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ON THE BEHAVIOUR OF TIME-VARIANT SEMIAUTOMATA

BY

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1. Introduction. A well known result from Formal Language Theory states that every regular language is the image through a very fine homomorphism of a local language (see, for instance [1]).

We shall recall briefly the definition of a local language. Let I be a finite alphabet, I_1 and I_2 two non-void subsets of I and K a finite language, $K \subseteq I^{(2)}$, where $I^{(m)}$ is the set of all words from I^*, the free monoid on I, having the length $m, m \in N$. The local language determined by I_1, I_2 and K is the language:

$$E = (I_1 I^* \cap I^* I_2) \setminus I^* K I^*.$$

To generalize this notion we will prefer a positive form of this definition. Let us consider the binary relation $\alpha \subseteq I \times I$ defined by:

$$\alpha = \{(i_1, i_2) \mid i_1 i_2 \notin K\}.$$

It is obvious that:

$$E = I_1 I^* \cap I^* I_2 \cap (I \cup E_\alpha),$$

where

$$E_\alpha = \{p \mid p = i_1 i_2 \dots i_n, \quad n \geqslant 2, \ (i_1, i_2), (i_2, i_3), \dots, (i_{n-1}, i_n) \text{ are pairs from } \alpha\} \subseteq I^*.$$

It is clear that every local language can be written as an intersection between a regular language, namely $U = I_1 I^* \cap I^* I_2$ and a language defined by a relation $K_\alpha = I \cup E_\alpha$.

Let now $\Omega = \{\alpha_n \mid n \geqslant 1\}$ be a sequence of binary relations on I. We shall consider the language $L_\Omega = \{p \mid p = i_1 i_2 \dots i_n, \ n \geqslant 2, \ (i_j, i_{j+1}) \in \alpha_j, 1 \leqslant j \leqslant n-1\}$. The *polylocal language* defined by Ω and the language U is given by:

$$L(\Omega, U) = [U \cap (I \cup L_\Omega)].$$

When Ω is a constant sequence and U is a regular language having the special form $U = I_1 I^* \cap I^* I_2$ then $L(\Omega, U)$ is a local language.

Theorem 1. *The intersection of two polylocal languages is a polylocal language. The preimage of a polylocal language through a very fine homomorphism is a polylocal language.*

Proof. Let $\Omega = \{\alpha_n \mid n \in N\}$ and $\Omega' = \{\alpha'_n \mid n \in N\}$ be two sequences of binary relations on I. We define the sequence $\Omega \wedge \Omega' = \{\alpha_n \cap \alpha'_n \mid n \in N\}$. It is straightforward to see that:

$$L(\Omega \wedge \Omega',\ U \cap U') = L(\Omega, U) \cap L(\Omega', U'),$$

which justifies the first part of the theorem.

Let $L(\Omega, U) \subseteq I^*$ be a local language and $h: I'^* \to I^*$ be a very fine homomorphism, which means that $h(i') = i$, $i \in I$, $\forall i' \in I$. Let us consider on I' the relations α'_n given by:

$$\alpha'_n = \{(i'_1, i'_2) \mid (h(i'_1), h(i'_2)) \in \alpha_n\}.$$

If U' is the language $U' = h^{-1}(U)$, then we have: $h^{-1}(L(\Omega, U)) = L(\Omega', U')$, where $\Omega' = \{\alpha'_n \mid n \in N\}$.

A time-variant semiautomaton (tvs) is a triple $A = (I, S, \{f_n \mid n \in N\})$, where I, S, are non-void sets representing respectively the input alphabet and the set of states and $f_n: S \times I \to S$ is the transition function at time n, $n \in N$. If, at time n the state is s and the input is i at $n+1$ the state will be $f_n(s, i)$. Every function can be extended until a function $f_n: S \times I^* \to S$; namely $f_n(s, p)$ will give the state at the moment $n + l(p)$, if the state at the n-th moment was s and we have applied the symbols of the word p beginning with this moment. Therefore, we shall take:

$$f_n(s, e) = s,\ \forall s \in S,\ n \in N,$$
$$f_n(s, pi) = f_{n+l(p)}\,(f_n(s, p)),\quad \forall p \in I^*,\ s \in S,\ n \in N.$$

Let $T: N \to \mathcal{P}(S)$ be a sequence of subsets of the set of states of A. The value of T in m will be denoted by T_m. The language represented in A by the initial state s_0 and the sequence T is the language $L(A, s_0, T)$ defined by:

$$L(A, s_0, T) = \{p \mid p \in I^*,\ f_0(s_0, p) \in T_{l(p)}\}.$$

Using a result of S a l o m a a [Th. 4 from [2]] for every language $L(A, s_0, T)$ there exists a tvs A' and a constant sequence of states T' such that $L(As_0, T) = L(A', s'_0, T')$, where s_0 is a suitable state of A. That means that there exists a subset S_1 of the set of states of A' for which

$$L(A, s_0, T) = \{p \mid f'_0(s'_0, p) \in S_1\},$$

where f is the extended initial transition function of A'. To simplify the notation this language will be denoted by $L(A, s_0, S_1)$.

II. Polylocal Languages and Languages Representable in Time-Variant Semiautomata. Let $A = (I, S, \{f_n \mid n \in N\})$ be a tvs. We shall consider on the set $\sum = S \times I \times S$ the sequence of binary relations $\Omega = \{\alpha_n\ n \geqslant 1\}$ defined by:

$$\alpha_n = \{(\sigma, \sigma') \mid \sigma = (s_1, i, s_2),\quad \sigma' = (s_3, i', s_4),\ s_3 =$$
$$= s_2 = f_{n-1}(s_1, i),\ s_4 = f_n(s_3, i')\},\ n \geqslant 1.$$

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Let us consider the regular language $W_{s_1} = \sum_0 \sum^* \cap \sum^* \sum_1$, where $\sum_0 = \{(s_0, i, s) \mid s \in S,\ i \in I\}$, $\sum_1 = \{(s, i, s_1) \mid s \in S, i \in I,\ s_1 \in S_1\}$ and $\sum^*$ is the whole free monoid generated by $\sum$. The very fine homomorphism $h: \sum^* \to I^*$ will be defined by $h((s, i, s')) = i$, $\forall (s, i, s') \in \sum$.

Theorem 2. *Every lynguage representable in a tvs, excluding the empty word e is the image through a very fine homomorphism of a suitable polylocal longuage $L(\Omega, W_{s_1})$, where W_{s_1} is the regular language previously defined.*

Proof. Without restricting the generality we can suppose that the language $L(A, s_0, T)$ is represented using a constant set of final states S_1. Let p be a word from $L(A, s_0, T) \setminus \{e\}$. If $l(p) = 1$ then $p = i \in I$ and $f_0(s_0, i) \in S_1$. Then $(s_0, i, f(s_0, i)) \in \sum_0 \cap \sum_1 \subseteq \sum_0 \sum^* \cap \sum^* \sum_1$, hence $p = i = h((s_0, i, f(s_0, i))) \in h((\sum \cup L_\Omega) \cap W_{s_1}) = h(L(\Omega, W_{s_1}))$.

Let now $p = i_1 i_2 \dots i_n$, $n \geqslant 2$ be a word from $L(A, s_0, T)$. We have $s_1 = f_0(s_0, i_1)$, $s_2 = f_1(s_1, i_2), \dots, s_n = f_{n-1}(s_{n-1}, i_n)$, hence $((s_0, i_1, s_1), (s_1, i_2, s_2)) \in \alpha_1, ((s_1, i_2, s_2), (s_2, i_3, s_3)) \in \alpha_2, \dots, ((s_{n-2}, i_{n-1}, s_{n-1}), (s_{n-1}, i_n, s_n)) \in \alpha_{n-1}$. Besides, the word $w = (s_0, i_1, s_1), (s_1, i_2, s_2) \dots (s_{n-1}, i_n, s_n)$ belongs to $\sum_0 \sum^* \cap \sum^* \sum_1$, hence $w \in L_\Omega \cap W_{s_1} \subseteq (\sum \cup L_\Omega) \cap W_{s_1}$. We infer that $p = h(w) \in h(L(\Omega, W_T))$.

To obtain the converse inclusion we have to follow conversely the arguments.

Theorem 3. *Every polylocal language is representable in a tvs.*

Proof. We shall prove that for any sequence $\Omega = \{\alpha_n \mid n \geqslant 1\}$ of binary relations, the language L_Ω is representable in a tvs. The result will follow directly, since the class of languages which are representable in a tvs is closed with respect to set theoretical operations.

Let us consider the tvs $A = (I, I \cup \{s_0\}, \{f_n \mid n \in N\})$, where the transition functions f_n are defined by:

$$f_0(s_0, i) = i,\ f_0(i, i') = i,\ \forall i, i' \in I,$$

and, for $n > 0$,

$$f_n(t, i) = \begin{cases} i, \text{ if } t \in I \text{ and } (t, i) \in \alpha_n \\ s_0, \text{ otherwise (if } t = s_0 \text{ or } t = i', \text{ but } (i', i) \notin \alpha_n) \end{cases}$$

for every $i \in I$.

It is obvious that $L(A, s_0, I) = L_\Omega$, hence $L(\Omega, U)$ is a language representable in a tvs.

Now, let P be an infinite word on the alphabet I and let Ω_P be the sequence of relations $\{\alpha_n \mid n \in N\}$, where $\alpha_n = (i_n, i_{n+1})$, i_n being the n-th symbol of the word P. Using Theorem 3 we can obtain a part of Theorem 2 from [2] as a special case. Indeed, it is clear that the set of all initial subwords of P, L_P is equal to the polylocal language $L(\Omega_P, I^*)$, hence L_P is representable in a tvs.

III. Tvs and Nerode-like Equivalences. Let $A = (I, S, \{f_n \mid n \in N\})$ be a tvs and $s^{(m)}p$ the state which is reached by A starting at the moment m from s and feding to the input wire of the tvs the symbols belonging to p.

The Nerode-like equivalence defined by A and the state s_0 is the equivalence ε given by: $\varepsilon = \{(p, q) \mid p, q \in I^*,\ s^{0}_{(0)}p = s^{0}_{(0)}q\}$.

Theorem 4. *The equivalence ε on I^* has the following properties:*

i) ε has a finite rank (which means that the quotient set I^*/ε consists of a finite number of classes);

ii) the equivalence $\lambda \cap \varepsilon$ (where λ is the congruence $\lambda = \{(p, q) \mid p, q \in I^*,\ l(p) = l(q)\}$ is a right congruence on I^*.

Proof. Let us consider the mapping $\psi : I^* \to S$ given by $\psi(p) = s^{(0)}_0 p$. Since S is a finite set the equivalence $\ker\psi = \{(p, q) \mid p, q \in I^*,\ \psi(p) = \psi(q)\}$ has a finite rank, not exceeding $|S|$, the cardinal of S. By remarking that $\ker\psi = \varepsilon$, we obtain the first part of the theorem.

For the second part, we shall start with the equality

$$s^{(m)}(pq) = (s^{(m)}p)^{(m+l(p))}q,$$

which holds for every $s \in S$, $p, q \in I^*$ and $m \in N$. Suppose that $(p, q) \in \lambda \cap \varepsilon$. We have:

$$s^{(0)}_0(pr) = (s^{(0)}_0 p)^{(l(p))}r = (s^{(0)}q)^{(l(q))}r = s^{(0)}_0(qr).$$

because $l(p) = l(q)$ and $s^{(0)}_0\, p = s^{(0)}_0\, q$. Besides, we have $l(pr) = l(qr)$, hence $(pr, qr) \in \lambda \cap \varepsilon$ and the proof is completed.

Let us consider now the right congruence ν_E on I^* generated by the language $E \subseteq I^*$ and defined by:

$$\nu_E = \{(p, q) \mid p, q \in I^*,\ \chi_E(pr) = \chi_E(qr),\ \forall r \in I^*\},$$

where χ_E is the characteristic function of the language E, $\chi_E : I^* \to \{0, 1\}$.

Theorem 5. *The language E is representable in a tvs $A = (I, S, \{f_n \mid n \in N\})$, $E = L(A, s_0, S_1)$ iff $\lambda \cap \varepsilon \subseteq \nu_E$ and $\varepsilon \subseteq \chi_E$, where ε is the Nerode-like equivalence induced by A and the initial state s_0, and χ_E is the equivalence $\chi_E = \mathrm{Ker}\chi_E$.*

Proof. Suppose that E is representable in the tvs A, $E = L(A, s_0, S_1)$ and let (p, q) be a pair of words from $\lambda \cap \varepsilon$.

If $pr \in E$ for a word $r \in I^*$ then $s^{(0)}_0(pr) = (s^{(0)}_0 p)^{l(p)}\, r \in S_1$. Since $s^{(0)}_0 p = s^{(0)}_0 q$ and $l(p) = l(q)$ we obtain $s^{(0)}_0(qr) = (s^{(0)}_0 q)^{l(q)}\, r = (s^{(0)}_0 p)^{l(p)} r \in S_1$, hence $qr \in E$. Therefore $pr \in E$ iff $qr \in E$ for every $r \in I^*$, which points that $(p, q) \in \nu_E$.

If $(p, q) \in \varepsilon$ we have $s^{(0)}_0 p = s^{(0)}_0 q$, hence $s^{(0)}_0 p \in S_1$ iff $s^{(0)}_0 q \in S_1$. Therefore $p \in E$ iff $q \in E$, which points that $(p, q) \in \ker\chi_E$. We obtain $\varepsilon \subseteq \chi_E$.

Conversely, suppose that $\lambda \cap \varepsilon \subseteq \nu_E$ and $\varepsilon \subseteq \chi_E$. Let us consider the set of states $S_1 = \{s_1 \mid s_1 = s^{(0)}_0 p,\ p \in E\}$. We shall prove that $E = L(A, s_0, S_1)$. Since the inclusion $E \subseteq L(A, s_0, S_1)$ is obvious, we have to prove only the converse inclusion. Suppose that $q \in L(A, s_0, S_1)$. There exists a word $p \in E$ for which $s^{(0)}_0 p = s^{(0)}_0 q \in S_1$, hence $(p, q) \in \varepsilon$. Using the inclusion $\varepsilon \subseteq \chi_E$ it follows that $(p, q) \in \chi_E$ and, since $p \in E$ we obtain that $q \in E$ and we have completed the proof.

Theorem 6. *Let ε be an equivalence on I^* having the finite rank. If $\lambda \cap \varepsilon$ is a congruence, $\lambda \cap \varepsilon \subseteq \nu_E$ and $\varepsilon \subseteq \chi_E$ then E is representable in a tvs.*

Proof. Let us consider the tvs $A = (I, I^*/\varepsilon, \{f_n \mid n \in N\})$, where

$$f_n([p]_\varepsilon, i) = \begin{cases} [qi]_\varepsilon, \text{ if } [p]_\varepsilon \text{ contains a word } q \text{ of length } n, \\ \text{arbitrary, otherwise.} \end{cases}$$

The function f_n is well defined for, if q' is another word from $[p]_\varepsilon$ with $l(q') = n$ we have $(q, q') \in \lambda$ and $(q, q') \in \varepsilon$, hence $(q, q') \in \lambda \cap \varepsilon$. Therefore $(qi, q'i) \in \lambda \cap \varepsilon$, hence $[qi]_\varepsilon = [q'i]_\varepsilon$.

We shall prove that $E = L(A, [e]_\varepsilon, S_1)$, where $S_1 = \{[p]_\varepsilon \mid p \in E\}$, Suppose that $p \in L(A, [e]_\varepsilon, S_1)$. We have $[e]^{(0)}_\varepsilon p = [q]_\varepsilon$, where $q \in E$, hence $[p]_\varepsilon = [q]_\varepsilon$. From $(p, q) \in \varepsilon$, it follows that $(p, q) \in \chi_E$, hence $q \in E$ because $\varepsilon \subseteq \chi_E$ and we have $L(A, [e]_\varepsilon, S_1) \subseteq E$.

To prove the converse inclusion let q be a word from E. Since $[e]^{(0)}_\varepsilon q = [q]_\varepsilon \in S_1$ it follows immediately that $q \in L(A, [e]_\varepsilon, S_1)$, hence $E \subseteq L(A, [e]_\varepsilon, S_1)$. Therefore $E = L(A, [e]_\varepsilon, S_1)$ and E is representable in the tvs A.

The last two theorems give the following:

Corollary. *The language E is representable in a tvs iff there exists a finite rank equivalence ε for which*

i) $\lambda \cap \varepsilon$ *is a right congruence on* I^*;

i) $\lambda \cap \varepsilon \subseteq \nu_E$;

iii) $\varepsilon \subseteq \chi_E$.

Remark. Let C be the set of finite rank congruences satisfying the previous three conditions. If $r(\alpha)$ is the rank of the equivalence α let $k = \min\{r(\alpha) \mid \alpha \in C\}$. The minimal number of states of a tvs in which the language E is representable is equal to k.

The following result will allow us to recapture the characterization of languages representable in tvs obtained by A. Salomaa in [2].

Theorem 7. *There exists an equivalence ε satisfying the conditions* i) – iii) *from the last corollary iff there exists a number k for which $|I^{(m)}/\nu_E| \leqslant k$, for every $m \in N$.* Here $I^{(m)}$ is the set of words having length m.

Proof. Suppose that there exists the equivalence ε with properties i) – iii) and let p be a word of length m. The λ-class of p contains all words having the length m. Let us suppose that this class $I^{(m)}$ contains h ε-nonequivalent words $q_1, \ldots, q_n$. Obviously $h \leqslant |I^*/\varepsilon|$, hence we have at most $|I^*/\varepsilon|$-classes contained in $[p]_\lambda = I^{(m)}$. Since $\lambda \cap \varepsilon \subseteq \nu_E$, it follows that $|I^{(m)}/\nu_E| \leqslant |I^{(m)}/(\lambda \cap \varepsilon)| \leqslant |I^*/\varepsilon|$.

Conversely, suppose that $|I^{(m)}/\nu_E| \leqslant k$ for every $m \in N$. Each set $I^{(m)}$ is partitioned by ν_E in classes $K^{(m)}_1, \ldots, K^{(m)}_{h_m}$, where $h_m \leqslant k$, $\forall m \in N$. It is clear that every set $K^{(m)}_j$ is included either in E or in $E^c = I^*/E$. We shall denote by $U^{(m)}_0$ the language:

$$U^{(m)}_0 = \bigcap \{K^{(m)}_j \mid K^{(m)}_j \subseteq E\};$$

the remaining classes $K^{(m)}_j$ will be re-denoted by $U^{(m)}_1, \ldots, U^{(m)}_{c_m}$, where $0 \leqslant$

$\leqslant c_m \leqslant h_m$ (the value $c_m = 0$ being reserved for the case when all $K_j^{(m)}$ are included in E and the value $c_m = h_m$ being prescribed when every class $K_j^{(m)}$ is disjoint from E).

The collection $\{E, W_1, \ldots, W_k\}$, where $W = \bigcup \{U_d^{(m)} \mid m \in N, U_d^{(m)} \neq \emptyset\}$, $1 \leqslant d \leqslant k$, is obviously a partition of I^*; let ε be the equivalence induced by this partition.

From the definition of ε, it follows that $\lambda \cap \varepsilon \subseteq \nu_E$. If $(p, q) \in \lambda \cap \varepsilon$ then p and q have the same length m and there exists a block $U_e^{(m)}$ which contains both p and q. Since pi and qi have again the same length and $(pi, qi) \in \nu_E$ it follows that $(pi, qi) \in \lambda \cap \varepsilon$ and, moreover $\{pi, qi\} \subseteq$ $\subseteq U_d^{(m+1)}$ for a certain $d \leqslant k$. Therefore $(pi, qi) \in \lambda \cap \varepsilon$, hence $\lambda \cap \varepsilon$ is a right congruence.

From the definition of ε it follows that for a pair $(p, q) \in \varepsilon$, we have either $\{p, q\} \subseteq E$ or $\{p, q\} \cap E = \emptyset$ which points that $\varepsilon \subseteq \varkappa_E$.

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EQUILIBRIUM POINTS FOR $2 \times n$ BIMATRIX GAMES

BY

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It is well known that for two-person zero-sum games, in which one of the players has only two pure strategies, it is possible to find all optimal strategies by using a graphical method, [1]. In this note we shall describe a simple procedure for finding an equilibrium point for two-person nonzero-sum games in which one of the players has only two pure strategies.

1. Let $\Gamma = (A, B)$ be a bimatrix game, given by the payoff matrices $A = (a_{ij})$, $B = (b_{ij})$, each of them with m rows and n columns. As usually one denotes by X and Y the mixed strategies for the two players, i.e.

(1) $$X = \{x \mid x \in R^m, x \geqslant 0, \textstyle\sum x_i = 1\}, \quad Y = \{y \mid y \in R^n, y \geqslant 0, \textstyle\sum y_j = 1\}.$$

Definition 1 [2]. *A pair of strategies* (x^0, y^0), $(x^0, y^0) \in X \times Y$, *is called an equilibrium point* (*EP*) *for the game* $\Gamma = (A, B)$ *if*

(2) $$xAy^0 \leqslant x^0Ay^0, \text{ for all } x \in X, \quad x^0By \leqslant x^0By^0, \text{ for all } y \in Y.$$

Definition 2. *A saddle point for a game* $\Gamma = (A, B)$ *is called an equilibrium point* (x^0, y^0) *such that* x^0 *and* y^0 *are pure strategies.*

Definition 3. *Let* $\Gamma = (A, B)$ *be a bimatrix game; the graph associated with* Γ *is a mixed graph* $G = (V, U)$, *where*

i) $V = \{(i, j) \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}$,

ii) *the arc* $((i, j), (k, l)) \in U$ *iff* $i = k$ *and* $b_{ij} > b_{kl}$, *or* $j = l$ *and* $a_{ij} > a_{kl}$,

iii) *the edge* $\{(i, j), (k, l)\} \in U$ *iff* $i = k$, $j \neq l$ *and* $b_{ij} = b_{kl}$, *or* $j = l$, $i \neq k$, *and* $a_{ij} = a_{kl}$.

If $G = (V, U)$ is the associated graph to a bimatrix game Γ, we shall write $(i, j) \succ (k, l)$ when $((i, j), (k, l)) \in U$ and $(i, j) \sim (k, l)$ when $\{(i, j), (k, l)\} \in U$.

Definition 4. *Let* $G = (V, U)$ *be the graph associated with a bimatrix game; then case call the row* i *of* G *the subgraph of* G *the vertices of which are* (i, j), $j = 1, \ldots, n$, *and the column* j *of* G *is the subgraph of* G *the vertices of which are* (i, j), $i = 1, \ldots, m$.

Definition 5. *A nondominated vertex in a mixed graph* $G = (V, U)$ *is a vertex* $v, v \in V$, *with the property that the arc* $(v', v) \notin U$ *for all* $v' \in V$.

It is clear that a bimatrix game has a saddle point iff his associated graph has a nondominated vertex, and every row and every column of the associated graph to a bimatrix game has at least one nondominated vertex.