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GLOBAL SLICES IN G-SPACES

BY

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1. Let X be a Hausdorff space and G a compact Lie group. Suppose that G operates continuously on X, If H is a closed subgroup of G we denote:

$$X^H = \{x \in X/hx = x \text{ for all } h \in H\},$$

(the subspace of fixed points of H), X^H is a closed subspace of X[2].

For $x \in X$ let G_x denote the isotropy subgroup of x and $G(x)$ the orbit of x. Let A be a G_x-space and $G \times_{G_x} A$ the twist product of G and A. There is a G-tube about $G(x)$, more precisely, there is a map

$$\varphi : G \times_{G_x} A \to X,$$

such that $\varphi(G \times_{G_x} A)$ is an open neighbourhood of $G(x)$ and φ is an equivariant embeding. The image of $[e, A] \in G \times_{G_x} A$ is called the slice at x and we denote it by $\varphi[e, A] = S$.

Theorem 1. [2] *Let X be a G-space, let $x \in S \subset X$ and put $G_x = H$. The following statements are equivalent:*

1. There exists a G-tube about $G(x)$: $\varphi : G \times_{G_x} A \to X$;

2. S is a slice at x;

3. $G(S)$ is an open neighbourhood of $G(x)$ and there is an equivariant retraction $f : G(S) \to G(x)$ such that $f^{-1}(x) = S$.

Corollary 1. [2] *If S is a slice at x, then $g.S$ is a slice at $g.x$.*

Theorem 2. [2] *Let X be a G-space and let $x \in S$. Suppose that:*

1. S is closed in $G. S$;

2. $G \cdot S$ is an open neighbourhood of $g \cdot x$;

3. $G_x \cdot S = S$;

4. $g \cdot S \cap S \neq \varnothing$ then $g \in G_x$.

Then S is a slice at x. Conversely, every slice satisfies these conditions.

2. The next hypotesis is very important in what follows:

(A). If U_1 and U_2 are two G-tubes, there exists two slices S_1 and S_2 such that: $G \cdot S_1 = U_1$, $G \cdot S_2 = U_2$ and $U_1 \cap U_2 = G \cdot (S_1 \cap S.)$

Corollary 2. *Let S be a slice at x and let $y \in S$ such that* type $G(y) =$ type $G(x) =$ type G/H. *If $G_x = H$ then $G_y = H$.*

Proof. Let $f : G.S \to G(x)$ be the equivariant retraction such that $f^{-1}(x) = S$. Thus, for every $y \in S$, $G_y \subset G_x = H$. As type $G(y) =$ type G/H it follows that G_y is conjugate with H. hence $G_y = H$.

Lemma 1. *Let S be a slice at $x_0 \in X$ and $x \in S$ such that* type $G(x) =$ type $G(x_0)$. *Then S is a slice at x.*

Proof. $G \cdot S$ is a G-tube about $G(x_0)$. Since $x \in S$ we have $G(x) \subset$ $\subset G \cdot S$ hence $G \cdot S$ is an open neighbourhood of $G(x)$. Since type $G(x) =$ $=$ type $G(x_0)$, by Corollary 2, $G_x = G_{x0}$. Let α be the map $\alpha : g \cdot x_0 \in G(x_0) \to$ $\to \alpha(g \cdot x_0) = g \cdot x \in G(x)$. Thus α is an equivalence of G-spaces (equi-variant homeomorphism). Let $f_1 = \alpha \circ f$. Thus f_1 is clearly equivariant and $f_1^{-1}(x) = (\alpha \circ f)^{-1}(x) = f^{-1}(\alpha^{-1}(x)) = f^{-1}(x_0) = S$. The statement 3) of Theorem 1 is satisfied hence S is a slice at x.

Theorem 3. *Let $x_1, x_2 \in X$. Suppose that:*

1. type $G(x_1) =$ type $G(x_2) =$ type G/H.
2. *U_1 and U_2 are G-tubes about $G(x_1)$ and $G(x_2)$ respectively, such that the hypotesis (A) is satisfied.*
3. *There is $x \in U_1 \cap U_2$ such that* type $G(x) =$ type G/H,

Thus $U_1 \cup U_2$ is a G-tube about $G(x)$.

Proof. Let $f_i : U_i \to G(x_i)$ be the equivariant retraction of $U_i (i = 1, 2)$. Let $x_0 \in G(x)$ such that $G_{x_0} = H$. We denote $\bar{x}_i = f_i(x_0)$; then $x_0 \in S_1 \cap S_2$ where $S_i = f_i^{-1}(\bar{x}_i)$. Obviously $U_i = G.S_i$. From Lemma 1 we obtain that S_i is slice at x_0 and the map $\alpha_i \circ f_i$ are equivariant retraction of G-tube U_i about $G(x_0)$; here α_i is defined by $\alpha_i(g \cdot \bar{x}_i) = g \cdot x_0$. Let $\tilde{x} \in G \cdot (S_1 \cap$ $\cap G \cdot S_2 = G \cdot (S_1 \cap S_2)$. There is a point $\tilde{x}_0 \in S_1 \cap S_2$ and $g \in G$ such that $\tilde{x} = g \cdot \tilde{x}_0$. We obtain $h_i(\tilde{x}) = h_i(g \cdot \tilde{x}_0) = g \cdot h_i(\tilde{x}_0) = g \cdot x_0 (h_i = \alpha_i \circ f_i)$ hence h_1 and h_2 coincide on $G \cdot (S_1 \cap S_2)$. It follows the existence of a map $h : G \cdot (S_1 \cup S_2) \to G(x_0)$ such that $h | G \cdot S_i = h_i$ $(i = 1, 2)$ which is cle-arly an equivariant retraction of the open set $G \cdot (S_1 \cup S_2)$ to the orbit $G(x_0)$ and which satisfies $h^{-1}(x_0) = S_1 \cup S_2$. By Theorem 1, $G.S_1 \cup G.S_2$ is a G-tube about $G(x_0)$.

Remark 1. Under the conditions of Theorem 3, Lemma 1 implies that $G \cdot S_1 \cup G \cdot S_2 = G \cdot (S_1 \cup S_2)$ is a G-tube about the orbits $G(\bar{x}_1)$ and $G(\bar{x}_2)$ with the equivariant retractions $\bar{f}_i = \alpha_{i0}^{-1} \circ h$.

Theorem 4. *Let X be a connected paracompact G-space. If all orbits have the same type G/H and (A) is verified for every pair of a covering of G-tubs then there is an equivariant map $f : X \to G/H$.*

Proof. Since X is paracompact and connected it is a Lindelöf space. Let $\{U_\alpha / \alpha \in I\}$ be an open covering which consists of G-tubes and with verific (A). We can extract a countable covering $\{U_i / i \in I_1\}$. As X is connected, we may suppose that:

$$\left(\bigcup_{k=1}^{p} U_{i_k} \right) \cap U_{i_{p+1}} \neq \emptyset \quad \text{for all } i_p \in I_1$$

Let us consider the open covering $\{V_n\}$, where $V_n = \bigcup_{k=1}^{n} U_{i_k}$. We obtain the increasing sequence of open sets:

$$V_1 \subset V_2 \subset \ldots\ldots \subset V_n \subset \ldots\ldots$$

Let $\bar{x}_1 \in V_1 = U_{i_1}$. By Remark 1, $V_2 = U_{i_1} \cup U_{i_2}$ is a G-tube about $G(x_1)$ with the retraction (Theorem 3) $\bar{f}_1 : V_2 \to G(\bar{x}_1)$. We remark that $\bar{f}_1$ coincides with f_1 on $V_1 = U_{i_1}$. Indeed, if $\tilde{x} \in V_1$ there is $\tilde{x}_0 \in S_1$ and $g \in G$ such that $\tilde{x} = g\, \tilde{x}_0$. It follows that:

$$\bar{f}(\tilde{x}) = \bar{f}_1(g \cdot \tilde{x}_0) = g \cdot \bar{f}_1(\tilde{x}_0) = g \cdot (\alpha_1^{-1} \circ h)(\tilde{x}_0) = g \cdot \alpha_1^{-1}(\tilde{x}_0) = g \cdot \bar{x}_1.$$

$$f_1(\tilde{x}) = f_1(g \cdot \tilde{x}_0) = g.\, f_1(\tilde{x}_0) = g.\, \bar{x}_1.$$

We denote, for symmetry, f_1 by $\bar{f}_1$ and $\bar{f}_1$ by $\bar{f}_2$. By induction, we suppose that V_n is a G-tube about $G(\bar{x}_1)$ with the retraction $\bar{f}_n : V_n \to G \cdot \bar{x}_1$ and let $U_{i_{n+1}}$ with the retraction $f_{n+1} : U_{i_{n+1}} \to G(\bar{x}_{n+1})$. By Theorem 3 and Remark 1 it follows that $V_{n+1} = V_n \cup U_{i_{n+1}}$ is a G-tube about $G(\bar{x}_1)$ with the retraction $\bar{f}_{n+1} : V_{n+1} \to G(\bar{x}_1)$ and by the preceding argument, this coincides with $\bar{f}_n$ on V_n.

Let $\bar{f} : X \to G(\bar{x}_1)$ defined by $\bar{f}(x) = f_k(x)$ where $k = \min \{n / x \in V_n\}$. This map is well defined and continuous. Let $x \in X$ and $g \in G$, then $k = \min \{n / x \in V_n\} = \min \{n / g \cdot x \in V_n\}$ hence $\bar{f}(x) = \bar{f}_k(x)$ and $\bar{f}(g \cdot x) =$ $= \bar{f}_k(g \cdot x) = g \cdot \bar{f}_k(x) = g \cdot \bar{f}(x)$; it follows that $\bar{f}$ is an equivariant map. To end the proof we consider $f = \alpha \circ \bar{f}$ where α is the equivalence of G-spaces $\alpha : G(\bar{x}_1) \to G/H$, $\alpha(g \cdot \bar{x}_1) = g, H$.

Definition. *A slice S is called a global slice if $G.S = X$.*

Theorem 5. *$S = \bar{f}^{-1}(\bar{x}_1)$ is a global slice at $\bar{x}_1$.*

Proof. Let $x \in X$ and $k = \min \{n / x \in V_n\}$. Then $x \in V_k$. Let $S_k =$ $= \bar{f}_k^{-1}(\bar{x}_1)$. Since $G \cdot S_k = V_k$ there is an $x_1 \in S_k$ and $g \in G$ such that $x =$ $= g \cdot x_1$. But $\bar{f}(x_1) = \bar{f}_k(x_1) = \bar{x}_1$, hence $x_1 = \bar{f}^{-1}(\bar{x}_1) = S$ where $x = g \cdot x_1 \in$ $\in g \cdot S \subset G.S$ and hence $X \subset G \cdot S$. By condition 3) of Theorem 3 we conclude that $G \cdot S$ is a G-tube about $G(\bar{x}_1)$.

Remark 2. If X is a connected C^∞-differentiable G-space, then the map f of Theorem 3 is differentiable and f^{-1} $(e \cdot H)$ is a closed submanifold of X.

3. Let X be a G-space and $H \subset K \subset G$ closed subgroups of G.

Lemma 2. $X^K \subset X^H$.

Proof. If $x \in X^K$ thus $k.\, x = x$ for all $k \in K$, hence for all $h \in H$ and hence $x \in X^H$.

Theorem 6. *Let X be a G-space with all orbits of the same type G/H. X^H is a global slice if and only if $H = N$ ($N =$ the normalizer of H in G).*

Proof. Suppose that $H = N$. We prove that X^H is a global slice. If $x \in X$ there is $y \in X$ such that $y = g \cdot x$ and $G_y = H$ (since type $G \cdot x =$ $=$ type G/H). Then $y \in X^H$ and $x \in g^{-1} \cdot X^H \subset G \cdot X^H$, hence $X \subset G \cdot X^H$ and it follows $X = G \cdot X^H$. Obviously, $H \cdot X^H = X^H$. Let $x \in X^H$ such that $g \cdot x = y \in X^H$; then $G_y = H = g \cdot G_x g^{-1} = g \cdot H \cdot g^{-1}$ and $g \in N = H$.

Conversely, let X^H be a global slice. We prove that, for every $g \in N$ we have $g \in H$. Let $g \in N$, and $y = g \cdot x$. Thus $G_y = g \cdot G_x \cdot g^{-1} = g \cdot H \cdot g^{-1} = H$ and hence $y \in X^H$. We conclude that for every $g \in N$, $g \cdot X^H \cap X^H \neq \emptyset$. Since X^H is a slice we obtain $g \in H$, hence $N = H$.

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SUR LE CALCUL DIFFÉRENTIEL DE MARINESCU DANS LES ESPACES LOCALEMENT CONVEXES

PAR

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Le calcul différentiel dans les espaces localement convexes (e.l.c) développé par G. Marinescu dans [3], utilise essentiellement la topologie raffinée de e.l.c. définie par un sous-ensemble de semi-normes qui définissent la convergence des suites dans les espaces considérés (topologie qui dépend de ce sous-ensemble de semi-normes), dans le but d'obtenir un théorème d'inversion locale pour des applications de classe C^k.

Dans cette note nous considérons la classe des e.l.c. métrisables et complets (les espaces de Fréchet) et définissons la topologie raffinée d'un espace de Fréchet qui ne dépend pas du sous-ensemble de semi-normes continues qui définit la topologie l.c. de l'espace (Définition 5). Puis, on démontre la proposition 1 qui est utilisée pour la définition d'une classe particulière d'applications de classe C^k dans le sens de Marinescu, d'applications pour lesquelles on donne un théorème d'inversion locale et un théorème des fonctions implicites, qui soient applicables dans l'étude des variétés différentiables modelées sur des espaces de Fréchet (voir [6]). On donne aussi un théorème d'existence pour les équations différentielles ordinaires dans les espaces de Fréchet qui sera utilisé pour résoudre le problème de l'intégration locale des champs de vecteurs sur des variétés modelées sur des espaces de Fréchet.

Soient E et F deux e.l.c. sur le champ R des nombres réels. Notons par Γ_E (respectivement Γ_F) l'ensemble de toutes les semi-normes continues sur E (resp. F). Evidemment, l'ensemble Γ_E est filtrant par l'ordre naturel et nous identifierons Γ_E avec l'ensemble des indices des semi-normes continues sur E, c'est-à-dire, nous notons $|\cdot|_\alpha = \alpha$ pour toute semi-norme $|\cdot| \in \Gamma_E$.

Désignons par $L(E, F)$ l'espace vectoriel de toutes les applications linéaires et continues de E dans F et par Φ l'ensemble de toutes les applications de Γ_F dans Γ_E. L'ensemble Φ est aussi filtrant par l'ordre : $\varphi_1 \leq \varphi_2$ si $\varphi_1(\beta) \leq \varphi_2(\beta)$ pour tout $\beta \in \Gamma_F$, où $\varphi_1, \varphi_2 \in \Phi$. Pour chaque application $\varphi \in \Phi$ nous considérons le sous-espace vectoriel. $L_\varphi(E, F) = \{u \in L(E, F) \mid |x|_{\varphi(\beta)} = 0$ implique $|u(x)|_\beta = 0$ et $|u|_{\beta, \varphi(\beta)} = \sup\limits_{|x|_{\varphi(\beta)} \neq 0} [|u(x)|_\beta / |x|_{\varphi(\beta)}] < \infty$, pour toute semi-norme $\beta \in \Gamma_F\}$.