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SEMI-SIMPLE INFRA-NEAR RINGS

BY

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The concept of left infra-near ring was introduced and studied in [11]—[14], as a generalization of left near-rings and weak rings (see Climescu [6]). In this paper we introduce the notion of the semi-simplicity for left infra-near rings N satisfying the minimum condition for right N-subgroups. Our results generalize the definition and the properties of simple and semi-simple near-rings, given by Blackett [3]—[4] and studied by Berman and Silverman [2] and Deskins [7]. We obtain structure theorems for semi-simple left infra-near rings (Propositions 2.1—2.5). In Section 3. we give a characterization of semi-simplicity, by using a special right ideal of N built up with „right annihilators" of the left distributive elements of the minimal nonzero right N-subgroups of N.

1. **Definitions and notations.** Recall some definitions and notations in the theory of infra-near rings. For further informations see Ștefănescu [11] — [14].

Definition 1.1. *A left infra-near ring is a triple $(N,+,\cdot)$, where $(N,+)$ is a group, $(N,\cdot)$ is a semigroup and the multiplication „$\cdot$" is left infra-distributive over the addition „$+$", i.e. the following identity holds:*

$$x\cdot(y+z) = x\cdot y - x\cdot 0 + x\cdot z\,,\ \text{for all } x,\ y,\ z \in N. \tag{1.1}$$

If $x\cdot 0 = 0$, for all $x \in N$, then N is a left near-ring. If $(x+y)\cdot z = x\cdot z - 0\cdot z + y\cdot z$ is valid for all $x, y, z \in N$, and the addition is commutative, then N is a weak ring. If $x\cdot y = x\cdot 0$, for all $x, y \in N$, then N is called a *trivial left infra-near ring*.

Examples. 1°. The set of all affine mappings of a linear space V over a field K is a left infra-near ring with respect to pointwise addition and mapping composition. It is also a right near-ring (a generalization of this infra-near ring is given in [12]).

2°. Let R be a ring. Then $N = R^6$ is a left infra-near ring with respect to the following binary operations:

$$x + y = (x_1+y_1,\ x_2+y_2,\ x_3+y_3,\ x_4+y_4,\ x_5+x_1\cdot y_4+x_3\cdot y_2+y_5),$$
$$x \cdot y = (x_1\cdot y_1,\ x_2\cdot y_2,\ x_3,\ x_4,\ x_4),\ \text{for all } x,\ y \in N.$$

3°. The cyclic group of order 3, $Z_3 = \{0, 1, 2\}$, is a left infra-near ring with the multiplication given by the equalities:

$$0 \cdot x = 0, \quad 1 \cdot x = 1, \quad 2 \cdot x = x, \text{ for all } x \in Z_3.$$

4°. The symmetric group, $S_3 = \{0, a, b, x, y, z\}$ (with $2 \cdot a = b$, $3 \cdot a = 2 \cdot x = 0$, $x + a = y$, $x + b = z$), can be structured as a left infra-near ring with respect to one of the following multiplications:

(I) $0 \cdot s = 0$, $a \cdot s = x \cdot s = y \cdot s = z \cdot s = a$, $b \cdot s = s$, for all $s \in S_3$;

(II) $0 \cdot s = 0$, $a \cdot s = x \cdot s = y \cdot s = a$, $b \cdot s = z \cdot s = s$, for all $s \in S_3$;

(III) $0 \cdot s = 0$, $a \cdot s = x \cdot s = a$, $b \cdot s = y \cdot s = z \cdot s = s$, for all $s \in S_3$.

The elements of a left infra-near ring verify some identities, as:

$$x \cdot (-y) = x \cdot 0 - x \cdot y + x \cdot 0, \tag{1.2}$$

$$x \cdot (y - z + t) = x \cdot y - x \cdot z + x \cdot t, \tag{1.3}$$

the last one leading to a ternary interpretation for the addition ([11], § 2).

In the examples and *everywhere in this paper, we assume that N satisfies also the identity*:

$$0 \cdot x = 0, \tag{1.4}$$

i.e. N is a left C-infra-near ring (Definition 4.1, [11]).

Definition 1.2. (Definition 3.1, [11]). *A normal subgroup $(A, +)$ of $(N, +)$ is a left ideal of N, if it satisfies the property:*

$$x \cdot a - x \cdot 0 \in A, \text{ for all } a \in A, \ x \in N, \tag{1.5}$$

and it is a right ideal of N, if it satisfies the property:

$$(a + x) \cdot y - x \cdot y \in A, \text{ for all } a \in A, \ x, y \in N. \tag{1.6}$$

A left and a right ideal of N is called an *ideal of N*.

N and 0 are ideals of N.

Definition 1.3. (Definition 3.2, [11]). *Let N be a left infra-near ring, $(M, +)$ be a group. M is called a right N-group, if there exists a mapping $\mu : M \times N \to M$, denoted by $\mu(m, x) = m \cdot x$, for all $m \in M$ and $x \in N$, such that the following axioms hold:*

$$m \cdot (x + y) = m \cdot x - m \cdot 0_N + m \cdot y, \tag{1.7}$$

$$(m \cdot x) \cdot y = m \cdot (x \cdot y), \tag{1.8}$$

$$0_M \cdot x = 0_M, \text{ for all } m \in M, \ x, y \in N. \tag{1.9}$$

Remark. Obviously, N is a right N-group. Moreover, each right ideal of N is a right N-subgroup of N. To prove it, we take $x = 0$ in (1.6). However, a right N-subgroup T of N is not always a right ideal, since $(T, +)$ is not a normal subgroup and (1.6) is not valid in T.

Definition 1.4. *Let M, M' be two right N-groups. A homomorphism of right N-groups is a mapping $\varphi : M \to M'$, which satisfies the properties:*

$$\left\{\begin{array}{l} \varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2), \text{ for all } m_1, m_2 \in M, \\ \varphi(mx) = \varphi(m)x, \text{ for all } m \in M \text{ and } x \in N. \end{array}\right. \tag{1.10}$$

The image of φ, $\operatorname{Im} \varphi = \{\varphi(m) \mid m \in M\}$, is a right N-subgroup of M', while the kernel of φ, $\operatorname{Ker} \varphi = \{m \mid m \in M, \varphi(m) = 0_{M'}\}$, is a normal right N-subgroup of M, having the property:

$$(m_0 + m) \cdot x - m \cdot x \in \operatorname{Ker} \varphi, \text{ for all } m_0 \in \operatorname{Ker} \varphi, \ m \in M, \ x \in N. \tag{1.11}$$

A bijective homomorphism of right N-groups is an isomorphism, hence φ is an isomorphism if and only if $\operatorname{Im} \varphi = M'$, $\operatorname{Ker} \varphi = 0$.

If we take N as a right N-group, then the kernels of the homomorphisms of right N-groups $\Psi : N \to M$ are just the right ideals of N. Indeed, if B is a right ideal of N, then $N/B = \{x + B \mid x \in N\}$ can be structured as a right N-group with respect to binary operations given by: $(x + B) + (y + B) = (x + y) + B$, $(x + B) \cdot y = x \cdot y + B$, for all $x + B$, $y + B \in N/B$ and $y \in N$. The mapping $\eta : N \to N/B$, $\eta(x) = x + B$, for all $x \in N$, is a surjective homomorphism of right N-groups with $\operatorname{Ker} \eta = B$.

Consider the following subsets of a left infra-near ring N, formed by all left distributive elements of N and, respectively, all weakly left distributive elements of N:

$$D = \{d \mid d \in N, \ d \cdot 0 = 0\}, \tag{1.12}$$

$$W = \{w \mid w \in N, \ w \cdot 0 = w\}. \tag{1.13}$$

It is obvious that: 1). $(D, \cdot)$ is a semigroup of $(N, \cdot)$; 2) $W \cap D = 0$; 3) $w \cdot x = w$, $x \cdot w = W$, for all $x \in N$, $w \in W$. If the element 0 is right distributive, then D is a left subnear-ring of N, $(D, +)$ is a normal subgroup of $(N, +)$ and W is a right N-subgroup of N. In Example 1°, $D(N) = \operatorname{Hom}_K(V, V)$, $W(N) = T_c(V)$, the set of all constant functions on V; in Example 2°, $D(N) = \{(x_1, x_2, 0, 0, 0) \mid x_1, x_2 \in R\}$, $W(N) = \{(0, 0, x_3, x_4, x_5) \mid x_i \in R, i = 3, 4, 5\}$; in Example 3°, $D(N) = \{0, 2\}$, $W(N) = \{0, 1\}$; in Example 4°, $D_I = \{0, b\}$, $D_{II} = \{0, b, z\}$, $D_{III} = \{0, b, y, z\}$ and $W_I = W_{II} = W_{III} = \{0, a\}$.

We need the following:

Proposition 1.1. ([11], Proposition 5.1) *Let N be a left infra-near ring and $e \in D$ be an idempotent of N. Then the mapping $\varphi_e : N \to N$, defined by $\varphi_e(x) = e \cdot x$, for all $x \in N$, is an endomorphism of the right N-group N. Moreover, $N = e \cdot N + \operatorname{Ker} \varphi_e$, a semidirect sum of additive subgroups, $e \cdot N$ is a right N-subgroup and a left subinfra-near ring of N with e as a left identity.*

Definition 1.5. *Let M be a right N-group. The sets:*

$$\operatorname{Ann}_r(m) = \{a \mid a \in N, \ m \cdot a = m \cdot 0_N\}, \tag{1.14}$$

and

$$\operatorname{Ann}_r(M) = \{a \mid a \in N, \ m \cdot a = m \cdot 0_N, \text{ for all } m \in M\} \tag{1.15}$$

are called, respectively, *the right annihilator of m and the right annihilator of M.*

We can easily verify the following properties of the right annihilators: (*i*) $\operatorname{Ann}_r(m)$ is a right ideal of N; (*ii*) $\operatorname{Ann}_r(M)$ is an ideal of N; (*iii*)

$\mathrm{Ann}\,(w) = \mathrm{Ann}_r(W) = N$, for all $w \in W$; (iv) $\mathrm{Ann}\,(D(P))$ is a right ideal, for each right N-subgroup P of N, where $D(P)=\{p \mid p \in P,\ p\cdot 0=0\}= P \cap D$.

Proposition 1.2. *The intersection of a family of right N-subgroups (right ideals, ideals) of N is a right N-subgroup (respectively, a right ideal, an ideal) of N. If A and B are two right ideals (ideals) of N, then $A + B = \{a+b \mid a \in A,\ b \in B\}$ is a right ideal (an ideal) of N.*

Proof. The two statements can be verified by straightforward calculations. For example, let A and B be ideals of N. We know that $(A+B,+)$ is a normal subgroup of $(N,+)$. As A and B are right ideals, then for any $a+b \in A+B$ and $x,\ y \in N$, we have : $((a+(b+x))\cdot y - (b+x)\cdot y) + ((b+x)\cdot y - x\cdot y) \in A+B$. If A and B are also left ideals, then, for any $x \in N$ and $a+b \in A+B$, we have : $x\cdot(a+b) - x\cdot 0 = (x\cdot a - x\cdot 0) + (x\cdot b - x\cdot 0) \in A+B$. Therefore $A+B$ is an ideal of N.

2. Semi-simplicity. Now we are able to define the notion of semi-simplicity for left infra-near rings.

As usual, a nonzero right N-subgroup M of N is called *minimal*, if M does not contain any other nonzero right N-subgroup M', $M' \neq M$. We denote the set of all minimal nonzero right N-subgroups of N by $\mathfrak{M}$. We say that N *satisfies the decreasing chain condition for right N-subgroups* (shortly, $(D.C.C.)$), if every strictly decreasing chain of right N-subgroups of N is finite. This conditions is equivalent to *the minimum condition for right N-subgroups*, i.e. every nonempty collection of right N-subgroups of N (ordered by set-inclusion) has a minimal element.

First we have to prove two lemmas :

Lemma 2.1. *If M is a minimal nonzero right N-subgroup of N, containing a left distributive element m, such that $m\cdot M \neq 0$, then M contains also a left identity e, such that $M = e\cdot M = e\cdot N$.*

Proof. Consider the mapping $\tau : M \to M$, defined by $\tau(x) = m\cdot x$, for all $x \in M$, ; this mapping is an endomorphism of the right N-group M. Since $m\cdot M \neq 0$, $\mathrm{Im}\ \tau = m\cdot M \neq 0$ is a nonzero right N-subgroup of M, which is minimal. Therefore $\mathrm{Im}\ \tau = M$. Now $\mathrm{Ker}\ \tau$ is a right N-subgroup of M, different from M, hence $\mathrm{Ker}\ \tau = 0$ and τ is an isomorphism of right N-groups. Therefore there exists $e \in M$, such that $m = \tau(e) = m\cdot e$. By multiplying this equality by e at the right, we obtain : $m = m\cdot e = m\cdot e^2$, hence $\tau(e^2)=\tau(e)$ and $e^2=e$, because of the injectivity of τ. In the same manner, we prove that $e\cdot 0 = 0$ and $e\cdot x = x$, for any $x \in M$, hence e is a left identity of M. We have $M = e\cdot M = e\cdot N$.

Lemma 2.2. *If M is a right N-subgroup of N containing a minimal nonzero right N-subgroup M_0 which satisfies the condition in Lemma 2.1, then $M = M_0 + (\mathrm{Ker}\ \varphi_{e_0} \cap M)$ — a semidirect sum of groups — where $M_0 = e_0\cdot M$, with e_0 a left identity of M_0 and φ_{e_0} the endomorphism of the right N-group N given in Proposition 1.1.*

The proof is straightforward and we pass over it, noting only that $m = e_0\cdot m + (-e_0\cdot m + m)$, for all $m \in M$, where $e_0\cdot m \in M_0$, $-e_0\cdot m + m \in \mathrm{Ker}\ \varphi_{e_0} \cap M$.

Definition 2.1. *A nontrivial left infra-near ring N is called semi-simple, if each minimal nonzero right N-subgroup M of N contains a left distributive element m such that $m\cdot M \neq 0$ and N satisfies the $(D.C.C.)$ for right N-subgroups. A semi-simple left infra-near ring N without any proper right N-subgroup is called simple.*

Remark. In a semi-simple left infra-near ring N, each nonzero right N-subgroup M satisfies the hypotheses of lemma 2.2; therefore M has a decomposition in a semidirect sum of subgroups which are right N-subgroups, namely : $M = e_0\cdot M + (\mathrm{Ker}\ \varphi_{e_0} \cap M)$.

Proposition 2.1. *A nonzero right N-subgroup M of a semi-simple left infra-near ring N is a finite direct sum of minimal nonzero right N-subgroups of N contained in M.*

Proof. If $M \in \mathfrak{M}$, the statement is proved. If $M \notin \mathfrak{M}$, then there exists a minimal nonzero right N-subgroup $M_0 \subseteq M$. Let e_0 be the distributive idempotent in M_0 (Lemma 2.2) and $M = M_0 + (\mathrm{Ker}\ \varphi_{e_0} \cap M)$. If M_0 is a normal subgroup of M, then we take $M_1 = M_0$, $e_1 = e_0$, $M'_1 = \mathrm{Ker}\ \varphi_{e_1} \cap M$. If M_0 is not normal, we consider $M' = \mathrm{Ker}\ \varphi_{e_0} \cap M$, which is a normal nonzero right N-subgroup of M. If M' is a minimal nonzero right N-subgroup, then we take $M_1 = M'$, $e_1 = e'$ — the distributive idempotent of M', such that $M' = e'\cdot M$ — and $M'_1 = M_0$. If M' is not in $\mathfrak{M}$, then there exists a minimal nonzero normal right N-subgroup M_1 of M contained in M'. M_1 is also in $\mathfrak{M}$, because, in other case, $M_1 = \overline{M}_1 + \mathrm{Ker}\ \varphi_{\bar{e}_1} \cap M_1$, with $\overline{M}_1 \in \mathfrak{M}$. But $\mathrm{Ker}\ \varphi_{\bar{e}_1} \cap M_1$ is a normal right N-subgroup (in M) properly included in the minimal normal right N-subgroup M_1. Therefore $\overline{M}_1 = M_1$. Now we have $M = M_1 \oplus M'_1$, where $M_1 = e_1\cdot M_1 = e_1\cdot M$, $e_1\cdot M'_1 = 0$. By applying the same argument to M'_1, we obtain $M'_1 = M_2 + M'_2$, hence $M = M_1 \oplus M_2 \oplus M'_2$, where M_1, M_2 are minimal nonzero right N-subgroups of N and $M_2 = e_2\cdot M_2 = e_2\cdot M$, $e_2\cdot M_1 = e_2\cdot M'_2 = 0$. By repeating the argument, we obtain a decreasing chain of right N-subgroups $M'_1 \supseteq M'_2 \supseteq \ldots \supseteq M'_k \supseteq \ldots$, which must be finite. Therefore there exists a minimal M'_{p-1} and we take $M_p = M'_{p-1} = e_p\cdot M$. We have $M = M_1 \oplus M_2 \oplus \ldots \oplus M_p$, where $M_i = e_i\cdot M$, $i = 1, 2, \ldots, p$, $e_i\cdot M_j = 0$, if $i \neq j$, $i, j \in \{1, 2, \ldots, p\}$.

Such a direct sum of minimal nonzero right N-subgroups of N giving M is called a *complete direct decomposition of M in right N-subgroups*.

By taking $M = N$ in Proposition 2. 1, we obtain :

Corollary 2.1. *If N is a semi-simple left infra-near ring, then it has a complete direct decompositon in right N-subgroups :*

$$N = N_1 \oplus N_2 \oplus \ldots \oplus N_s,$$

where $N_i = e_i\cdot N_i = e_i\cdot N$, $e_i\cdot N_j = 0$, *for* $i \neq j$, $i, j \in \{1, 2, \ldots, s\}$.

Lemma 2.3. *If N is a semi-simple left infra-near ring, then each minimal nonzero right ideal of N (respectively, each minimal normal right N-subgroup of N) is also a minimal nonzero right N-subgroup of N.*

Proof. It was noted that a right ideal R of N is a right N-subgroup of N. Assume R is a minimal nonzero right ideal, but it is not a minimal right N-subgroup of N, hence there is $M_0 \in \mathfrak{M}$, contained in R. Let e_0 be the idempotent of M_0 given in Lemma 2.1 and $M_0 = e_0\cdot M_0 = e_0\cdot N$. The mapping $\psi : R \to M_0$, given by $\psi(x) = e_0\cdot x$, for all $x \in R$, is a homomorphism of right N-groups and it has $\mathrm{Im}\ \psi = M_0$, $\mathrm{Ker}\ \psi = A$, with A a right ideal of N contained in R and different from R. As R is minimal, $\mathrm{Ker}\ \psi = 0$ and ψ is an isomorphism of right N-groups. But $\mathrm{Im}\ \psi = M_0 \subseteq R$ and $\psi_{/M_0} = I_{M_0}$.

Therefore $M_0 = R$ and R is in $\mathfrak{M}$. The second statement has a similar proof. In the Proposition 2.1 and in its proof, we may take a right ideal $J \neq 0$ (instead of the right N-subgroup M) and minimal right ideals (instead of minimal right N-subgroups). So we obtain:

Proposition 2.2. *In a semi-simple left infra-near ring N, a nonzero right ideal J is a finite direct sum of minimal nonzero right ideals: $J = J_1 \oplus J_2 \oplus \ldots \oplus J_p$, where $J_i = e_i \cdot J = e_i \cdot N$, $i = 1, 2, \ldots, p$. Moreover $e = e_1 + e_2 + \ldots + e_p$ is a left identity of J.*

We have to prove only that e is a left identity of J. We need the following equality: $(e_1 + e_2 + \ldots + e_h) \cdot a = e_1 \cdot a + e_2 \cdot a + \ldots + e_h \cdot a$, for any $a \in J$, $1 \leqslant h \leqslant p$. We can obtain it by using induction on h. For $h = 1$, the equality is trivial. Assume the equality holds for all h, $h \leqslant k < p$ and prove it for $h = k+1$. The sets $e_1 \cdot J \oplus \ldots \oplus e_k \cdot J$ and $e_{k+1} \cdot J$ are right ideals of N. Therefore we have: $(e_1 + \ldots + e_k + e_{k+1}) \cdot a - (e_1 + \ldots + e_k) \cdot a \in e_{k+1} \cdot J$, $(e_1 + \ldots + e_k + e_{k+1}) \cdot a - e_{k+1} \cdot a \in e_1 \cdot J \oplus e_2 \cdot J \oplus \ldots \oplus e_k \cdot J$. Because of the direct sum, $(e_1 + \ldots + e_k) \cdot a + e_{k+1} \cdot a = e_{k+1} \cdot a + (e_1 + \ldots + e_k) \cdot a$ and we have: $(e_1 + \ldots + e_k + e_{k+1}) \cdot a - (e_1 + \ldots + e_k) \cdot a - e_{k+1} \cdot a \in \left(\bigoplus_{i=1}^{k} e_i \cdot J\right) \cap e_{k+1} \cdot J = 0$, hence it is equal to 0. We have then $(e_1 + \ldots + e_k + e_{k+1}) \cdot a = (e_1 + \ldots + e_k) \cdot a + e_{k+1} \cdot a = e_1 \cdot a + \ldots + e_k \cdot a + e_{k+1} \cdot a$. But $a \in J$ can be uniquely written as a sum: $a = a_1 + \ldots + a_p$ with $a_i \in J_i$, $i = 1, 2, \ldots, p$. Then we have: $e \cdot a = (e_1 + \ldots + e_p) \cdot a = e_1 \cdot a + e_2 \cdot a + \ldots + e_p \cdot a = a_1 + a_2 + \ldots + a_p = a$ and e is a left identity.

Now by taking $J = N$ in the last proposition, we have:

Corollary 2.2. *If N is a semi-simple left infra-near ring, then it is a finite direct sum of minimal nonzero right ideals, $N = J_1 \oplus \ldots \oplus J_t$, with $J_i = e_i \cdot N$, $i = 1, 2, \ldots, t$, $e_i \cdot J_k = 0$ for $i \neq k$, $i, k \in \{1, 2, \ldots, t\}$. Moreover, $e = e_1 + \ldots + e_t$ is a left identity of N.*

Proposition 2.3. *If a right ideal J of a semi-simple left infra-near ring N is written as a finite sum of nonzero right ideals: $J = J_1' \oplus J_2' \oplus \ldots \oplus J_s'$, then there exist left distributive idempotents $e_i' \in J_i'$, with the properties: $e_i' \cdot J_i' = e_i' \cdot J = J_i'$, $e_i' \cdot e_k' = 0$, for $i \neq k$, $i, k \in \{1, \ldots, s\}$.*

Proof. Let e be the left identity of J, existing from Proposition 2.2. Then $e = e_1' + \ldots + e_s'$, where $e_i' \in J_i'$, $e \cdot e_i' = e_i'$, $i = 1, 2, \ldots, s$. By using the same argument as in the proof of the Proposition 2.2, we obtain: $(e_1' + \ldots + e_s') \cdot a = e_1' \cdot a + \ldots + e_s' \cdot a$, for any $a \in J$, and $e_i' \cdot a \in J_i'$, $i = 1, \ldots, s$, as J_i' is a right N-subgroup of N. Then $e \cdot e_i' = e_1' \cdot e_i' + \ldots + e_i' \cdot e_i' + \ldots + e_s' \cdot e_i' = e_i'$, for $i = 1, 2, \ldots, s$. By the uniqueness of the sum giving e_i', we have $e_i' \cdot e_i' = e_i'$, $e_k' \cdot e_i' = 0$, for $i \neq k$, i and k from $\{1, 2, \ldots, s\}$. Taking $a_i \in J_i'$, $i = 1, 2, \ldots, s$, we obtain: $e \cdot a_i = a_i = e_1' \cdot a_i + \ldots + e_s' \cdot a_i$, therefore $e_i' \cdot a_i = a_i$, $e_k' \cdot a_i = 0$, for $i \neq k$, i and k from $\{1, 2, \ldots, s\}$.

The normal nonzero right N-subgroups of a semi-simple left infra-near ring N play the part of the submodules of a semi-simple module over a ring or the part of the left (right) ideals in a semi-simple ring. We shall show it in the following:

Proposition 2.4. *Let N be a semi-simple left infra-near ring. Then: (i) N has only a finite number of nonisomorphic minimal nonzero right N-subgroups; (ii) any normal minimal nonzero right N-subgroup R of N is a member of a complete direct decomposition of N in right N-subgroups.*

Proof. (i) Indeed, let M' be a minimal nonzero right N-subgroup of N and $M' = e' \cdot M' = e' \cdot N$ Let $N = N_1 \oplus \ldots \oplus N_t$ be a complete direct decomposition of N in right N-subgroups. Because $M' \neq 0$, there exists $i \in \{1, 2, \ldots, t\}$ so that $e' \cdot N_i \neq 0$ (in other case, for all $x = x_1 + \ldots + x_t \in N$, $e' \cdot x = e' \cdot x_1 + \ldots + e' \cdot x_t = 0$). The mapping $\tau_i : N_i \to M'$, given by $\tau_i(x_i) = e' \cdot x_i$, for any $x_i \in N_i$, is an isomorphism of right N-groups. Therefore any minimal nonzero right N-subgroup of N is isomorphic to some N_i in a complete direct decomposition of N in right N-subgroups. (ii) In the proof of the Proposition 2.1. we take $M = N$ and we start with R, which is a normal minimal nonzero right N-subgroup of N.

Now we analyse the part of nonzero ideals in this theory.

Lemma 2.4. *A nonzero ideal B of a semi-simple left infra-near ring N is the direct sum of those right ideals J_{i_k} in a complete direct decomposition of N in right ideals, $N = J_1 \oplus \ldots \oplus J_t$, which have the property $J_{i_k} \cap B \neq 0$.*

Proof. If $b \in B$, $b \neq 0$, is written as $b = b_{i_1} + \ldots + b_{i_p}$, with $b_{i_h} \in J_{i_h}$, $h = 1, 2, \ldots, p$ (we consider only the nonzero components of the sum giving b), then, for $h = 1, 2, \ldots, p$, $0 \neq b_{i_h} \in B \cap J_{i_h}$, because of the equality $b_{i_h} = e_{i_h} \cdot b - e_{i_h} \cdot 0$ and the fact that B is a left ideal. But $B \cap J_{i_h} (\neq 0)$ is a right ideal contained in J_{i_h} which is minimal, hence $B \cap J_{i_h} = J_{i_h}$. Therefore $J_{i_h} \subseteq B$, $h = 1, 2, \ldots, p$ and $B = \bigoplus_{h=1}^{p} J_{i_h}$.

Obviously the ideal B of N is a direct summand of N, as $N = B \oplus J$, where $J = \bigoplus_{\substack{i=1 \\ i \neq i_h}}^{t} J_i$.

Lemma 2.5. *If the semi-simple left infra-near ring N is a direct sum of groups, $N = B \oplus A$, where B is an ideal of N and A is a right N-subgroup of N, then any right A-subgroup of A is a right N-subgroup.*

Proof. Indeed, if C is a right A-subgroup, then $(C, +)$ is a subgroup of N and, for any $x \in N$, $x = b + a$, $b \in B$, $a \in A$, $c \in C$, we have $c \cdot x = (c \cdot b - c \cdot 0) + c \cdot a$, where $c \cdot b - c \cdot 0 \in B \cap A = 0$, since A is a right N-subgroup, $c \in A$ and B is a left ideal of N. Therefore $c \cdot b = c \cdot 0$ and $c \cdot x = c \cdot a \in C$, since C is a right A-subgroup of A. Hence C is a right N-subgroup of N.

We note that, by taking $C = A$, we obtain $a \cdot b = a \cdot 0$, for all $a \in A$ and $b \in B$, hence $B \subseteq \mathrm{Ann}_r(A)$.

Proposition 2.5. *Let N be a left infra-near ring satisfying the following conditions: (i) For any nonzero right N-subgroup M of N, $\mathrm{Ann}_r(M) \neq N$; (ii) $N = \bigoplus_{i=1}^{s} B_i$, with B_i, $i = 1, 2, \ldots, s$, nonzero ideals of N, which are (semi-)simple left infra-near rings. Then N is semi-simple. The multiplication of N is given by: $(x_1 + \ldots + x_s) \cdot (y_1 + \ldots + y_s) = x_1 \cdot y_1 + \ldots + x_s \cdot y_s$, for all $x_1 + \ldots + x_s$, $y_1 + \ldots + y_s \in N$.*

Proof. Obviously, (i) implies that any nonzero right N-subgroup M of N is not trivial and $M_i = M \cap B_i$ is different from 0 for some i in $\{1, 2, \ldots, s\}$. If $M_i = 0$, for all $i = 1, 2, \ldots, s$, then for each $m \in M$, $x \in N$, $x = x_1 + \ldots + x_s$, $m \cdot x = (m \cdot x_1 - m \cdot 0) + \ldots + (m \cdot x_s - m \cdot 0) + m \cdot 0 = m \cdot 0$, because $m \cdot x_i - m \cdot 0 \in M_i = 0$, for $i = 1, 2, \ldots, s$. We obtain a contradiction. If M_i is different from 0, it is a nonzero right B_i-subgroup. But B_i is (semi-)simple. Therefore M_i has a distributive element m_i, such that $m_i \cdot M_i \neq 0$. We have also $m_i \cdot M \neq 0$, and $m_i \in M$. Let $M^1 \supseteq M^2 \supseteq \ldots \supseteq M^t \supseteq \ldots$ be a decreasing chain of right N-subgroups of N. Then the chains $M_i^1 \supseteq M_i^2 \supseteq \ldots \supseteq M_i^t \supseteq \ldots$, where $M_i^h = M^h \cap B_i$, $h = 1, 2, \ldots$, $i = 1, 2, \ldots, s$, are decreasing chains of right B_i-subgroups in B_i, $i = 1, 2, \ldots, s$. They are all finite. Let l_i be the length of the chain number i, $i = 1, 2, \ldots, s$ and take $L = \max\limits_{1 \leq i \leq s} l_i$. Then $M^L = M^{L+1} = \ldots$ and the given chain is finite. Therefore N is semi-simple. To verify the equality giving the multiplication in N, we use induction on h for proving the equality: $(x_1 + \ldots + x_h) \cdot (y_1 + \ldots + y_h) = x_1 \cdot y_1 + \ldots + x_h \cdot y_h$, $1 \leqslant h \leqslant s$. Remark that $x_i \cdot y_j = x_i \cdot 0$, for $i \neq j$, since $x_i \cdot y_j - x_i \cdot 0 \in B_i \cap B_j = 0$. Assume that the equality holds for all k, $1 \leqslant k \leqslant h$, and prove it for $h+1$. We have: $(x_1 + \ldots + x_h + x_{h+1}) \cdot (y_1 + \ldots + y_{h+1}) - (x_1 + \ldots + x_h) \cdot (y_1 + \ldots + y_{h+1}) - x_{h+1} \cdot (y_1 + \ldots + y_{h+1}) \in (\bigoplus_{i=1}^{h} B_i) \cap B_{h+1} = 0$, hence: $(x_1 + \ldots + x_h + x_{h+1}) \cdot (y_1 + \ldots + y_h + y_{h+1}) = (x_1 + \ldots + x_h) \cdot (y_1 + \ldots y_h) - (x_1 + \ldots + x_h) \cdot 0 + ((x_1 + \ldots + x_h) \cdot y_{h+1} - (x_1 + \ldots + x_h) \cdot 0) + (x_1 + \ldots + x_h) . 0 + (x_{h+1} \cdot (y_1 + \ldots + y_h) - x_{h+1} \cdot 0) + x_{h+1} \cdot y_{h+1} = x_1 \cdot y_1 + \ldots x_h \cdot y_h + x_{h+1} \cdot y_{h+1}$, because the inner terms are equal to 0. Indeed, for example, $(x_1 + \ldots + x_h) \cdot y_{h+1} - (x_1 + \ldots + x_h) \cdot 0 \in (\bigoplus_{i=1}^{h} B_i) \cap B_{h+1} = 0$, since $\bigoplus_{i=1}^{h} B_i$ is a right ideal and B_{h+1} is a left ideal of N.

Remark. If $W \neq 0$ and 0 is a right distributive element of N, then W is a nonzero right N-subgroup of N which does not have a distributive nonzero element inside. Hence the left infra-near rings of affine type (see [12]) are not simple.

Having some simple left infra-near rings, we can build up a semi-simple left infra-near ring in the following manner:

Proposition 2.6. *If N_i, $i = 1, 2, \ldots, s$ are simple left infra-near rings, then $N = N_1 \times N_2 \times \ldots \times N_s$ is a semi-simple left infra-near ring with respect to componentwise addition and multiplication.*

Proof. We can verify immediately that N is a left infra-near ring. In N, the subsets $N_i' = \{(0, 0, \ldots, x_i, \ldots, 0) \mid x_i \in N_i\}$, $i = 1, 2, \ldots, s$ are ideals. But $N = \bigoplus_{i=1}^{s} N_i'$ (as an additive group). For any N_i', which is a simple left infra-near ring isomorphic to N_i, we have $\mathrm{Ann}_r (N_i') \neq N$, because of its simplicity (there is an element $e_i' \in N_i'$, $e_i' \cdot 0 = 0$, $e_i' \cdot N_i' \neq 0$). By Proposition 2.5, we obtain the semi-simplicity of N. (Of course, one can prove this proposition by straightforward checking).

The left infra-near rings in Examples 3° and 4° are simple. For the three left infra-near rings on the group S_3 given in Example 4°, consider the left distributive multiplication associated to „·" by:

$$x * y = x \cdot y - x \cdot 0, \text{ for all } x, y \in N.$$

We obtain left near-rings, but they are not simple in the sense of B l a c k e t t ([3], Definition on p. 780). There are two nonisomorphic left near-rings which are simple (see C l a y [5], p. 370, classes 20 and 34), on the group S_3.

3. A characterization of semi-simplicity. For a left infra-near ring N, denote $S = \bigcap_{M \in \mathfrak{M}} \mathrm{Ann}_r(D(M))$, where $\mathfrak{M}$ is the set of minimal nonzero right N-subgroups of N and $D(\mathfrak{M}) = \{d \mid d \in M,\ d \cdot 0 = 0\}$. Assume that $D(M) \neq 0$, for any $M \in \mathfrak{M}$.

Proposition 3.1. *If N is a semi-simple left infra-near ring, then $S = 0$. Conversely, if the left infra-near ring N satisfies (D.C.C.) for right N-subgroups and has $S = 0$, then N is semi-simple.*

Proof. If N is semi-simple, then for any $M \in \mathfrak{M}$, there exists $d \in D(M)$, such that $d \cdot M \neq 0$. As S is a right N-subgroup (because it is a right ideal), if $S \neq 0$, then there exists $T \subseteq S$, $T \in \mathfrak{M}$. But T does not have any $d \in D(T)$, such that $d . T \neq 0$, because $T \subseteq S \subseteq \mathrm{Ann}_r\ (D(T))$. Hence S must be equal to 0. Conversely, let $S = 0$. Assume there exists $M_1 \in \mathfrak{M}$, such that for all $d_1 \in D(M_1)$, $d_1 \cdot M_1 = 0$. If for any $M \in \mathfrak{M}$ and for any $d \in D(M)$, $d \cdot M_1 = 0$, then, by definition of S, $M_1 \subseteq S$ and S is different from 0. Therefore there is a minimal nonzero right N-subgroup M_2, such that for a distributive element of M_2, $d_2 \in D\ (M_2)$, $d_2 \cdot M_1 \neq 0$. Define $\psi : M_1 \to M_2$, by $\psi\ (m_1) = d_2 \cdot m_1$, for all $m_1 \in M_1$. This mapping is an isomorphism of N-groups, as one can easily verify, because M_1 and M_2 belong to $\mathfrak{M}$. But, from the surjectivity of ψ, there exists an element $d \in M_1$, which must be left distributive, such that $\psi\ (d) = d_2 \cdot d = d_2$. We have $(d_2 \cdot d) \cdot M_1 = d_2 \cdot M_1 \neq 0$ and also $(d_2 \cdot d) \cdot M_1 = d_2 \cdot (d \cdot M_1) = d \cdot 0 = 0$, a contradiction. Hence our assumption on M_1 is false and N is semi-simple.

We remind that D e s k i n s [7] has given a similar characterization of the semi-simplicity of the left near-rings, by using a set S which is 0 if and only if N is semi-simple. When we consider the multiplication „*" associated to „·", we obtain a left distributive multiplication which is associative or not. But even if „*" is associative, the obtained left near-ring $(N, +, *)$ is not always semi-simple, as one can see in Example 4°, Section 1.

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SUR QUELQUES STRUCTURES ALGEBRIQUES DANS LES CATÉGORIES

PAR

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Soit $\mathcal{C}$ une catégorie et A un objet de $\mathcal{C}$. Par définition [1, cap. 3; 2. cap. 4] on suppose que *A est doté d'une certaine structure algébrique si $\forall X \in |\mathcal{C}|$, ($|\mathcal{C}|$ désigne la classe des objets de $\mathcal{C}$) l'ensemble $\mathcal{C}(X, A)$ est doté de la structure considérée et pour chaque morphisme $u: X \to Y$ ($Y \in |\mathcal{C}|$), $\mathcal{C}(u, A)$ est un homomorphisme pour cette structure.* On peut montrer [1, cap. 3; 2. cap. 4] que cette condition est équivalente à une condition de factorization pour le foncteur contravariant $\mathcal{C}(-, A): \mathcal{C} \to$ Ens (Ens désigne la catégorie des ensembles).

Nous allons supposer dès maintenant que, sauf mention spéciale, $\mathcal{C}$ a des produits finis et un objet final, cas dans lequel on peut expliciter, comme ci-dessous, la condition antérieure.

Soient p_1, p_2, p_3 les projections pour $A \times A \times A$, q_1, q_2 les projections pour $A \times A$. Notons par $<.>$ le morphisme qui correspond par la propriété d'universalité du produit à un ensemble adequat de morphismes (de même source).

Soit un morphisme $f: A \times A \to A$, nommé opération sur l'objet A. Pour chaque $X \in |\mathcal{C}|$ on peut définir sur $\mathcal{C}(X, A)$ une opération$*_X$ par $u *_X v = f < u, v >$, où $u, v \in \mathcal{C}(X, A)$.

Remarque 1: $\forall X \in |\mathcal{C}|$, $u, v \in \mathcal{C}(X, A)$ et $\forall q: Y \to X$, $(u *_X v)q = (uq) *_X (vq)$.

Lemme 1. *Les conditions suivantes sont équivalentes:*

a) $f(1_A \times f) < p_1, < p_2, p_3 >> = f(f \times 1_A) << p_1, p_2 >, p_3 >$ *(condition d'associativité pour f)*;

b) *$\forall X \in |\mathcal{C}|$, l'opération $*_X$ définie sur $\mathcal{C}(X, A)$ est associative;*

c) *l'opération $*_{A\times A\times A}$ définie sur $\mathcal{C}(A \times A \times A, A)$ satisfait à l'égalité:* $p_1 *_{A\times A\times A} (p_2 *_{A\times A\times A} p_3) = (p_1 *_{A\times A\times A} p_2) *_{A\times A\times A} p_3$.

Lemme 2. *Les conditions suivantes sont équivalentes:*

a) $f < q_2, q_1 > = f$ *(condition de commutativité pour f)*$(< q_2, q_1 > \neq < q_1, q_2 > = 1_{A\times A})$;

b) *$\forall X \in |\mathcal{C}|$, l'opération $*_X$ définie sur $\mathcal{C}(X, A)$ est commutative;*

c) *l'opération $*_{A\times A}$ définie sur $\mathcal{C}(A \times A, A)$ satisfait à l'égalité:* $q_1 *_{A\times A} q_2 = q_2 *_{A\times A} q_1$.

Soit 1 l'objet final de $\mathcal{C}$ et les produits