tion de g) et $h \circ c(h)$ est l'application indentique de $pr_2 h$ (domaine des valeurs de h). En choisissant $h = g = c^2(f)$, les formules déduites assurent que $f = c^2(f)$.

Corollaire. *Si $\mathcal{C}$ est une* C.A. *contenue dans $\mathcal{R}$, il y a une seule* S.A. *sur $\mathcal{C}$.*

En effet, la condition A.2 appliquée pour f et $c(f)$, nous assure que $c(f) = f$.

Théorème 8. *Si $\mathcal{C}$ est une* C.A, *il y a une seule* S.A. *sur* c.

Démonstration. Supposons par absurde que c et d sont des S.A. distinctes sur $\mathcal{C}$: c'est à dire qu'il existe $u \in \mathcal{C}(A, B)$ tel que $c(u) \neq d(u)$. Soit $\mathcal{D}$ la sous-catégorie pleine de $\mathcal{C}$ ayant seulement les objets A et B. $\mathcal{D}$ après le théorème 4 il y a un foncteur bijectif $\kappa : \mathcal{D} \to \mathcal{E}$ où $\mathcal{E}$ est une sous-catégorie de $\mathcal{R}$. Evidemment $\mathcal{E}$ est une C.A. Mais $r(c(u))$ et $\kappa(d(u))$ sont des inverses généralisés de $\kappa(u)$ donc $\kappa(c(u)) = \kappa(d(u))$, égalité contredite par $c(u) \neq d(u)$.

Une démonstration analogue permet de renoncer à la condition que $\mathcal{C}$ est contenue dans $\mathcal{R}$ dans le théorème 7.

Théorème 9. *La condition nécessaire et suffisante afin que la catégorie $\mathcal{C}$ soit une* C.A. *est l'existence d'une* S.A. *involutive sur $\mathcal{C}$.*

Démonstration. La nécessité de la condition énnoncée a été prouvée plus haut.

La condition A.2 assure que si c est une S.A involutive sur $\mathcal{C}$ alors quel que soit un morphisme u de $\mathcal{C}$, $c(u)$ est un inverse généralisé de u. Suposons que u admet encore un inverse généralisé; puisque c est involutif on peut la désigner par $c(v)$. On a donc: $u \cdot c(v) \cdot u =$ $u \wedge c(v)$ $u \cdot c(v) = c(v)$. Ces égalités deviennent, grâce à l'injectivité de c: $\tau(u, v, u) = u \wedge \tau(v, u, v) = v$ ou $v < u \wedge u < v$, donc $u = v$. Il résulte ainsi que si c est une S.A. involutive sur $\mathcal{C}$ alors quel que soit le morphisme u de $\mathcal{C}$, le seule inverse généralisé de u est $c(u)$.

Soit x un idempotent de $\mathcal{C}$; on déduit que l'inverse généralisé de x est x, donc $c(x) = x$ et par conséquent: $x = x \cdot c(x) = c(x) \cdot x$.

Si x et y sont des idempotents et appartiennent à $\mathcal{C}(A, A)$ on déduit à l'aide de A.3: $x \cdot y = 1_A \cdot c(x) \cdot x \cdot c(y) \cdot y = \tau(\tau(1_A, x, x), y, y) = \tau(\tau(1_A, y, y), x, x) = 1_A \cdot c(y) \cdot y \cdot c(y) \cdot c(x) \cdot x = y \cdot x$. Nous avons prouvé ainsi la conditions C.A.2 et donc que $\mathcal{C}$ est une C.A.

Nous considérons ce dernier résultat une justification pour l'utilisation du terme proposé plus haut de catégorie-amas; le théorème 4 donne ainsi simultanément une réalisation binaire *propre* pour un *ensemble* d'amas.

BIBLIOGRAPHIE

1. B r â n z e i D. — *Structures affines et opérations ternaires.* An. șt. Univ. Iași, t. XXIII s. I a, 1977, f. 1.
2. B r â n z e i D. — *On some Categories of Miron Spaces.* Lucrările Colocv. Naț. de Geometrie și Topologie, Timișoara 26—28 aprilie 1977.
3. C r o i s o t R. — *Demi-groupes et axiomatique des groupes.* C. R. Acad. Sc. Paris 237 (1953), p. 778—780.
4. V a g n e r V. V. — *Teorija obsceonîh grup i obsceonîh grud.* Mat. sb. 32 (1953), 545—632

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A GENERALIZATION OF THE CONCEPT OF NEAR RING: INFRA-NEAR RINGS

BY

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In this paper, we begin the study of an algebraic system which generalizes the concepts of ring, near-ring, weak ring (quasiring); we call it a *left infra-near ring*. It is a (nonabelian) group with respect to the addition, having also an associative multiplication, that satisfies a condition of left generalized distributivity over the addition.

The generalization of the distributivity in the definition of the linear algebra and the ring is an old question, whose roots can be found in D i c k s o n's paper: „On finite algebras" (1905). The substantial developments in the theory of near-rings and near-fields (initiated by L. E D i c k s o n himself and H. Z a s s e n h a u s in the years thirty) suggest it.

In Section 1, we give basic definitions and properties, as well as some examples of left infra-near rings. We discuss there the place of the element giving the „deviation" from the left distributivity in the right-hand side of the equality in the definition. Section 2 obtains the left infra-distributivity of an infra-near ring from the left distributivity of its binary multiplication over the composition of a ternary group (Prop. 2.3). Here we consider ternary groups in C e r t a i n e's sense ([6], Def. 3, Th. 4) for defining ternary near rings of first type and a new kind of ternary groups (Def. 2.2) for defining ternary near rings of second type. The axioms defining a ternary group of second type are independent (Prop. 2.2) and there exists a group associated canonically to each such a ternary group (Prop. 2.1). We distinguish the regular cases, namely the cases when ternary composition is induced by the binary group addition (see the equalities (2.6) and (2.16)). We show that the binary systems associated with the left ternary near rings are exactly the left infra-near rings introduced in Section 1. In the following sections, we choose the binary approach to the theory of infra-near rings. In Section 3, we present congruences, homomorphisms, ideals of the left infra-near rings. As usual, there is a bijection between the set of the congruences and the set of the two-sided ideals of a left infra-near ring I. We define right and left I-groups, left ternary I-groups and we show that there is an equivalence between the category of the left regular ternary groups over I and the category of the left I-groups. Section 4 presents a decomposition of a left infra-near ring I as a semidirect sum (as an additive group) of its maximal Z-infra-near ring and C-infra-near ring. This is a generalization of a result of B e r m a n and S i l v e r m a n [3] for near rings. In the last Section, we study the idempotents in left infra-near rings and the decomposition of a left infra-near ring associated to them.

In our other papers [13], [14], we have studied the set of the left infra-distributive multiplications on a group G, respectively some special classes of left infra-near rings, for example the class of those infra-near rings which have a great analogy with the left infra-near ring of the affine transformations of a linear space (see Example 5°, Section 1).

In the development which follows, elementary ring and near ring properties and constructions which can be readily translated to infra-near rings are used without comment. Any standard results in near ring theory used without quotation may be found in [4], [5] and [10].

§ 1. **Definitions, examples. Definition 1.1.** *We call a left infra-near ring* (l.i-n.r.) *a triple* $(I, +, .)$, *where* I *is a nonempty set,* $+$ *and* $.$ *are binary compositions on* I, *addition and multiplication, satisfying the following conditions: i)* $(I, +)$ *is a group (in general, nonabelian); ii)* $(I, .)$ *is a semigroup; iii) the multiplication is left infra-distributive over the addition,* i.e. *the equality*:

$$x\cdot(y+z)= x\cdot y - x\cdot 0 + x\cdot z \tag{1.1}$$

holds for all x, y, z *in* I (0 *is the neutral element of* $(I, +)$).

If the law of „right infra-distributivity", i.e. the equality:

$$(x+y)\cdot z= x\cdot z - 0\cdot z + y\cdot z \tag{1.2}$$

holds for all x, y, z in I (instead of (1.1)), then $(I, +, .)$ will be called a *right infra-near ring.* A left and simultaneously right infra-near ring will be called a *two-sided infra-near ring* or, shorter, an *infra-near ring.*

In a general l.i-n.r., $0\cdot 0 \neq 0$, $0\cdot x \neq 0$ and $x\cdot 0 \neq 0$, as one can see in some of the following examples given to clarify the definition.

Remark 1.1. A left near ring is a l.i.-n.r. for which:

$$x\cdot 0= 0, \qquad \forall x \in I. \tag{1.3}$$

In general, if an element x in I satisfies the equality (1.3), then we call it a *left distributive element* of I.

Remark 1.2. A weak ring (see C l i m e s c u [7, 8]) is an i.-n.r. for which the group $(I, +)$ is abelian and the elements are *weakly left* and *right distributive*, i.e. they satisfy the equalities:

$$x\cdot 0= x, \quad 0\cdot x= x \quad \text{(for the first type, see [7])}, \tag{1.4}$$

or

$$x\cdot 0= \varphi(x) + \psi(0),\ 0\cdot x= \varphi(0) + \psi(x) \ \text{(for the II}^{nd}\text{ type, see [8])}, \tag{1.4$'$}$$

where $\varphi(\cdot) - \varphi(0)$, $\psi(\cdot) - \psi(0) \in \mathrm{End}\ (I, +)$.

We remark here that the definition of the „quasiring", given by G. C u p o n a [9] in 1969, is analogous to C l i m e s c u's definition of the „weak ring" [7], given in 1961. We quote it after Math. Rev.: the left and right distributivity of the multiplication over the addition were

replaced by a generalized distributivity: $x \circ (y + z) + x= x \circ y + x \circ z$ and $(x + y) \circ z + z= x \circ z + y \circ z$. If $(I, +, .)$ is a ring, then $(I, +, \circ)$, where $x \circ y= x + y - x\cdot y$ (for C u p o n a) or $x \circ y= x\cdot y + x + y$ (for Climescu's first type) is a weak ring or a quasiring).

Example 1°. Let $(R, +, .)$ be a ring; then $I= R^5$ is a l.i.-n.r. with respect to one of the following pairs of binary compositions:

$$x + y = (y_1 + x_1,\ y_2 + x_2,\ y_3 + x_3,\ y_4 + x_4,\ y_5 + x_2\cdot y_3 + x_4\cdot y_1 + x_5),$$

$$x\cdot y = (x_1\cdot y_1,\ x_2\cdot y_2,\ x_3,\ x_4,\ x_5),$$

or

$$x + y= \text{the same as above},$$

$$x\cdot y= (x_1\cdot y_1,\ x_2\cdot y_2,\ a_3,\ a_4,\ a_5), \text{ with } a_3, a_4, a_5 \in R.$$

If $(R, +, .)$ is a distributive near ring with all products in the centre of $(R, +)$, then $(R^5, +, .)$ — as above — is a l.i.-n.r.

Example 2°. Let $(G, +)$ be a group. Then $I= G^2$ is a l.i.-n.r. with respect to the operations:

$$x + y= (x_1 + y_1,\ x_2 + y_2),$$

$$x\cdot y= (x_1 + y_1,\ y_2), \qquad \forall x, y \in I.$$

Example 3°. Let $(I, +)$ be a group. Then $(I, +, +)$ or $(I, +, +_a)$, or $(I, +, \perp_a)$, where: $x +_a y := x + a + y$, $\forall x, y \in I$ and $x \perp_a y := a$, $\forall x, y \in I$, $a \in I$ fixed, are infra-near rings.

Example 4°. Every binary composition on the cyclic group of order 2 is infra-distributive, but some of them are nonassociative.

Example 5°. The set of the affine transformations, $A(V)$, of a linear space V over a field K (namely, the functions $f: V \to V$, $f= h + c$, with $h \in \mathrm{Hom}_K(V, V)$ and $c \in T_c(V)$ a constant function on V, $c(x)= c$, $\forall x \in V$, $c \in V$, fixed) is a l.i.-n.r. (and a right near ring) with respect to the point addition and the usual composition of functions.

In a l.i.-n.r., one can easily verify the identities:

$$x\cdot(-y)= x\cdot 0 - x\cdot y + x\cdot 0, \qquad \forall x, y \in I, \tag{1.5}$$

$$x\cdot\left(\sum_{i=1}^{n} y_i\right)=\sum_{i=1}^{n-1}(x\cdot y_i - x\cdot 0) + x\cdot y_n, \qquad \forall x, y_1, \ldots, y_n \in I. \tag{1.6}$$

Remark 1.3. The multiplication defined on I by the formula:

$$x \circ y= x\cdot y - x\cdot 0, \qquad \forall x, y \in I, \tag{1.7}$$

is left distributive (with respect to addition), but sometimes it is not associative. We have discussed the conditions under which the multiplication (1.7) is associative in [13].

Remark 1.4. The two-sided infra-distributivity implies the permutability (with respect to the addition) of the elements of the form:

$$a= -0\cdot z + y\cdot z - y\cdot 0 + 0\cdot 0,\quad b= -x\cdot 0 + x\cdot t - 0\cdot t + 0\cdot 0, \text{ as}$$

one can see by developing in two ways the product $(x+y)\cdot(z+t)$, for all x, y, z, t in I. If I is a distributive near ring, $a = y\cdot z$, $b = x\cdot t$, and we obtain a classical result [11].

Remark 1.5. Replace the condition (1.1) by the following one:

$$(1.8) \qquad x\cdot(y+z) + x\cdot 0 = x\cdot y + x\cdot z, \qquad \forall x, y, z \in I.$$

Then, by taking $y = 0$ in (1.8), we obtain:

$$(1.9) \qquad x\cdot z + x\cdot 0 = x\cdot 0 + x\cdot z, \qquad \forall x, z \in I$$

Thus, by (1.8) and (1.9), we have: $x\cdot(y+z) = x\cdot y + x\cdot z - x\cdot 0 = x\cdot y - x\cdot 0 + x\cdot z$, $\forall x, y, z \in I$ hence we reobtain the left infra-distributivity given by (1.1). However, this time the condition (1.9) is necessary, whilst by the first definition, the condition (1.9) is not necessary, as one can see in Examples 1°, 2°, 3° (when the given additive group is not commutative). Hence there are l.i.-n.r.—s which do not satisfy (1.9).

§ 2. Left ternary near rings. What happens when the group $(I, +)$ is replaced by a ternary group having a left distributive multiplication over its ternary composition? We recall that the notion of ternary group is known from the papers of B a e r [1], C e r t a i n e [6] etc. C e r t a i n e's ternary group operation was a slight generalization of the ternary operation of a „Schar", used by B a e r in connexion with the Erlangen Program on groups, more precisely, in connexion with what can be called „the affine geometry on groups". We shall explain the left infra-distributivity of the multiplication of a l.i.-n.r. as a „reflection" of its left distributivity over a ternary group composition defined by means of the binary group composition.

We give first the basic concepts:

Definition 2.1. *Let I be a nonempty set and $[.,.,.]: I^3 \to I$ be a ternary composition on I. If there exists 0 in I, such that the following axioms are fulfilled:*

$$(2.1) \qquad [x, 0, [y, z, t]] = [[x, 0, y], z, t], \qquad \forall x, y, z, t \in I,$$

$$(2.2) \qquad [x, 0, 0] = x, \qquad \forall x \in I,$$

$$(2.3) \qquad [x, x, 0] = 0, \qquad \forall x \in I,$$

then we call I a ternary group of first type. If, in addition, the equality

$$(2.4) \qquad [x, y, z] = [z, y, x], \qquad \forall x, y, z \in I,$$

holds, then I is called semiabelian.

It is known that, by defining a binary operation:

$$(2.5) \qquad x + y := [x, 0, y], \qquad \forall x, y \in I,$$

we obtain a group $(I, +)$, with 0 as its neutral element (C e r t a i n e [6], Def. 3, Th. 4). When the group composition and the ternary group composition are related by the equality:

$$(2.6) \qquad [x, y, z] = x - y + z, \qquad \forall x, y, z \in I,$$

then the ternary group of first type will be called *regular*. In every ternary group of first type, the following identities hold:

$$(2.7) \qquad [0, 0, x] = x, \qquad \forall x \in I,$$

$$(2.8) \qquad [[0, x, 0], 0, x] = 0, \qquad \forall x \in I,$$

since 0 is the neutral element of the group $(I, +)$ and $[0, x, 0]$ is the additive inverse of $x \in I$ in $(I, +)$.

If in I, there are two elements 0 and 0_1 which satisfy Definition 2.1, then the two corresponding groups $(I, +, 0)$, $(I, +_1, 0_1)$ are isomorphic and the isomorphism between them is given by: $x \to [x, 0, 0_1]$, $\forall x \in I$ (C e r t a i n e [6]). Moreover, the same binary group is obtained from distinct ternary groups of first type by means of (2.5). In this sense, one can see C e r t a i n e's example for the cyclic group $I = \{0, a\}$. But we have also other examples. Let $I = \{0, a, b\}$ be the cyclic group of order 3, hence $b = 2a$. We have found three ternary groups of first type, namely:

1°. the regular one, i.e. with the ternary composition given by (2.6);

2°. with the following special products: $[a, a, a] = [0, b, b] = b$, $[b, b, b] = [0, a, a] = a$, $[b, a, a] = [a, b, b] = 0$, the other ternary products are regular;

3°. with the special products: $[a, a, b] = [b, b, a] = 0$, $[0, b, a] = [b, a, b] = a$, $[0, a, b] = [a, b, a] = b$, the other ternary products being regular.

Definition 2.2. *Let I be a nonempty set and $(.,.,.): I^3 \to I$ be a ternary operation on I. We call $(I, (.,.,.))$ a ternary group of second type, if there exists $0 \in I$, such that the following axioms hold:*

$$(2.9) \qquad ((x, y, 0), z, 0) = (x, (y, z, 0), 0), \qquad \forall x, y, z \in I,$$

$$(2.10) \qquad (0, x, 0) = x, \qquad \forall x \in I,$$

$$(2.11) \qquad ((0, 0, x), x, 0) = 0, \qquad \forall x \in I.$$

Proposition 2.1. *If $(I, (.,.,.))$ is a ternary group of second type, then $(I, +)$, where $+$ is the binary operation, given by:*

$$(2.12) \qquad x + y := (x, y, 0), \qquad \forall x, y \in I,$$

is a group.

Proof. The axiom (2.9) implies the associativity of $+$, the axiom (2.10) implies 0 is the left neutral element of $(I, +)$ and the axiom (2.11) implies that $(0, 0, x)$ is the left additive inverse of x. We have also the equalities:

$$(2.13) \qquad (x, 0, 0) = x, \qquad \forall x \in I,$$

$$(2.14) \qquad (x, (0, 0, x), 0) = 0, \qquad \forall x \in I.$$

Remark 2.1. $(I, +)$ is an abelian group if and only if:

$$(2.15) \qquad (x, y, 0) = (y, x, 0), \qquad \forall x, y \in I.$$

Remark 2.2. If $(I, +)$ is a group, then $(I, (\cdot, \cdot, \cdot))$, where the ternary composition on I is given by:

$$(x, y, z) := x + y - z, \qquad \forall x, y, z \in I, \tag{2.16}$$

is a ternary group of second type. We call it *regular*. But there are ternary groups of second type which are not regular. Consider, for example, the ternary group of second type on $I = \{0, a\}$, given by: $(0, 0, 0) = (a, 0, a) = (a, a, 0) = (a, a, a,) = 0$, $(0, a, 0) = (0, 0, a) = (a, 0, 0) = (0, a, a) = a$ where $(a, a, a) = 0 \neq a + a - a = a$. *We denote this example by* $(\mathcal{J})$. We have (2.16) if and only if the equality:

$$(x, y, z) = ((x, y, 0), (0, 0, z), 0) \tag{2.17}$$

is verified for all x, y, z in I.

Proposition 2.2. *The axioms* (2.9) — (2.11) (*from Definition* 2.2) *are independent.*

Proof. We have the following ternary operations on $I = \{0, a\}$ which satisfy only two of these three axioms:

$$(\mathcal{J}_1) \begin{cases} (0, 0, 0) = (a, 0, 0) = (0, a, a) = (a, a, 0) = 0, \\ (0, 0, a) = (a, 0, a) = (0, a, 0) = (a, a, a) = a \end{cases}$$

(satisfying only (2.10) and (2.11)),

$$(\mathcal{J}_2) \begin{cases} (0, 0, 0) = (0, 0, a) = (0, a, 0) = (0, a, a) = 0, \\ (a, 0, 0) = (a, 0, a) = (a, a, 0) = (a, a, a) = a \end{cases}$$

(satisfying only (2.9) and (2.11)),

$$(\mathcal{J}_3) \begin{cases} (0, 0, 0) = (a, 0, a) = (0, a, a) = 0. \\ (0, 0, a) = (a, 0, 0) = (0, a, 0) = (a, a, 0) = (a, a, a,) = a \end{cases}$$

(satisfying only (2.9) and (2.10)).

Remark 2.3. Like the ternary groups of first type, there are many ternary groups of second type associated with the same group by virtue of the identity (2.12) (see, for example, the regular ternary group of second type on $(I, +)$, with $I = \{0, a\}$, and the example $(\mathcal{J})$, which is not regular).

Now consider a binary multiplication on I, which is left distributive with respect to the ternary composition on I, — denote it by $[\cdot, \cdot, \cdot]$ — i.e. the equality:

$$x \cdot [y, z, t] = [x \cdot y, x \cdot z, x \cdot t], \tag{2.18}$$

holds for all x, y, z in I.

Definition 2.3. *A ternary group (of first or second type), together with an associative binary multiplication on it, which satisfies* (2.18), *will be called a left ternary near-ring (of first or second type).*

Remark 2.4. If $(I, +, \cdot)$ is a l.i.-n.r., then the regular ternary group of first type is a left ternary near ring of first type with respect to the same multiplication. Indeed, $x \cdot [y, z, t] = x \cdot (y - z + t) = x \cdot y + (- x \cdot 0 + x \cdot (- z) - x \cdot 0) + x \cdot t = x \cdot y - x z + x \cdot t = [x \cdot y, x \cdot z, x \cdot t]$, for all x, y, z, t in I (by using (1.5)). Obviously, a *regular* left ternary near ring of first type can be „canonically" organized as a l.i.-n.r. with the binary addition given by (2.5) and the same multiplication. This statement is true for *all* left ternary near rings of first type.

Proposition 2.3. *If $(I, [\cdot, \cdot, \cdot], \cdot)$ is a left ternary near ring of first type, then $(I, +, \cdot)$, with $+$ given by* (2.5), *is a l.i.-n.r.*

Proof. We know (C e r t a i n e [6], Th. 4) that $(I, +)$ is a group. So we must prove only the left infra-distributivity of the multiplication over the addition. We have, for all x, y, z in I: $x \cdot (y + z) = x \cdot [y, 0, z] = [x \cdot y, x \cdot 0, x \cdot z] = [[x \cdot y, 0, 0], x \cdot 0, x \cdot z] = [x \cdot y, 0, [0, x \cdot 0, x \cdot z]] = x \cdot y + [0, x \cdot 0, x \cdot z]$; but $x \cdot 0 + [0, x \cdot 0, x \cdot z] = [x \cdot 0, 0, [0, x \cdot 0, x \cdot z]] = [[x \cdot 0, 0, 0], x \cdot 0, x \cdot z] = [x \cdot 0, x \cdot 0, x \cdot z] = x \cdot [0, 0, z] = x \cdot z$, hence $[0, x \cdot 0, x \cdot z] = - x \cdot 0 + x \cdot z$, and $x \cdot (y + z) = x \cdot y - x \cdot 0 + x \cdot z$, the left infradistributivity.

Remark 2.5. If we take a left regular ternary near ring of second type, we obtain, for the addition given by (2.12), the equality:

$$x \cdot (y + z) = x \cdot y + x \cdot z - x \cdot 0, \quad \forall x, y, z \in I. \tag{2.19}$$

But this is the relation (1.8) and it implies that:

$$x \cdot z - x \cdot 0 = - x \cdot 0 + x \cdot z, \quad \forall x, z \in I, \tag{2.20}$$

hence the left infra-distributivity (nothing new!).

For a left ternary near ring of second type, we have (2.19) if and only if the following identity is fulfilled:

$$((x \cdot y, x \cdot z, x \cdot 0), x \cdot 0, 0) = (x \cdot y, x \cdot z, 0), \quad \forall x, y, z \in I. \tag{2.21}$$

But this equality implies that:

$$(x \cdot 0, x \cdot z, 0) = (x \cdot z, x \cdot 0, 0), \quad \forall x, z \in I,$$

hence $x \cdot 0 + x \cdot z = x \cdot z + x \cdot 0$, $\forall x, z \in I$, and the l.i.-n.r. corresponding to the left ternary near ring of second type with the property (2.21) is a usual l.i.-n.r. (see Remark 1.5).

We note that the concept of left ternary near ring generalizes the notion of „préanneau" (prering), given by P. J a n i n [11] and used by A. B a t b e d a t [2, 3], because, in their papers, the ternary group is supposed to be semiabelian or moreover idempotent, while the distributivity of the multiplication over the ternary composition is supposed to be two-sided.

§ 3. Congruences, ideals, homomorphisms of left infra-near rings. The theory of congruences of a l.i.-n.r. is connected with that of ideals (or homomorphisms), as usual.

Definition 3.1. *A two-sided ideal of a l.i.-n.r. I is a nonempty subset J of I, which satisfies the conditions: (i) $(J, +)$ is a normal subgroup of $(I, +)$; (ii) $x \cdot j - x \cdot 0 \in J$, $\forall x \in I$, $\forall j \in J$; (iii) $(j + x) \cdot y - x \cdot y \in J$, $\forall x, y \in I$, $\forall j \in J$. If J satisfies only the conditions (i) and (ii) (respectively, (i) and (iii)), J is called a left ideal (respectively, a right ideal) of I.*

It is obvious that I and $\{0\}$ are ideals of I.

Remark 3.1. Taking $x = 0$ in (iii), Definition 3.1, we obtain:

$$j \cdot y - 0 \cdot y \in J, \quad \forall j \in J, \quad \forall y \in I, \tag{3.1}$$

when J is a right ideal; this identity does not imply the condition (*iii*). This last implication holds, for example, in the case when $(I, +, .)$ is an i.-n.r. The condition (*iii*), Definition 3.1, is equivalent to the condition:

$$(x + j) \cdot y - x \cdot y \in J, \quad \forall j \in J, \quad \forall x, y \in I, \tag{3.2}$$

as it is easily proved.

Remark 3.2. If $(I, +, .)$ is a left near ring, then $x \cdot 0 = 0$ for all x in I, and (*ii*), Definition 3.1, becomes:

$$x \cdot j \in J, \quad \forall j \in J, \quad \forall x \in I. \tag{3.3}$$

In this case, Definition 3.1 is the definition of the ideal in a near ring (see [10], [5]).

Remark 3.3. In general, a left (right or two-sided) ideal J of a l.i.-n.r. I is not a subinfra-near ring of I, since the product of two elements of $J, j.j'$, belongs to J if and only if $j \cdot 0 \in J$ (respectively, $0 \cdot j \in J$, respectively $0 \cdot 0 \in J$). for all j in J.

The set of the left (right, two-sided) ideals of a l.i.-n.r. I – we denote it by $\mathcal{L}_l(I)$ (resp. $\mathcal{L}_r(I)$. $\mathcal{L}(I)$) – is a complete (modular) lattice with respect to the set-theoretical intersection and the addition of two ideals. There exists a bijection between $\mathcal{L}(I)$ and the set of the congruences of I.

If J is a two-sided ideal of I, then $I/J = \{x + J / x \in I\}$ is a l.i.-n.r. (usual operations between classes).

Define a *homomorphism* between two left infra-near rings, I, I', as a mapping φ from I to I' which preserves the two operations, i.e. $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$, $\forall x, y \in I$. The class of the left infra-near rings and the class of their homomorphisms form a category, $\mathcal{J}_l$. The *kernel of* φ, $\operatorname{Ker} \varphi = \{x / x \in I, \varphi(x) = 0' \in I'\}$, is a two-sided ideal of I, and the *image of* φ, $\operatorname{Im}\varphi = \{\varphi(x) / x \in I\}$, is a subinfra-near ring of I'. Thus, the two sided ideals of a l.i.-n.r. I are kernels of the homomorphisms from I to some l.i.-n.r. The isomophism theorems and many other theorems from the theory of rings are valid for left infra-near rings.

Let us define now a *homomorphism* between two left regular ternary near rings, $(I, [.,.,.], 0, .)$, $(I', [.,.,.], 0', .)$ as a mapping $\varphi : I \to I'$, with the properties: $\varphi([x, y, z]) = [\varphi(x), \varphi(y), \varphi(z)]$, $\varphi(0) = 0'$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$, $\forall x, y, z \in I$. The class of the regular left ternary near rings, together with their homomorphisms, forms a category, let us denote it by $\mathcal{J}_t$. By using results of Section 2 (Proposition 2.3, Remarks 2.4, 2.5), we prove:

Proposition 3.1. *There exists an equivalence between the categories* $\mathcal{J}_t$ *and* $\mathcal{J}_l$.

Now, we generalize the notion of module over a ring (or a near ring), giving only the basic concepts:

Definition 3.2. *Let* $(M, +)$ *be a group and* $(I, +, .)$ *be a l.i.-n.r. If there exists a multiplication* $\circ : M \times I \to M$, *such that the following conditions are fulfilled:* $m \circ (x + y) = m \circ x - m \circ 0 + m \circ y$, $m \circ (x \cdot y) = (m \circ x) \circ y$, $\forall m \in M$, $\forall x, y \in I$, *then we call* M *a right* I*-group. If there exists a multiplication* $\circ : I \times M \to M$, *such that the conditions:* $x \circ (m_1 + m_2) = x \circ m_1 - x \circ 0 + x \circ m_2$, $(x \cdot y) \circ m_1 = x \circ (y \circ m_1)$, *hold for all* $x, y \in I$ *and* $m_1, m_2 \in M$, *then we call* M *a left* I*-group.*

We note that, because of the behaviour of the multiplication over the addition of M or I, the I-groups cannot be considered groups with operators.

We can also define a *left ternary* I*-group*, when I is a l.i.-n.r., $(M, [.,.,.])$ is a ternary group of first or second type, and the multiplication $\circ : I \times M \to M$ is left distributive over the ternary composition in M, i.e. $x \circ [m_1, m_2, m_3] = [x \circ m_1, x \circ m_2, x \circ m_3]$, $\forall m_1, m_2, m_3 \in M$, $\forall x \in I$, and $(x \cdot y) \circ m_1 = x \circ (y \circ m_1)$, $\forall x, y \in I$, $\forall m_1 \in M$. Then we obtain (for the first type):

Proposition 3.2. *Let* I *be a l.i.-n.r. If* M *is a left ternary* I*-group of first type, then* $(M, +)$, *with* $+$ *defined by* (2.5) *is a left* I*-group with respect to the given multiplication. Conversely, if* $(M, +)$ *is a left* I*-group then the regular ternary group of first type, together with the same multiplication, is a left ternary* I*-group of first type.*

Proof. To prove the first statement, we have only to show that $x \circ (m_1 + m_2) = x \circ m_1 - x \circ 0 + x \circ m_2$, $\forall x \in I$, $\forall m_1, m_2 \in M$. But $x \circ (m_1 + m_2) = x \circ [m_1, 0, m_2] = [x \circ m_1, x \circ 0, x \circ m_2] = [[x \circ m_1, 0, 0], x \circ 0, x \circ m_2] = x \circ m_1 + [0, x \circ 0, x \circ m_2]$. As $x \circ 0 + [0, x \circ 0, x \circ m_2] = x \circ m_2$, we have $x \circ (m_1 + m_2) = x \circ m_1 - x \circ 0 + x \circ m_2$. The second statement can be proved easily.

Define a homomorphism between two left I-groups (left ternary I-groups) as a mapping preserving all the operations (and the distinguished element of the ternary group). The class of the left I-groups (left ternary I-groups) and the class of their homomorphisms form a category, $\mathcal{G}_I$ (respectively, $\mathcal{T}_I$). By using Proposition 3.2, we can prove:

Proposition 3.3. *The categories* $\mathcal{T}_I$ *and* $\mathcal{G}_I$ *are equivalent.*

The kernel of a homomorphism of right (left) I-groups, $\varphi : M \to M'$, is a normal subgroup of M, satisfying the condition:

$$(a + m) \circ x - m \circ x \in \operatorname{Ker} \varphi, \quad \forall a \in \operatorname{Ker} \varphi, \quad \forall m \in M, \quad \forall x \in I \tag{3.4}$$

respectively:

$$x \circ a - x \circ 0 \in \operatorname{Ker} \varphi, \quad \forall a \in \operatorname{Ker} \varphi, \quad \forall x \in I. \tag{3.5}$$

Hence, the kernels of the homomorphisms of right (left) I-groups, $\varphi : I \to M$, are exactly the right (left) ideals of I.

We call *the right annihilator* of a right I-group M (of an element $m \in M$) in I, the subset of I given by:

$$\operatorname{Ann}_r(M) = \{x / x \in I, \ m \circ x = m \circ 0, \quad \forall m \in M\} \tag{3.6}$$

(respectively:

(3.7) $\mathrm{Ann}_r(m) = \{x/x \in I,\ m \circ x = m \circ 0\}$)

and it is a two-sided (respectively, right) ideal of I.

§ 4. A decomposition of a left infra-near ring. The definitions an the constructions of this section generalize those obtained by G. B e r m a and R. J. S i l v e r m a n [4] concerning the decomposition of a left nea ring as a sum of its maximal Z-ring and C-ring:

$$N = Z(N) + C(N), \text{ where } Z(N) = \{n/n \in N,\ 0\cdot n = n\},\ C(N) = \{n/n \in N,\ 0\cdot n = 0\}.$$

Let $(I, +, .)$ be a l.i.-n.r. that satisfies the condition:

$$0\cdot 0 = 0, \tag{4.1}$$

i.e. 0 is a left distributive element.

Consider the subsets of I:

$$C(I) = \{x/x \in I,\ 0\cdot x = 0\},\ Z(I) = \{x/x \in I,\ 0\cdot x = x\}. \tag{4.2}$$

We have:

Proposition 4.1. *If I is a l.i.-n.r. satisfying (4.1), then $C(I)$ is subinfra-near ring and a right ideal of I, $Z(I)$ is a right I-group, $I = C(I) + Z(I)$ and $C(I) \cap Z(I) = \{0\}$. Moreover, 0 defines an endomo phism φ_0 of I considered as a right I-group.*

Proof. Define $\varphi_0 : I \to I$, by $\varphi_0(x) = 0 . x$, $\forall x \in I$. Because of (4.1 $\varphi_0(x + y) = \varphi_0(x) + \varphi_0(y)$ and $\varphi_0(x\cdot y) = 0\cdot(x\cdot y) = (0\cdot x)\cdot y = \varphi_0(x)\cdot$ $\forall x, y \in I$, i.e. φ_0 is an endomorphism of the right I-group I.

But $\mathrm{Ker}\,\varphi_0 = C(I)$, hence $C(I)$ is a right ideal of I, which is clos under multiplication. Because of the equality $\mathrm{Im}\,\varphi_0 = Z(I)$, $Z(I)$ is right I-group. We have also $x = (x - 0\cdot x) + 0 . x \in C(I) + Z(I)$, $\forall x \in$ and this decomposition is unique ($C(I) \cap Z(I) = \{0\}$).

Definition 4.1. *If the l.i.-n.r. I is equal to $C(I)$ (respectively, $Z(I$ we call it a left C-infra-near ring (respectively a left Z-infra-near rin In this case, φ_0 is trivial. When I is a left near ring, we obtain the sults of* [4].

The l.i.-n.r. $(I, +, .)$ in Example 1°, as well as $A(V)$ in Examp 5°, are left C-i.n.r.s., while that in Example 2° is a left Z-i.n.r.

Remark 4.1. In a left C-i.n.r., a right ideal J is a normal rig sub-I-group, since, by taking $x = 0$, $j \in J$, $y \in I$, in the relati $(j + x)\cdot y - x\cdot y \in J$, which characterizes the right ideal, we have $j\cdot y \in$ $\forall j \in J$, $\forall y \in I$.

Remark 4.2. If I is a left C-i.n.r. with weakly left distributive e ments (i.e. $0\cdot x = 0$, $x\cdot 0 = x$, $\forall x \in I$), then its multiplication is given b

$$x\cdot y = x, \qquad \forall x, y \in I, \tag{4.4}$$

since $x\cdot y = (x\cdot 0)\cdot y = x\cdot(0\cdot y) = x\cdot 0 = x$. Then $(I, +, .)$ is a right Z-ri Moreover, its two-sided ideals are just the normal subgroups of $(I, +$ It is obvious that every group $(I, +)$ can be organized as a left C-i.n with weakly left distributive elements, by using the multiplication (4.

The set of the constant functions on a group $(I, +)$, $T_c(I)$, is such a left C-i.n.r.

Remark 4.3. When we give up the condition (4.1), by defining $C(I) = \{x/x \in I,\ 0\cdot x - 0\cdot 0 = 0\}$, $Z(I) = \{x/x \in I,\ 0\cdot x - 0\cdot 0 = x\}$, we have: $C(I)$ and $Z(I)$ are subgroups of $(I, +)$, $C(I) \cap Z(I) = \{0\}$, $0^n\cdot x - 0^n\cdot 0 = x$, $n \geqslant 1$, $\forall x \in Z(I)$ and $0^n\cdot y = 0^n\cdot 0$, $\forall y \in C(I)$. If $(I, +)$ is a group, a is a fixed element in I, and we define the multiplication on I by $x\cdot y = x + a + y$, $\forall x, y \in I$, then $(I, +, .)$ is an i.-n.r., $Z(I)$ is the normalizer of a in $(I, +)$, while $C(I)$ is $\{0\}$.

§ 5. Idempotents in left infra-near rings. We consider some types of idempotents in a l.i.-n.r. I, with the condition (4.1), namely: a) *left distributive idempotents* ($e\cdot e = e$, $e\cdot 0 = 0$); b) *right distributive idempotents* ($e\cdot e = e$, $(x + y)\cdot e = x\cdot e + y\cdot e$, $\forall x, y \in I$); c) *central distributive idempotents* ($e\cdot e = e$, $e\cdot 0 = 0$, $e\cdot x = x\cdot e$, $\forall x \in I$).

In Example 1°, the left distributive idempotents are those of the form $e = (e_1, e_2, 0, 0, 0)$, with e_1, e_2 idempotents in R. 0 is a left distributive idempotent, while the elements of the form $s = (0, 0, s_3, s_4, s_5)$ are nondistributive idempotents in I, with the property $s\cdot x = s$, for all x in I. In Example 5°, every constant function is a right distributive idempotent, but it is not left distributive. In Example 2°, the elements of the form $y = (0, y_2)$ are left distributive idempotents, and $0 = (0, 0)$ is a left and right distributive idempotent, but it is not central.

The following three propositions give the decompositions associated to every type of idempotents:

Proposition 5.1. *Let e be a left distributive idempotent of I. Then the function $\varphi_e : I \to I$, $\varphi_e(x) = e\cdot x$, $\forall x \in I$, is an idempotent endomorphism of the right I-group, I, $\mathrm{Im}\,\varphi_e$ is a right I-group, on which φ_e is the identity function and e is a left identity, $\mathrm{Ker}\,\varphi_e$ is a right ideal of I and I is a semidirect sum (as an additive group) of $\mathrm{Ker}\,\varphi_e$ and $\mathrm{Im}\,\varphi_e$.*

Proof. Because it is easy to verify directly all the statements of Proposition 5.1, we give only the decomposition of an element $x \in I$. This is: $x = (x - e\cdot x) + e\cdot x$, where $x - e\cdot x \in \mathrm{Ker}\,\varphi_e$ and $e\cdot x \in \mathrm{Im}\,\varphi_e$.

By taking $e = 0$ in Proposition 5.1, we reobtain Proposition 4.1.

Proposition 5.2. *If e is a right distributive idempotent of I, then the function $\psi_e : I \to I$, $\psi_e(x) = x\cdot e$, $\forall x \in I$, is an idempotent endomorphism of the left I-group, I, $\mathrm{Im}\,\psi_e$ is a left I-group with e as a right identity, $\mathrm{Ker}\,\psi_e$ is a left ideal of I and I is a semidirect sum (as an additive group) of $\mathrm{Im}\,\psi_e$ and $\mathrm{Ker}\,\psi_e$.*

Proof. As above, we give only the decomposition of an element $x \in I$, namely: $x = (x - x\cdot e) + x\cdot e$, where $x - x\cdot e \in \mathrm{Ker}\,\psi_e$, $x\cdot e \in \mathrm{Im}\,\psi_e$.

When 0 is a left and a right distributive idempotent, but it is not a central one, we have: $I = C(I) + Z(I)$ and $I = D + W$, where $D = \mathrm{Ker}\,\psi_0$ is the set of left distributive elements and $W = \mathrm{Im}\,\psi_0$ is the set of left weakly distributive elements of I. Here D is a left near-ring, $D \cap C(I)$ is a left C-ring, $D \cap Z(I)$ is a left Z-ring.

Proposition 5.3. *If e is a central distributive idempotent of I, then, using the notations of Proposition 5.1, we have: φ_e is an idempotent endomorphism of the l.i.-n.r. I; $\mathrm{Im}\,\varphi_e$ is a right and left I-group, with e*

as its identity; $\mathrm{Ker}\,\varphi_e$ *is a two-sided ideal of* I, *which is a left C-ring*; $\mathrm{Im}\,\varphi_e \cdot \mathrm{Ker}\,\varphi_e = \mathrm{Im}\,\varphi_e \cdot \{0\}$, $\mathrm{Ker}\,\varphi_e \cdot \mathrm{Im}\,\varphi_e = \{0\}.\mathrm{Im}\,\varphi_e$, $I = \mathrm{Ker}\,\varphi_e + \mathrm{Im}\,\varphi_e$ (*a semidirect sum of groups*).

We have also the identities: $b \cdot x = b \cdot x_2$, $x \cdot b = x_2 \cdot b$, $a \cdot x = a \cdot x_1 + 0 \cdot x_2$, for all $x = x_1 + x_2 \in I$, a, $x_1 \in \mathrm{Ker}\,\varphi_e$, b, $x_2 \in \mathrm{Im}\,\varphi_e$.

REFERENCES

1. Baer R. — *Zur Einführung des Scharbegriffs*, J. reine angew. Math., 160 (1929), 199—207.
2. Batbedat A. — *Une généralisation de la notion de module*, C. R. Acad. Sci. Paris, 276 (1973), A 95—96.
3. Batbedat A. — *Préanneaux idempotents*, Atti Accad. Nazionale Lincei, 60 (1973), 325—330.
4. Berman G., Silverman R. J. — *Near rings*, Amer. Math. Monthly, 66 (1959), 23—34.
5. Blackett D. W. — *Simple and semi-simple near rings*, Proc. Amer. Math. Soc., 4 (1953), 772—785.
6. Certaine J. — *The ternary operation* $(abc) = ab^{-1}c$ *of a group*, Bull. Amer. Math. Soc. 49 (1943), 869—877.
7. Climescu Al. — *Anneaux faibles*, Bul. Inst. Politehn. Iaşi, 7 (9) (1961), 1—6.
8. Climescu Al. — *A new class of weak rings* (Romanian), idem, 10 (14), (1964), 1—4.
9. Cupona G. — *On quasirings* (Macedonian), Bull. Soc. Math. Phys. Macédoine, 20 (1969), 19—22 (1971), MR 44 # 1 703.
10. Fröhlich A. — *Distributively generated near rings. I: Ideal theory*, Proc. London Math. Soc., 8 (3) (1959), 74—94.
11. Furtwängler Ph., Taussky Olga — *Über Schiefringe*, S. B. Akad. Wiss. Wien, 145 (1936), 525.
12. Janin P. — *Une généralisation de la notion d'anneau: Préanneaux*, C. R. Acad. Sci. Paris, 269 (1969), A 62—64.
13. Ştefănescu Mirela — *Multiplications infra-distributives sur un groupe* (to appear).
14. Ştefănescu Mirela — *Infra-near rings of affine type*. An. Şt. Univ. Iaşi 24(1978), 5—14.

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SUR LES ESPACES BITOPOLOGIQUES s-CONNEXES

PAR

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Dans un espace topologique X, une partie A de X est dite *semiouverte* [3] s'il existe un ensemble O ouvert dans X tel que $O \subset A \subset \mathrm{cl}\,O$, où $\mathrm{cl}\,O$ dénote l'adhérence de O dans X. Un ensemble est dit *semifermé* si son complémentaire est semiouvert. L'intersection de tous les ensembles semifermés contenant A est dite *semiadhérence* de A et est *notée* $\mathrm{scl}\,A$; elle est toujours semifermé [1]. L'ensemble A est semifermé si et seulement si $A = \mathrm{scl}\,A$ [1]. Afin que $p \in \mathrm{scl}\,A$, il faut et il suffit que tout ensemble semiouvert contenant p rencontre A [1]. De plus, $A \subset \subset \mathrm{scl}\,A \subset \mathrm{cl}\,A$, et $A \subset B$ implique $\mathrm{scl}\,A \subset \mathrm{scl}\,B$. Un espace bitopologique (X, T_1, T_2) est un ensemble X muni de deux topologies T_1 et T_2.

1. Ensembles $<i, j>$-semiséparés. Définition 1. *Deux ensembles* A *et* B *dans un espace bitopologique* (X_1, T_1, T_2) *sont* $<i,j>$*-semiséparés si* $A \cap T_i - \mathrm{scl}\,B = \emptyset = T_j - \mathrm{scl}\,A \cap B$, $i, j \in \{1, 2\}$.

Remarque 1. Deux ensembles $<i, j>$-semiséparés sont nécessairement disjoints. La réciproque n'est pas, en général, vraie.

Remarque 2. Soient les ensembles A, B $<i, j>$-semiséparés, $C \subset A$ et $D \subset B$. Alors, C et D sont $<i, j>$ semiséparés.

Remarque 3. Soient A un ensemble T_i-semiouvert et B un ensemble T_j-semiouvert. Alors, A et B sont $<i, j>$-semiséparés si et seulement si $A \cap B = \emptyset$.

Proposition 1. *Si* A, B *sont* $<i, j>$*-semiséparés et* $A \cup B$ *est biouvert, alors* A *est* T_i*-semiouvert et* B *est* T_j*-semiouvert.*

Démonstration. On utilise les relations $(A \cup B) \cap (X \sim T_i - \mathrm{scl}\,B) = A$. et $(A \cup B) \cap (X - T_j - \mathrm{scl}\,A) = B$.

Proposition 2. *Soient* (Y, T_1^*, T_2^*) *un sous-espace biouvert d'un espace bitopologique* (X, T_1, T_2), $A \subset Y$ et $B \subset Y$. *Alors,* A *et* B *sont* $<i, j>$*-semiséparés dans* X *si et seulement si ils sont* $<i, j>$*-semiséparés dans* Y.

La démonstration est immédiate.

2. Espaces bitopologiques s-connexes. Definition 2. *On dit que l'espace bitopologique* (X, T_1, T_2) *est s-connexe, s'il ne peut pas s'exprimer comme réunion de deux ensembles non vides* $<i, j>$*-semiséparés.*

W. J. Pervin [6] a introduit la notion d'espace bitopologique connexe. On a,

Proposition 3. *Un espace bitopologique s-connexe est connexe.*