$\leqslant c_m \leqslant h_m$ (the value $c_m = 0$ being reserved for the case when all $K_j^{(m)}$ are included in E and the value $c_m = h_m$ being prescribed when every class $K_j^{(m)}$ is disjoint from E).

The collection $\{E, W_1, \ldots, W_k\}$, where $W = \bigcup \{U_d^{(m)} \mid m \in N, U_d^{(m)} \neq \emptyset\}$, $1 \leqslant d \leqslant k$, is obviously a partition of I^*; let ε be the equivalence induced by this partition.

From the definition of ε, it follows that $\lambda \cap \varepsilon \subseteq \nu_E$. If $(p, q) \in \lambda \cap \varepsilon$ then p and q have the same length m and there exists a block $U_e^{(m)}$ which contains both p and q. Since pi and qi have again the same length and $(pi, qi) \in \nu_E$ it follows that $(pi, qi) \in \lambda \cap \varepsilon$ and, moreover $\{pi, qi\} \subseteq U_d^{(m+1)}$ for a certain $d \leqslant k$. Therefore $(pi, qi) \in \lambda \cap \varepsilon$, hence $\lambda \cap \varepsilon$ is a right congruence.

From the definition of ε it follows that for a pair $(p, q) \in \varepsilon$, we have either $\{p, q\} \subseteq E$ or $\{p, q\} \cap E = \emptyset$ which points that $\varepsilon \subseteq \varkappa_E$.

REFERENCES

1. Eilenberg S. — *Automata, Languages and Machines*, vol. A, Academic Press, New York, London, 1974.
2. Salomaa A. — *On Finite Automata with a Time-Variant Structure*, Inf. and Control 13 (1968), pp. 85—98.
3. Salomaa, A. — *Theory of Automata*, Pergamon Press, Oxford, 1968.

Received 6.III.1978

Faculty of Mathematics
University of Iaşi
R. S. Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXV, s. I a, f. 1

EQUILIBRIUM POINTS FOR $2 \times n$ BIMATRIX GAMES

BY

CONSTANTIN SMADICI

It is well known that for two-person zero-sum games, in which one of the players has only two pure strategies, it is possible to find all optimal strategies by using a graphical method, [1]. In this note we shall describe a simple procedure for finding an equilibrium point for two-person nonzero-sum games in which one of the players has only two pure strategies.

1. Let $\Gamma = (A, B)$ be a bimatrix game, given by the payoff matrices $A = (a_{ij})$, $B = (b_{ij})$, each of them with m rows and n columns. As usually one denotes by X and Y the mixed strategies for the two players, i.e.

$$X = \{x \mid x \in R^m, x \geqslant 0, \sum x_i = 1\}, \quad Y = \{y \mid y \in R^n, y \geqslant 0, \sum y_j = 1\}. \tag{1}$$

Definition 1 [2]. *A pair of strategies* (x^0, y^0), $(x^0, y^0) \in X \times Y$, *is called an equilibrium point (EP) for the game* $\Gamma = (A, B)$ *if*

$$xAy^0 \leqslant x^0Ay^0, \text{ for all } x \in X, \quad x^0By \leqslant x^0By^0, \text{ for all } y \in Y. \tag{2}$$

Definition 2. *A saddle point for a game* $\Gamma = (A, B)$ *is called an equilibrium point* (x^0, y^0) *such that* x^0 *and* y^0 *are pure strategies.*

Definition 3. *Let* $\Gamma = (A, B)$ *be a bimatrix game; the graph associated with* Γ *is a mixed graph* $G = (V, U)$, *where*

i) $V = \{(i, j) \mid i = 1, \ldots, m, j = 1, \ldots, n\}$,

ii) *the arc* $((i, j), (k, l)) \in U$ *iff* $i = k$ *and* $b_{ij} > b_{kl}$, *or* $j = l$ *and* $a_{ij} > a_{kl}$,

iii) *the edge* $\{(i, j), (k, l)\} \in U$ *iff* $i = k$, $j \neq l$ *and* $b_{ij} = b_{kl}$, *or* $j = l$, $i \neq k$, *and* $a_{ij} = a_{kl}$.

If $G = (V, U)$ is the associated graph to a bimatrix game Γ, we shall write $(i, j) \succ (k, l)$ when $((i, j), (k, l)) \in U$ and $(i, j) \sim (k, l)$ when $\{(i, j), (k, l)\} \in U$.

Definition 4. *Let* $G = (V, U)$ *be the graph associated with a bimatrix game; then case call the row* i *of* G *the subgraph of* G *the vertices of which are* (i, j), $j = 1, \ldots, n$, *and the column* j *of* G *is the subgraph of* G *the vertices of which are* (i, j), $i = 1, \ldots, m$.

Definition 5. *A nondominated vertex in a mixed graph* $G = (V, U)$ *is a vertex* v, $v \in V$, *with the property that the arc* $(v', v) \notin U$ *for all* $v' \in V$.

It is clear that a bimatrix game has a saddle point iff his associated graph has a nondominated vertex, and every row and every column of the associated graph to a bimatrix game has at least one nondominated vertex.

Now, let us consider $G = (V, U)$ the associated graph to a bima[trix] game $\Gamma = (A, B)$ and $p: U \to R$ defined by

(3) $$p((i,j) \succ (k,l)) = \begin{cases} b_{ij} - b_{kl}, & i = k,\\ a_{ij} - a_{kl}, & j = l.\end{cases}$$

Definition 6. *If $G = (V, U)$ is the graph associated with a bima[trix] game $\Gamma = (A, B)$ and $p: U \to R$ is defined as above we call marked gra[ph] associated with Γ the couple (G, p).*

The following lemma was proved in [3].

Lemma 1. *The pair of strategies (x, y) is an equilibrium point [for] the game $\Gamma = (A, B)$ iff it is an equilibrium point for the game $\Gamma_{kh} = (A_k, B^h)$, $k \in \{1, \dots, m\}$, $h \in \{1, \dots, n\}$, where A_k, (B^h), is the matr[ix] obtained from A, (B), by subtracting the row k, (column h), from each r[ow] of A, (column of B).* Using this lemma, one can easily prove the fol[lo]wing lemma:

Lemma 2. *Two bimatrix games with the same marked associa[ted] graph have the same equilibrium points.*

Definition 7. *Let $G = (V, U)$ be the graph associated with a bimatr[ix] game; we shall say that the column j of G is weakly dominated in G if th[ere] exists h, $h \in \{1, \dots, n\}$, such that $(i, h) \succeq (i, j)$ for all $i = 1, \dots, m$.*

Remark 1. Let $\Gamma = (A, B)$ be a bimatrix game with the associ[a]ted graph $G = (V, U)$ and let J be the set of all weakly domina[ted] columns in G, if $\tilde{\Gamma} = (\tilde{A}, \tilde{B})$ is the bimatrix game obtained from Γ [by e]lating all the columns j, $j \in J$, from the matrices A, B and $(\tilde{x}, \tilde{y})$ is [an] EP for $\tilde{\Gamma}$, then (x, y) is an *EP* for Γ, where

(4) $$x = \tilde{x};\quad y_j = \begin{cases} \tilde{y}_j, & j \notin J,\\ 0, & j \in J.\end{cases}$$

Remark 2. Taking into account the fact that (x, y) is an EP [for] a $2 \times n$ bimatrix game $\Gamma = (A, B)_{2 \times n}$ iff

(2') $$a_i y \leq (x_1 a_1 + x_2 a_2)\, y;\quad x_1 b_1 + x_2 b_2 \leq (x_1 b_1 + x_2 b_2)\, y \qquad i = 1, [2]$$

where a_i, b_i, $i = 1, 2$, are the i-th rows of the matrices A, B, respe[cti]vely, it results that, when Γ has no a saddle point, we have $x > 0$

Remark 3. If the graph associated with a $2 \times n$ bimatrix ga[me] has no circuits of length 4 then the game has a saddle point.

2. Let (G, p) be the marked associated graph to a $2 \times n$ bimatr[ix] game $\Gamma = (A, B)$ and $(1, j) \succ (1, j_1) \succ (2, j_1) \succ (2, j) \succ (1, j)$ a circ[uit] of length 4 in G; we shall say that the pair of strategies (x^0, y^0), $i = 1, \dots, n$, $i \neq j, j_1$, are generated by this circuit if

(5) $$x_i^0 = \begin{cases} \dfrac{a}{a+b}; & i = 1,\\[2mm] \dfrac{b}{a+b}; & i = 2,\end{cases}\quad ;\quad y_h^0 = \begin{cases} \dfrac{c}{c+d}; & h = j,\\[2mm] \dfrac{d}{c+d}; & h = j_1,\\[2mm] 0; & h \neq j, j_1\end{cases}$$

$$y_h^i = \begin{cases} 1; & h = i,\\ 0; & h \neq i;\ i \in \{1, \dots, n\} \setminus \{j, j_1\},\end{cases}$$

where $a = p((2, j_1) \succ (2, j))$, $b = p((1, j) \succ (1, j_1))$, $c = p((1, j_1) \succ (2, j_1))$, $d = p((2, j) \succ (1, j))$.

Now, let us consider a $2 \times n$ bimatrix game $\Gamma = (A, B)$ and let α, β be two real numbers, $\alpha, \beta > 0$, $\alpha + \beta = 1$; we shall denote by $\Gamma(s, \alpha, \beta)$, $s \in \{1, 2\}$, the $2 \times n$ bimatrix game with the payoff matrices $A(s) = (a_{ij}(s))$ $B(s) = (b_{ij}(s))$ defined by

(6) $$\begin{aligned} a_i(s) &= \delta_i^s(\alpha a_1 + \beta a_2) + \delta_i^{s+1} a_2 + \delta_{i+1}^s a_1;\ i = 1, 2,\\ b_i(s) &= \delta_i^s(\alpha b_1 + \beta b_2) + \delta_i^{s+1} b_2 + \delta_{i+1}^s b_1,\ i = 1, 2,\end{aligned}$$

where δ_h^k is Kronecker's symbol and $a_i, b_i, a_i(s), b_i(s)$ are the i-th rows in the matrices A, B, $A(s)$, $B(s)$, respectively.

Definition 8. *If $(x(s), y(s))$ is a pair of strategies for the game $\Gamma(s; \alpha, \beta)$, we shall say that the pair of strategies (x, y), for the game Γ, is generated by $(x(s), y(s))$ if*

(7) $$\begin{aligned} y &= y(s),\\ x &= [x_1(s)\delta_1^s + x_2(s)\delta_2^s](\alpha, \beta) + x_1(s)\delta_2^s(1, 0) + x_2(s)\delta_1^s(0, 1),\end{aligned}$$

where (α, β), $(1, 0)$, $(0, 1)$ are considered as vectors with two components.

Theorem 1. *Let $\Gamma = (A, B)$ be a $2 \times n$ bimatrix game and $(x(s), y(s))$, $x(s) > 0$, an EP for $\Gamma(s; \alpha, \beta)$; then the pair of strategies, (x, y), generated by $(x(s), y(s))$ is an EP for the given game Γ.*

Proof. Let $(x(s), y(s))$, $x(s) > 0$, be an EP for $\Gamma(s; \alpha, \beta)$; then

(8) $$a_i(s) y(s) = x(s) A(s)\, y(s);\ i = 1, 2,\ x(s) B(s) \leq x(s) B(s) y(s),$$

because $x_1(s) + x_2(s) = 1$ and $x_1(s)$, $x_2(s) > 0$. Taking into account the formulas (6), (7), α, $\beta > 0$, $\alpha + \beta = 1$, it follows

(9) $$a_i\, y(s) = xAy(s);\ i = 1, 2,\ xB \leq xBy(s).$$

Hence (x, y), generated by $(x(s), y(s))$, is an EP for the game Γ and the theorem is proved.

Corollary 1. *If Γ is a $2 \times n$ bimatrix game and x^0 is a mixed strategy for the first player, generated by a circuit of length 4 in the graph associated with Γ, then every equilibrium point, $(x(s), y(s))$, $x(s) > 0$, for $\Gamma(s; x_1^0, x_2^0)$, $s \in \{1, 2\}$, generates an EP for the game Γ.*

Remark 4. If $x^0 \in X$ is a mixed strategy generated by a circuit of length 4 in the graph associated with a $2 \times n$ bimatrix game Γ, then in the associated graph to $\Gamma(s; x_1^0, x_2^0)$, $s \in \{1, 2\}$, there exists at least one weakly dominated column; therefore, if a circuit of length 4 in the associated graph to Γ does not generate an EP for Γ, would be usefull to establish whether one of the games $\Gamma(s; x_1^0, x_2^0)$, $s = 1, 2$, has equilibrium points, $(x(s), y(s))$, such that $x(s) > 0$. Note that, by remark 1, we can consider only $2 \times n$ bimatrix games, Γ, with exactly one nondominated vertex in every row of the graph associated with Γ (the ve[r]tex (i,j) is a nondominated vertex in the row i if $(i,j) \succsim (i,h)$ for [all] $h, h \neq j$).

Lemma 3. *Let Γ be a $2 \times n$ bimatrix game and $\alpha, \beta \in R$, $\alpha, \beta >$ [0], $\alpha + \beta = 1$; Then, if $(i,j) \succsim (k,j)$ in the graph associated with Γ, we ha[ve] $(i,j) \succsim (k,j)$ in the graph associated with $\Gamma(s; \alpha, \beta)$, $s \in \{1, 2\}$, and co[n]versely.*

Proof: Let us consider $(i,j) \succsim (k,j)$ in the graph associated wi[th] $\Gamma = (A, B)$, i.e. $a_{ij} \geq a_{kj}$; we shall have $(i,j) \succsim (k,j)$ in the graph asso[ciated] with $\Gamma(s; \alpha, \beta)$ iff $a_{ij}(s) \geq a_{kj}(s)$, where $a_{ij}(s)$, $a_{kj}(s)$ are defin[ed] by formulas (6). Taking into account the formulas (6) it follows

$$a_{ij}(s) - a_{kj}(s) = [(\delta_i^s - \delta_k^s)\alpha + \delta_{i+1}^s - \delta_{k+1}^s]\, a_{1j} + \\ + [(\delta_i^s - \delta_k^s)\beta + \delta_i^{s+1} - \delta_k^{s+1}]\, a_{2j}, \tag{10}$$

and because $\alpha + \beta = 1$ it follows

$$a_{ij}(s) - a_{kj}(s) = (\delta_1^s \alpha + \delta_2^s \beta)(a_{ij} - a_{kj}). \tag{11}$$

Since $\alpha, \beta > 0$, it results that $a_{ij}(s) - a_{kj}(s) \geq 0$ iff $a_{ij} - a_{kj} \geq 0$ an[d the] lemma is proved.

In the following lemma we suppose the game Γ such that in eve[ry] row of the graph associated with Γ there exists exactly one nondom[i]nated vertex.

Lemma 4. *Let Γ be a $2 \times n$ bimatrix game and $(1,j) \succ (1,j_1) \succ (2,j_1) \succ (2,j) \succ (1,j)$ a circuit of length 4 in the graph associated wi[th] Γ, where $(1,j)$, $(2,j)$ are the nondominated vertices in the first and th[e] second row, respectively. Then, if $x^0 \in X$ is the strategy generated by th[is] circuit and both $\Gamma(s; x_1^0, x_2^0)$, $s = 1, 2$, have a saddle point, it follows th[at] there exists $y \in Y$ such that (x^0, y) is an EP for Γ.*

Proof: First, note that we can suppose $j_1 > j$. Since the grap[h] associated with Γ has only a nondominated vertex in every row, an[d] the first (second) row in the graph associated with $\Gamma(2; x_1^0, x_2^0)$, ($\Gamma(1; x_1^0, x_2^0)$), is the same as the first (second) row in the graph associated wi[th] Γ it follows that $\Gamma(s; x_1^0, x_2^0)$, $s \in \{1, 2\}$, can have a saddle point on[ly] in the row s of the graph associated with it.

Let us suppose now that both the games $\Gamma(s; x_1^0, x_2^0)$, $s = 1, 2$, hav[e] a saddle point and let $(1, j_2)$ be a saddle point for $\Gamma(1; x_1^0, x_2^0)$ and $(2, j_3)$ a saddle point for $\Gamma(2; x_1^0, x_2^0)$; if $j_2 \in \{j, j_1\}$ or $j_3 \in \{j, j_1\}$ it is cle[ar] that (x^0, y^0) is an EP for Γ and lemma is proved; if $j_2, j_3 \notin \{j, j_1\}$ an[d] $j_2 = j_3$ then, by lemma 4, it rezults that (x^0, y^{j_2}) is an EP for Γ an[d] lemma is proved too. Therefore it remains to prove the lemma in th[e] case $j_2, j_3 \notin \{j, j_1\}$ and $j_2 \neq j_3$. Since the first row in the graph associate[d] with $\Gamma(1; x_1^0, x_2^0)$ is the same as the second row in the graph associate[d] with $\Gamma(2; x_1^0, x_2^0)$, it follows that $(1, j_2) \sim (1, j_3) \succsim (1, h)$ for all $h, h \neq j_2, j_3$, in the graph associated with $\Gamma(1; x_1^0, x_2^0)$, and because $(1, j_3)$ is a saddle point for $\Gamma(1; x_1^0, x_2^0)$, and $(2, j_3)$ is a saddle point for $\Gamma(2; x_1^0, x_2^0)$, by lemma 4, we can write $(1, j_2) \succsim (2, j_2)$ and $(2, j_3) \succsim (1, j_3)$ in the graph associated with Γ. Now, if $(1, j_2) \sim (2, j_2)$ or $(2, j_3) \sim (1, j_3)$ in

[t]he graph associated with Γ, then (x^0, y^{j_2}) or (x^0, y^{j_3}), respectively, is [a]n EP for Γ and lemma is proved; if $(1, j_2) \succ (2, j_2)$ and $(2, j_3) \succ (1, j_3)$, [i]n the graph associated with Γ, consider $\bar{y} = (\bar{y}_h) \in Y$ defined by

$$\bar{y}_h = \begin{cases} \dfrac{\alpha}{\alpha + \beta}, & h = j_2, \\ \dfrac{\beta}{\alpha + \beta}, & h = j_3, \\ 0, & h \neq j_2, j_3, \end{cases} \tag{12}$$

where $\alpha = p((2, j_3) \succ (1, j_3))$ and $\beta = p((1, j_2) \succ (2, j_2))$.

Since in the graph associated with $\Gamma(1; x_1^0, x_2^0)$ we have $(1, j_2) \sim (1, j_3) \succsim (1, h)$, for all $h \neq j_2, j_3$, it results that $(x^0, \bar{y})$ is an EP for Γ and this fact completes the proof of the lemma.

Lemma 5. *Let Γ be a $2 \times n$ bimatrix game and $x^0 \in X$ a strategy generated by a circuit of length 4 in the graph associated with Γ; then at least one of the games $\Gamma(s; x_1^0, x_2^0)$, $s = 1, 2$, has a saddle point.*

Proof: Let $(1, j)$ be a nondominated vertex in the first row of the graph associated with $\Gamma(1; x_1^0, x_2^0)$; if in the graph associated with Γ we have $(1, j) \succsim (2, j)$, then, by lemma 4, it follows $(1, j) \succsim (2, j)$ in the graph associated with $\Gamma(1; x_1^0, x_2^0)$, hence $(1, j)$ is a saddle point for $\Gamma(1; x_1^0, x_2^0)$; if $(2, j) \succ (1, j)$, in the graph associated with Γ, then we have $(2, j) \succ (1, j)$ in the graph associated with $\Gamma(2; x_1^0, x_2^0)$, and because the first row in the graph associated with $\Gamma(1; x_1^0, x_2^0)$ is the same as the second row in the graph associated with $\Gamma(2; x_1^0, x_2^0)$, it follows that $(2, j)$ is a saddle point for $\Gamma(2; x_1^0, x_2^0)$ and the lemma is proved.

Taking into account lemmas 4,5 and remark 2, we can state now the following theorem:

Theorem 2. *Let Γ be a $2 \times n$ bimatrix game and $(1, j) \succ (1, j_1) \succ (2, j_1) \succ (2, j) \succ (1, j)$ a circuit of length 4 in the graph associated with Γ, such that $(1, j)$ and $(2, j_1)$ are the nondominated vertices in the first and the second row respectively; then, if $x^0 \in X$ is a strategy generated by this circuit, one and only one of the following statements is true:*

i) *There exists a pair of strategies (x^0, y^t), generated by this circuit, which is an EP for Γ.*

ii) *For each EP, $(x(1), y(1))$, of $\Gamma(1; x_1^0, x_2^0)$ we have $x(1) > 0$.*

iii) *For each EP, $(x(2), y(2))$, of $\Gamma(2; x_1^0, x_2^0)$ we have $x(2) > 0$.*

iv) *There exists $y \in Y$, $y \neq y^i$, $i \in \{0, 1, \ldots, n\} \setminus \{j, j_1\}$, such that (x^0, y) is an EP for Γ.*

Let $x^0 \in X$ be a strategy generated by the circuit $(1, j) \succ (1, j_1) \succ (2, j_1) \succ (2, j) \succ (1, j)$ in the graph associated with a $2 \times n$ bimatrix game Γ, where $(1, j)$ and $(2, j_1)$ are the nondominated vertices in the first and the second row, respectively; let us denote by $J(s)$ the set of weakly dominated columns in the graph associated with $\Gamma(s; x_1^0, x_2^0)$, $s \in \{1, 2\}$, by $\tilde{\Gamma}(s; x_1^0, x_2^0)$ the game obtained from $\Gamma(s; x_1^0, x_2^0)$ delating all the columns j, $j \in J(s)$, and by $\Gamma(s)$ the game obtained from Γ delating all the columns j, $j \in J(s)$.

Lemma 6. *If $\tilde\Gamma(s; x_1^0, x_2^0)$, $s \in \{1, 2\}$, has every EP such that $\tilde x(s)$, then for every EP $(\bar x(s), \bar y(s))$, of $\Gamma(s; x_1^0, x_2^0)$ we have $\bar x_{\bar s}(s) = 0$, wh[ere] $\bar\Gamma(s; x_1^0, x_2^0)$ is obtained from $\Gamma(s; x_1^0, x_2^0)$ delating all the columns j, $j \in J$ and $\bar s$ is defined by*

(13) $$\bar s = \begin{cases} 1, & s = 2, \\ 2, & s = 1. \end{cases}$$

Proof. Suppose that for every EP, $(\tilde x(1), \tilde y(1))$, of $\tilde\Gamma(1; x_1^0, x_2^0)$ we have $\tilde x(1) > 0$; then there exists $j_2 \neq j, j_1$ such that $(1, j_2) \succ (1, h)$ for all $h \neq j_2$ and $(2, j_2) \succ (1, j_2)$, in the graph associated with $\tilde\Gamma(1; x_1^0, x_2^0)$. Since $\tilde\Gamma(1; x_1^0, x_2^0)$ is obtained from $\Gamma(1; x_1^0, x_2^0)$ delating all the weakly dominated columns, it follows that in the graph associated with $\Gamma(1; x_1^0, x_2^0)$ we have $(1, j_2) \succ (1, h)$ and $(2, h) \succ (2, j_2)$ for all $h \neq j_2$, which implies $(1, j_2) \succ (1, h)$ and $(2, j_2) \succ (2, h)$ in the graph associated with $\tilde\Gamma(2; x_1^0, x_2^0)$; according to remark 2 it results that $\tilde\Gamma(2; x_1^0, x_2^0)$ has only one EP corresponding to the saddle point $(2, j_2)$. Following the same way one can prove that, when for every EP $(\tilde x(2), \tilde y(2))$, of $\tilde\Gamma(2; x_1^0, x_2^0)$ $\tilde x(2) > 0$, then every EP of $\tilde\Gamma(1; x_1^0, x_2^0)$ is a saddle point and our lemma is proved.

Theorem 3. *If the game $\Gamma(s; x_1^0, x_2^0)$, $s \in \{1, 2\}$, has no saddle points then every EP for $\tilde\Gamma(s)$ gives an EP for Γ.*

Proof. Suppose $\Gamma(s; x_1^0, x_2^0)$ has no saddle points; since in the graph associated with Γ there exists exactly one nondominated vertex in every row, it follows that for the EP, $(x(s), y(s))$, of $\Gamma(s; x_1^0, x_2^0)$ we have $y_j(s) = 0$, $j \in J(s)$ and, if $(\tilde x(s), \tilde y(s))$ is an EP of $\tilde\Gamma(s; x_1^0, x_2^0)$, then $(\tilde x(s) \tilde y_j(s))$ is an EP of $\Gamma(s; x_1^0, x_2^0)$ where

(14) $$y_j(s) = \begin{cases} 0, & j \in J(s), \\ \tilde y_j(s), & j \notin J(s). \end{cases}$$

Now, let us suppose $s = 1$ and $(\tilde x, \tilde y)$ an EP for $\tilde\Gamma(1)$, i.e.

(15) $$\tilde x_1 \tilde b_1 + \tilde x_2 \tilde b_2 \leqslant (\tilde x_1 \tilde b_1 + \tilde x_2 \tilde b_2) \tilde y, \quad \tilde a_i \tilde y \leqslant (\tilde x_1 \tilde a_1 + \tilde x_2 \tilde a_2) \tilde y; \; i = 1, 2,$$

and define the pair of strategies $(x(1), y(1))$ for $\tilde\Gamma(1; x_1^0, x_2^0)$ by

(16) $$\tilde y(1) = \tilde y, \; \tilde x_1(1) = (1/x_1^0)\tilde x_1, \; \tilde x_2(1) = \tilde x_2 - (x_2^0/x_1^0)\tilde x_1.$$

It is clear that $\tilde y(1)$ is a strategy for the second player in the game $\tilde\Gamma(1; \tilde x_1^0, \tilde x_2^0)$ and $\tilde x_1(1) + \tilde x_2(1) = 1$, $\tilde x_1(1) \geqslant 0$; we shall prove that if $\tilde x_2(1) \geqslant 0$ then $(\tilde x(1), \tilde y(1))$, defined by (16), is an EP for $\tilde\Gamma(1; x_1^0, x_2^0)$

(17) $$\begin{aligned} &\tilde x_1(1)[x_1^0 \tilde b_1 + x_2^0 \tilde b_2] + \tilde x_2(1) \tilde b_2 \leqslant [\tilde x_1(1)(x_1^0 \tilde b_1 + x_2^0 \tilde b_2) + \\ &+ \tilde x_2(1) \tilde b_2] \tilde y(1) \; \tilde a_i(1) \tilde y(1) \leqslant [\tilde x_1(1) \tilde a_1(1) + \tilde x_2(1) \tilde a_2(1)] \tilde y(1); \; i = 1, 2 \end{aligned}$$

Taking into account (15) and (16), it follows:

(18) $$\begin{aligned} &\tilde x_1(1) x_1^0 \tilde b_1 + x_2^0 \tilde b_2 + \tilde x_2(1) \tilde b_2 = \tilde x_1 \tilde b_1 + \tilde x_2 \tilde b_2, \\ &\tilde a_i(1) \tilde y(1) = (\tilde x_1 \tilde a_1 + \tilde x_2 \tilde a_2) \tilde y; \; i = 1, 2, \\ &[\tilde x_1(1)(x_1^0 \tilde b_1 + x_2^0 \tilde b_2) + \tilde x_2(1) \tilde b_2] \tilde y(1) = (\tilde x_1 \tilde b_1 + \tilde x_2 \tilde b_2) \tilde y, \\ &[\tilde x_1(1) \tilde a_1(1) + \tilde x_2(1) \tilde a_2(1)] \tilde y(1) = (\tilde x_1 \tilde a_1 + \tilde x_2 \tilde a_2) \tilde y, \end{aligned}$$

and formulas (17) follow from (15) and (18).

Now we shall prove that $\tilde x_2(1) \geqslant 0$; suppose $\tilde x_2(1) < 0$ and let us consider the pair of strategies $(\tilde x(2), \tilde y(2))$ for $\tilde\Gamma(2; x_1^0, x_2^0)$ defined by

(19) $$\tilde y(2) = \tilde y, \; \tilde x_1(2) = \tilde x_1 - (x_1^0/x_2^0)\tilde x_2, \; \tilde x_2(2) = (1/x_2^0)\tilde x_2.$$

Because $\tilde x > 0$ and $\tilde x_2(1) < 0$ it results $\tilde x(2) > 0$, and as above, we can prove that $(\tilde x(2), \tilde y(2))$ is an EP for $\tilde\Gamma(2; x_1^0, x_2^0)$, and this contradicts lemma 7; therefore $\tilde x_2(1)$, defined by (16), is a nonnegative number and $(\tilde x(1), \tilde y(1))$ is an EP for $\tilde\Gamma(1; x_1^0, x_2^0)$. Then Remark 1 shows that $(x(1), y(1))$, where

(20) $$x(1) = \tilde x(1) > 0, \quad y_h(1) = \begin{cases} \tilde y_h(1); & h \notin J(1) \\ 0; & h \in J(1), \end{cases}$$

is an EP for $\Gamma(1; x_1^0, x_2^0)$ and, by Theorem 1, it results that the pair of strategies (x, y) generated by $(x(1), y(1))$ is an EP for Γ. Note that from (7), (16), (20) it follows

(21) $$x = \tilde x, \quad y_h = \begin{cases} \tilde y_h; & h \notin J(1), \\ 0; & h \in J(1). \end{cases}$$

hence we can obtain the EP (x, y) directly from $(\tilde x, \tilde y)$, and the theorem is proved in the case $s = 1$. Similarly one can prove that it is true in the case $s = 2$, and this completes the proof.

Now we can prezent the procedure for finding an EP for a $2 \times n$ bimatrix game. Let $\Gamma = (A, B)$ be a $2 \times n$ bimatrix game.

Step 0: Has Γ a saddle point? If YES then we have found an EP for Γ and STOP; if NO go to step 1.

Step 1: Delate all the weakly dominated columns in Γ and obtain the game $\tilde\Gamma = (\tilde A, \tilde B)$ such that in every row of $\tilde B$ there exists only one maximal entry. Let b_{1j} the maximal entry in the first row of $\tilde B$ and b_{2j_1} the maximal entry in the second row of $\tilde B$; by lemma 1, we can consider $\tilde a_{1h} = 0$, $h = 1, \ldots, n$, and $\tilde b_{1j} = \tilde b_{2j} = 0$.

Step 2: Consider $x^0 = (x_1^0, x_2^0)$, where

(22) $$x_1^0 = \frac{\tilde b_{2j_1}}{\tilde b_{2j_1} - \tilde b_{1j_1}} \qquad x_2^0 = \frac{-\tilde b_{1j_1}}{\tilde b_{2j_1} - \tilde b_{1j_1}}$$

and the games $\tilde\Gamma(s; x_1^0, x_2^0) = (\tilde A(s), \tilde B(s))$, $s = 1, 2$, given by the payoff matrices:

$$\tilde A(1) = \begin{bmatrix} x_2^0 \tilde a_2 \\ \tilde a_2 \end{bmatrix}; \; \tilde B(1) = \begin{bmatrix} x_1^0 b_1 + x_2^0 \tilde b_2 \\ \tilde b_2 \end{bmatrix}; \; \tilde A(2) = \begin{bmatrix} 0 \\ x_2^0 \tilde a_2 \end{bmatrix}; \; \tilde B(2) = \begin{bmatrix} \tilde b_1 \\ x_1^0 \tilde b_1 + x_2^0 \tilde b_2 \end{bmatrix}.$$

Step 3: If both the games $\tilde{\Gamma}(s; x_1^0, x_2^0)$, $s = 1, 2$, have a saddle poi[nt] then, using lemma 4, we obtain an EP for $\tilde{\Gamma}$ and, by Remark 1, for [Γ]. If $\tilde{\Gamma}(s; x_1^0, x_2^0)$, $s \in \{1, 2\}$, has not a saddle point, consider $J(s)$ and $\tilde{\Gamma}($[s)] obtained from $\tilde{\Gamma}$ delating all the columns $j \in J(s)$ and apply Step [1] to $\tilde{\Gamma}(s)$.

Since after each iteration we shall obtain a game, the number [of] columns of which is less than the number of columns in the origina[l] game, it follows that in a finite number of iterations we shall obtain [an] EP for the given game Γ.

Example: Let $\Gamma = (A, B)$ be the bimatrix game given by the pa[-]yoff matrices

$$A = \begin{bmatrix} 0 & 1 & -1 & 2 & 2 \\ -2 & -3 & 1 & 6 & 3 \end{bmatrix}; \quad B = \begin{bmatrix} -1 & -3 & 1 & 2 & 1 \\ 1 & 3 & 0 & -2 & -3 \end{bmatrix}.$$

We observe that in every row of B there exists only one maximal ele[-]ment, b_{14} and b_{22} respectively, and Γ has not a saddle point.

Step 1: Consider the matresic

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -3 & -3 & 2 & 4 & 1 \end{bmatrix}; \quad B^4 = \begin{bmatrix} -3 & -5 & -1 & 0 & -1 \\ 3 & 5 & 2 & 0 & 3 \end{bmatrix};$$

Step 2:

$$x^0 = (x_1^0, x_2^0) = (1/2, 1/2); \quad y^0 = (0, 4/7, 0, 3/7, 0)$$

$$A_1(1) = \begin{bmatrix} -3/2 & -3/2 & 1 & 2 & 1/2 \\ -3 & -3 & 2 & 4 & 1 \end{bmatrix}; \quad B^4(1) = \begin{bmatrix} 0 & 0 & 1/2 & 0 & -2 \\ 3 & 5 & 2 & 0 & -3 \end{bmatrix},$$

$$A_1(2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -3/2 & -3/2 & 1 & 2 & 1/2 \end{bmatrix}; \quad B^4(2) = \begin{bmatrix} -3 & -5 & -1 & 0 & -1 \\ 0 & 0 & 1/2 & 0 & -2 \end{bmatrix};$$

Step 3: The game with the payoff matrices $A_1(1)$, $B^4(1)$ has not a saddle point; delate from the matrices A_1 and B^4 the columns corres[-]ponding to the weakly dominated columns in $\Gamma(1; x_1^0, x_2^0)$, i.e. columns 1, 4, 5 and obtain the game $\Gamma(1)$ with the payoff matrices

$$A_1 = \begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}, \quad B^4 = \begin{bmatrix} -5 & -1 \\ 5 & 2 \end{bmatrix}.$$

Applying Step 1 to this game we obtain $x^0 = (3/7, 4/7)$, $y^0 = (2/5, 3/5)$ and observe that (x^0, y^0) is an EP for $\Gamma(1)$; therefore (x, y), where $x = (3/7, 4/7)$ and $y = (0, 2/5, 3/3, 0, 0)$, is an EP for the given game Γ.

REFERENCES

1. Motzkin T. S., Raiffa H., Thompson G. L. — *The double description method*, Ann. of Math. Studies, 28 (1953), 51—73.
2. Nash J. — *Equilibrium points in n-Person Games*, Proc. Nat. Acad. Sci. U.S.A., 36 (1950), 48—49.
3. Smadici, C. — *Finding an equilibrium point for bimatrix games by Benders decomposition*, to appear.

Faculty of Mathematics
University of Iaşi
R. S. România

Received 25.V.1978

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXV, s. I a, 1979, f. 1

NOTE

LA GÉOMÉTRIE ET LA THÉORIE DES CATÉGORIES

PAR

V. CRUCEANU

Introduction. La théorie des catégories représente la synthèse la plus générale et profonde dans le développement de la mathématique et elle constitue l'instrument fondamental pour la structuration de la mathématique actuelle. En particulier, la notion de catégorie nous permet de définir avec beaucoup de rigueur et de clarté l'objet des nombreuses disciplines mathématiques et celui de foncteur, d'établir d'une manière précise les rapports entre ces disciplines.

Le but de ce travail est d'établir certaines implications de la théorie des catégories dans la géométrie classique. Le point de vue prèdominant dans ce domaine, qui a joué un rol fondamental pour le développement de la géométrie dans le dernier siècle, est celui de F. Klein, ou autrement dit celui de la théorie des groupes. Conformément à ce point de vue, étant donné un espace S doué d'une certaine structure et un groupe G d'automorphismes de cette structure, la géométrie de l'espace S, par rapport au groupe G, est l'étude de ces propriétés des figures de S qui sont invariantes aux transformations de G. Or, en géométrie nous n'étudions pas d'habitude un seul espace, mais une classe d'espaces d'un certain type. D'autre part, nous considérons non seulement des tranformations bijectives d'un espace en lui même, mais des transformations qui, en général, ne sont pas bijectives et qui transforment un espace donné en un autre de la même classe. Il en résulte que l'objet de la géométrie est beaucoup plus général que celui considéré par F. Klein. Le point de vue de la théorie des groupes est aussi limité en ce qui concerne le rapport entre les diverses géométries parcequ'il se réfère seulement à certaines relations d'isomorphisme ou de subordination assez restrictives. Nous allons montrer que la théorie des catégories nous offre un point de vue plus général et précis pour définir les diverses géométries et pour établir les rapports entre celles-ci, ainsi que, entre la géométrie et d'autres disciplines mathématiques.

Dans ce qui suit nous considérons seulement des catégories dont les objets sont des ensembles doués de certaines structures et les morphismes sont des applications qui conservent ces structures. Soit $\mathcal{C}$ et $\mathcal{D}$ deux catégories de ce type et supposons que les objets de $\mathcal{C}$ s'obtiennent des objets de $\mathcal{D}$ par l'adjonction d'une certaine structure supplémentaire σ et que les morphismes de $\mathcal{C}$ sont des morphismes de $\mathcal{D}$ qui satisfont à la condition de conserver la structure σ.

Un *foncteur d'oubli* de $\mathcal{C}$ vers $\mathcal{D}$ sera un foncteur qui associe à chaque objet de $\mathcal{C}$ l'objet de $\mathcal{D}$ avec le même ensemble support et à chaque morphisme de $\mathcal{C}$, le morphisme de $\mathcal{D}$ représenté par la même application mais en omettant le fait qu'il conserve la structure σ. Nous dirons que la catégorie $\mathcal{C}$ est *subordonnée* à la catégorie $\mathcal{D}$ si, et seulement si, il existe un foncteur d'oubli de $\mathcal{C}$ vers $\mathcal{D}$. Dans ce cas nous dirons aussi que la discipline mathématique qui a comme objet d'étude la catégorie $\mathcal{C}$ est subordonnée à la discipline correspondante à $\mathcal{D}$. Si $\mathcal{C}$ est une sous-catégorie de $\mathcal{D}$ alors le foncteur injection canonique $i: \mathcal{C} \to \mathcal{D}$ peut être regardé comme un foncteur d'oubli et par suite toute sous-catégorie de $\mathcal{D}$ est subordonnée à $\mathcal{D}$. Une catégorie $\mathcal{C}$ est *isomorphe* à une catégorie $\mathcal{D}$ s'il existe un foncteur de $\mathcal{C}$ vers $\mathcal{D}$ qui soit un isomorphisme, c'est à dire plein, fidel et bijectif. Les disciplines mathématiques correspondantes à deux catégories isomorphes seront appellées aussi isomorphes. Une catégorie $\mathcal{C}$ est *équivalente* à une catégorie $\mathcal{D}$ s'il existe un foncteur $F: \mathcal{C} \to \mathcal{D}$ qui soit plein et fidel et tel que chaque objet B de $\mathcal{D}$ soit isomorphe à un objet de la forme $F(A)$, avec A objet de $\mathcal{C}$. Les disciplines mathématiques correspondantes à deux catégories équivalentes seront appellées aussi équivalentes. Un *squelette* pour une