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ALMOST PRODUCT STRUCTURES AND SPECIAL ALGEBRAS

BY

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§ **0. Introduction.** The differentiable manifolds with an almost product structure, and the existence of linear connections related to such a structure, were investigated by many authors, as for example [1]—[4].

The purpose of the present paper is to generalize a result of Cruceanu [4] and to obtain a complete caracterisation of the pairs of connections compatible with an almost product structure.

Using the methods of [4], [5], [6] and also the results from [7] we define in §1. the notions of special and respectively *-special algebras of the vector fields (definition 3), special linear connections (definition 4) and special pairs of linear connections (definition 5) and investigate the basic properties of these objects.

§ **1. Definitions and statement of the main theorem.** All geometrical objects in this paper are assumed to be of class C^∞.

Let M be a m-dimensional differentiable manifold. We denote by $\mathcal{F}(M)$ the ring of all real functions on M and by $\mathcal{X}(M)$ the set of all vector fields.

Definition 1. *An almost product manifold of type* $(m_1, \ldots, m_r)$ *is a manifold* M *with a system of distributions* T_α, $\alpha = 1, \ldots, r$, *i.e.* r *differentiable plane fields, such that* [1]:

(i). *for* $\alpha \neq \beta$ *the planes* T_α, T_β *at any point are disjoint;*

(ii). $\sum_\alpha T_\alpha = T$ *where* T *is the distribution of tangent planes on* M;

(iii). *the distribution* T_α *is* m_α*-dimensional.*

From now on let the indices α, β run over the range $1, \ldots, r$.

Let us denote by π_α the projection tensor field corresponding to the distribution T_α, and for every α let us define the tensor fields π^*_α and J_α by the relations:

$$\pi^*_\alpha = I - \pi_\alpha = \textstyle\sum_{\beta \neq \alpha} \pi_\beta, \tag{1.1}$$

$$J_\alpha = \pi_\alpha - \pi^*_\alpha \tag{1.2}$$

where I is the Kronecker tensor field.

From (1.1) and (1.2) we obtain:

$$\pi_\alpha \circ \pi^*_\alpha = \pi^*_\alpha \circ \pi_\alpha = 0, \tag{1.3}$$

$$\pi_\beta \circ \pi^*_\alpha = \pi^*_\alpha \circ \pi_\beta = \pi_\beta, \quad (\beta \neq \alpha) \tag{1.4}$$

$$J^2_\alpha = I. \tag{1.5}$$

Let S be a tensor field of type $(1, s)$. Then we denote by $\pi_\alpha S$ the tensor field of type $(1, s)$ such that

$$\pi_\alpha S(X_1, \ldots, X_s) = \pi_\alpha[S(X_1, \ldots, X_s)]. \tag{1.6}$$

We can easily verify that

$$S = \sum_\alpha \pi_\alpha S. \tag{1.7}$$

Let M be an almost product manifold and let A be a tensor field of type $(1, 2)$. If we define the product of two vector fields X and Y as the vector field

$$X * Y = A(X, Y), \tag{1.8}$$

then the module $\mathscr{X}(M)$ becomes an algebra over $\mathscr{F}(M)$, denoted by $\mathscr{U}(M, A)$ (see [7] or [8]).

Definition 2. *Let us consider an algebra $\mathscr{U}(M, A)$. Then we call special and *-special parts of A those defined by the equations:*

$$\Omega A(X, Y) = \sum_\alpha \pi_\alpha A(X, \pi_\alpha Y), \tag{1.9}$$

and respectively

$$^*\Omega A(X, Y) = \sum_\alpha \pi_\alpha A(X, \pi_\alpha^* Y), \tag{1.10}$$

so that $A = \Omega A + {}^*\Omega A$

We shall now state two definitions

Definition 3. *The algebra $\mathscr{U}(M, A)$ is called a special algebra, respectively a *-special algebra, if ${}^*\Omega A = 0$, respectively $\Omega A = 0$.*

Definition 4. *A linear connection ∇ on the almost product manifold M is called a special connection if each projection π_α is a ∇-parallel tensor field, i.e. the equations:*

$$\nabla_X \pi_\alpha = \nabla_X \circ \pi_\alpha - \pi_\alpha \circ \nabla_X = 0 \tag{1.11}$$

are satisfied.

Let $\Gamma(M)$ and $\Gamma_0(M)$ be, respectively, the set of all linear connections and the set of all special linear connections on M.

Using the method of [4], we define a mapping:

$$\Phi : \Gamma(M) \to \Gamma_0(M), \quad \Phi(\nabla)_X = \sum_\alpha \pi_\alpha \circ \nabla_X \circ \pi_\alpha. \tag{1.12}$$

We will prove in section 2 that Φ is well defined (proposition 1) and we have also:

Proposition 2. *Let ∇^1 and ∇^2 be two linear connections. If the connection ∇^1 is special, then the connection ∇^2 is special if and only if the algebra $\mathscr{U}(M)\,\nabla^2 - \nabla^1)$ is special.*

Theorem 1. *Let ∇^0 be a fixed linear connection on M. Then the set $\Gamma_0(M)$ coincides with the set of all connections ∇ of the form.*

$$\nabla = \Phi(\nabla^0) + \Omega(A), \tag{1.13}$$

where A is a tensor field of the type $(1, 2)$.

Definiton 4. *A pair of two linear connections (∇^1, ∇^2) is called a special pair, if we have*

$$\nabla^2 = 2\Phi(\nabla^1) - \nabla^1. \tag{1.14}$$

We remark that a linear connection ∇ is special, if and only if the pair (∇, ∇) is special.

For the description of the special pairs we obtain:

Proposition 3. *The pair (∇^1, ∇^2) is special if and only if the linear connection $(\nabla^1 + \nabla^2)/2$ is special and the algebra $\mathscr{U}(M, \nabla^2 - \nabla^1)$ is *-special.*

Theorem 1 and proposition 3 give a complete description of the special pairs as follows:

Theorem 2. *Let ∇^0 be a fixed linear connection on M. Then the set of all special pairs of connections coincides with the set of all pairs (∇^1, ∇^2) of the form:*

$$\begin{aligned} \nabla^1 &= \Phi(\nabla^0) + \Omega(A') - {}^*\Omega(A''), \\ \nabla^2 &= \Phi(\nabla^0) + \Omega(A') + {}^*\Omega(A''), \end{aligned} \tag{1.15}$$

where A' and A'' are two tensor fields of the type (1.2).

§ 2. Proofs. 2.1. We shall give first a description of the special and *-special algebras by:

Lemma 1. *The algebra $\mathscr{U}(M, A)$ is a special algebra, respectively a *-special algebra if and only if $A = \Omega(A)$, respectively $A = {}^*\Omega(A)$.*

Lemma 2. *All algebras of the form $\mathscr{U}(M, \Omega(A))$, respectively $\mathscr{U}(M, {}^*\Omega(A))$ are special, respectively *-special algebras.*

Proof. It may be easily proved that we have ${}^*\Omega(\Omega(A)) = 0$ and $\Omega({}^*\Omega(A)) = 0$.

We are now able to prove the following result:

Proposition 1. The mapping defined by (1.12) is well defined is a surjection and

$$\Phi(\nabla + A) = \Phi(\nabla) + \Omega(A). \tag{2.1}$$

Proof. Clearly $\Phi(\nabla)$ is a linear connection. It is a special connection, because for every ∇ we have $(\nabla)_X \circ \pi_\alpha = \pi_\alpha \circ \Phi(\nabla)_X$.

Let ∇ be a special connection. Then, from (1.11) we obtain the relations: $\Phi(\nabla)_X = \sum_\alpha \pi_\alpha \circ \nabla_X \circ \pi_\alpha = \sum_\alpha \pi_\alpha \circ \pi_\alpha \circ \nabla_X = (\sum_\alpha \pi_\alpha) \circ \nabla_X = \nabla_X$; hence Φ is a surjection.

The formula (2.1) can be obtained from the definitions (1.12) and (1.9).

From the above arguments we get

Lemma 3. *A linear connection ∇ is special if and only if $\nabla = \Phi(\nabla)$.*

2.2. Proof of the proposition 2. Let ∇^1 be a special connection. If ∇^2 is a connection, then using (2.1) and lemma 3 then we obtain:

$$\Phi(\nabla^2) = \nabla^2 - {}^*\Omega(\nabla^2 - \nabla^1). \tag{2.2}$$

From lemma 3 and (2.2), we conclude that the conditions

(i) — ∇^2 is special

(ii) — ${}^*\Omega(\nabla^2 - \nabla^1) = 0$, i.e. $\mathscr{U}(M, \nabla^2 - \nabla^1)$ is special, are equivalent.

2.3. Proof of the theorem 2. Theorem 1 is a simple consequence of the proposition 2 and the lemma 3.

Let ∇^0 be a fixed connection and let ∇ be a special linear connection. If we put $\nabla = \Phi(\nabla^0) + A$ then $\mathcal{U}(M, A)$ must be special, hence $A = \Omega A$. Conversely, if $\nabla = \Phi(\nabla^0) + \Omega(A)$ then $\Phi(\nabla) = \nabla$ and ∇ is special.

2.4. Proof of the proposition 3. We use the following.

Lemma 4. *The pair (∇^1, ∇^2) is special if and only if the pair (∇^2, ∇^1) is also special.*

Proof. If we supposse that the pair (∇^1, ∇^2) is special, then we have $\nabla^2 = 2\Phi(\nabla^1) - \nabla^1$. It follows now from $\Phi(\Phi(\nabla^1)) = \Phi(\nabla^1)$ that

$$\nabla^1 = 2\Phi(\nabla^2) - \nabla^2. \tag{2.3}$$

Let (∇^1, ∇^2) be a special pair of connections, let $\nabla = (\nabla^1 + \nabla^2)/2$ be the median connection and let $A = \nabla^2 - \nabla^1$ be the deformation tensor field. From (1.14) and (2.3) we find

$$\nabla = \Phi(\nabla), \tag{2.4}$$

and subtracting (2.3) from (1.14) we obtain $\Phi(\nabla^2) = \Phi(\nabla^1)$ hence we have

$$\Omega(A) = 0, \tag{2.5}$$

Conversely, (2.4) and (2.5) implies (1.14) and the proof is completed.

§ 3. Some remarks. If the manifold M is an almost product manifold of type (m_1, m_2), then the almost product structure of M is completely determined by the tensor field $J = \pi_1 - \pi_2$.

In this case, a linear connection ∇ is special if and only if $\nabla_X J = 0$ and a pair of two connections (∇^1, ∇^2) is special if and only if $\nabla_X^{1,2} J = \nabla_X^1 \circ J - J \circ \nabla_X^2 = 0$, for every vector field X.

More generally, in [4] C r u c e a n u studied the case in which J is an arbitrary tensor field of type (1.1) which has the property that rank $J = \dim M$. For an almost product manifold of type (m_1, m_2) his general results ([4], the formulae (30), (34), (44), (46)) coincide with the relations (1.14), (2.1), (1.13), (1.15).

REFERENCES

1. W a l k e r A. G. — *Almost-product structures*, Proc. Symp. Pure Math. III (1961), 94—10
2. W i l l m o r e T. J. — *Connections for systems of parallel distributions*, Quart. J. Math (2) 7 (1956), 191—204.
3. K o b a y a s h i A. E. T. — *A remark on the existence of a G-structure*, Proc. Amer. Math Soc. 16 (1965), 1 329—1 331.
4. C r u c e a n u V. — *Connexions compatibles avec certaines structures sur un fibré vectori banachique*, Czech. Math. J., 24 (99) (1974), 126—142.
5. O b a t a M. — *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Jap. J. Math. 26 (1956), 43—47.
6. C r u c e a n u V. et M i r o n R. — *Sur les couples de connexions compatibles avec les structures presque-complexes.* An. șt. Univ. Iași, 13 (1967) 1, 79—88.
7. N i c o l e s c u L. et M a r t i n M. — *Sur l'algèbre associée à un champ tensoriel du ty (1, 2)*, Acta Math. Sc. Hungaricae, 31 (1—2) (1978), 27—35.
8. V a i s m a n I. — *Sur quelques formules du calcul de Ricci global*, Com. Math. Helv., (2) (1966—67), 73—87.

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EMBEDDINGS OF C^∞-SUBCARTESIAN SPACES

BY

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§ 1. Preliminaries. The C^∞-subcartesian spaces were introduced by N. A r o n s z a j n [1] and their theory was developed by Ch. M a r s h a l l [6].

In this paper, we show that $J^k(S, S')$ is a C^∞-subcartesian space and study the Whitney C^k-topology on $C^\infty(S, S')$. Then we characterize the C^∞-subcartesian spaces which can be embedded into euclidean spaces and prove that $\mathrm{Im}(S, S')$ and $\mathrm{Emb}\,(S, S')$ are open subsets of $C^\infty(S, S')$.

We consider only locally compact and second countable C^∞-subcartesian spaces.

We review some basic definitions [6].

Definition 1.1. *A C^∞-subcartesian space is a Hausdorff space S together with a maximal C^∞-atlas $\mathcal{A}_S = \{\varphi : U_\varphi \to R^{n_\varphi}\}$ such that: (A_1). $\{U_\varphi\}$ is an open cover of S and each φ is a homeomorphism. (A_2).* If $\varphi, \psi \in \mathcal{A}_S$ and $p \in U_\varphi \cap U_\psi$ *there exists a C^∞-mapping s extending $\psi\varphi^{-1}$: $\varphi(U_\varphi \cap U_\psi) \to \psi(U_\varphi \cap U_\psi)$ in a neighborhood of $\varphi(p)$.*

Remark 1.2. Each topological subspace of S is a C^∞-subcartesian space.

Definition 1.3. *$f : S \to S'$ is a C^∞-mapping if for every $p \in S$, and every $\varphi \in \mathcal{A}_S$ and $\psi \in \mathcal{A}_{S'}$ with $p \in U_\varphi$ and $f(U_\varphi) \subset U_\psi$, there exists a C^∞-extension $\tilde{f}$ of $\psi f \varphi^{-1}$ in a neighborhood of $\varphi(p)$. The set of C^∞-mappings is denoted by $C^\infty(S, S')$.*

It is well-known that such a C^∞-subcartesian space admits C^∞-partition of unity [6].

The structural dimension of S at p is the number $n_{S,p} = \min\{n_\varphi / \varphi \in \mathcal{A}_S$ and $p \in U_\varphi\}$. $\varphi \in \mathcal{A}_S$ is called tangential at p if $p \in U_\varphi$ and $n_\varphi = n_{S,p}$.

In the set of triples (φ, p, v), where $p \in S$, $\varphi \in \mathcal{A}_S$ is tangential at p and $v \in T_{\varphi(p)} R^{n_\varphi}$, we define the equivalence relation $(\varphi, p, v) \sim (\psi, q, w) \Leftrightarrow p = q$ and $w = s_*(v)$, where s is as in (A_2), Definition 1.1. The tangent space of S at p, T_pS, is the set of all equivalence classes of $\sim$ with footpoint p. For each $\varphi \in \mathcal{A}_S$ tangential at p we have the bijection $\varphi_{*p} : T_pS \to T_{\varphi(p)} R^{n_\varphi}$, $\varphi_{*p}([\varphi, p, v]) = v$ (where $[\varphi, p, v]$ is the equivalence class defined by (φ, p, v)).

If $f \in C^\infty(S, S')$ and $p \in S$, we define $f_{*p} : T_pS \to T_{f(p)}S'$ by $f_{*p} = \psi_{*f(p)}^{-1} \tilde{f}_{*p} \varphi_{*p}$.