal{RV}_0 \subset \mathcal{RV}_3$ hence we shall have

$$\mathcal{RV}_3 = \mathcal{RV}_2 = \mathcal{RV}_1 = \mathcal{RV}_0 = \mathcal{L}_0.$$

Thus, the study of time-varying grammars (G, φ), with mappings φ of an intermediate type between periodic and recursive function becomes of interest.

Theorem $\mathcal{L}_0 \subset \mathcal{RV}_3$.

Proof. It is known that any language in $\mathcal{L}_0$ can be generated b a type-3 grammar with a type-1 control language [1] (p. 184).

Thus, let $G = (V_N, V_T, S, P)$ be a right-linear grammar and l $C \in \mathcal{L}_1$ such that $L_C(G) = L$, for a given $L \in \mathcal{L}_0$. Let Lab (P) be th set of labels for rules in P and let

$M = \{p \in \mathrm{Lab}(P) \mid$ the rule with the label p is a terminal one

Then the language $C' = C \cap (\mathrm{Lab}\,(P) - M)^* M$ is also in $\mathcal{L}_1$ and we hav $L_C(G) = L_{C'}(G)$. Let us now construct the grammar $G' = (V_N \cup \{S'$ $V_T, S', P')$, where S' is a new symbol and $P' = P \cup \{S' \to S', S' \to S\}$.

Assume that $P' = \{r_1, r_2, \ldots, r_k\}$ and let Lab $(P') = \{p_1, p_2, \ldots, p$ be a set of labels for rules in P' such that p_1 is the label of the ru $r_1 :\ S' \to S'$, p_2 is the label of the rule $r_2 :\ S' \to S$ and p_i, $i \geqslant 3$ a labels for the rules in P. We consider also the language

$$C'' = \{p_2\}\, h(C')$$

where h is the homomorphism which renames the labels in Lab(P) b $p_3, p_4, \ldots, p_k$. Obviously, $C'' \in \mathcal{L}_1$ and $L_C(G) = L_{C'}(G) = L_{C''}(G'')$.

We consider the time-varying grammar (G', φ) with φ defined a follows:

Let us define on Lab$(P')^*$ the lexicographical order. We shall tak $x < y$ if one of the conditions written below is fulfilled:

i) $|x| < |y|$ ($|z|$ is the length of the string z),

ii) $|x| = |y|$ and $x = x_1 p_i x_2$, $y = x_1 p_j x_3$, $x_1, x_2, x_3 \in \mathrm{Lab}(P)^*$, $i <$

Let us consider the infinite string $\sum$ obtained by concatenating a the non-null strings in Lab$(P')^*$ in the lexicographical order.

For $n \geqslant 1$, $0 \leqslant r \leqslant k^{n+1} - 1$ we denote:

$$B(n, r) = \sum_{i=1}^{n} i k^i + (n + 1)\, r + 1,$$

(This is the position in $\sum$ of the first symbol in the $(r + 1)$-th word of length $n + 1$ from the list of Lab$(P')^*$ in the lexicographical order

$$E(n, r) = B(n, r) + n + 1$$

(This is the position of the last symbol of the $(r + 1)$-th word of length $n + 1$ in the string $\sum$).

Clearly, there is an algorithm which computes $w(n)$, the symb occurring in the n-th position of $\sum$ for any $n \geqslant 1$. Moreover, there i an algorithm which, for any n, r as above, answers the question: does the string which begins by $w(B(n, r))$ and ends by $w(E(n, r))$ (this strin will be denoted by $W(n, r)$) belong to C''_G (Indeed the string $W(n,$ can be effectively found and C'' is in $\mathcal{L}_1$, hence it is a recursive language

Now, we define

$$\varphi :\ N \to \mathcal{P}(P')$$

in the following way. For any $t \geqslant 1$, $\varphi(t)$ contains the rule $S' \to S'$ and also contains a rule corresponding to a label p_i, $i \geqslant 2$ if $w(t) = p_i$, $B(n, r) \leqslant t \leqslant E(n, r)$ for some $n \geqslant 1$, $0 \leqslant r \leqslant k^{n+1} - 1$, and $W(n, r) \in C''$. Clearly,

φ is a recursive mapping. The equality $L_{C''}(G') = L(G', \varphi)$ holds. Indeed, any derivation in (G', φ) begins with some applications of the rule $S' \to S'$, followed by an application of the rule $S' \to S$ when this rule is in $\varphi(t)$, $t = B(n, r)$ for some n, r as above. The derivation continues by using rules in P hence without any occurrence of S'. The string of these rules corresponds to a string in C' hence the derivation can be terminated only when the string in C' is terminated. The derivation in (G', φ) corresponds in this way to a derivation in G' with the control word in C''.

Conversely, any terminal derivation in G' with the control word in C'' can be simulated in (G', φ), therefore $L_{C''}(G') = L(G', \varphi)$ holds and $L \in \mathcal{RV}_3$. In conclusion, $\mathcal{L}_0 \subset \mathcal{RV}_3$.

Definition 2. *A time-varying grammar* (G, φ) *is said to be in n-canonical form iff* card $(\varphi(t)) \leqslant n$ *for any* $t \geqslant 1$.

Corollary. Any type-0 language can be generated by a recursively time-varying grammar of type-3 in the 2-canonical form.

Clearly, the above assertion does not hold if we replace "2-canonical form" by "1-canonical form". Indeed if $\varphi(t)$ contains at most one rule for any $t \geqslant 1$, then $L(G, \varphi)$, for G right-linear, contains at most one string.

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