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ON SOME FUNDAMENTAL DISCRETE INEQUALITIES OF THE WENDROFF TYPE

BY

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1. **Introduction.** In recent years several authors have successfully applied discrete analogue of Gronwall's inequality [3] and its variants to the study of many problems in the theory of finite difference equations and numerical analysis [4] — [11]. It is perhaps unfortunate that the discrete analogue of the two independent variable generalization of the Gronwall's inequality due to Wendroff given in [1, p. 154] seems to have been overlooked by the researches in discrete inequalities and their applications. Our objective here is to establish a number of discrete inequalities of the Wendroff type which can be used in the analysis of various problems in the theory of finite difference equations involving two independent variables.

2. **Main Results.** Before giving the main results in this section, we first recollect a few of the basic notions and definitions from [7] — [11]. Let N be the set of points $n_0 + k(k = 0, 1, 2, \ldots)$, where $n_0 \geqslant 0$ is a given integer. The expression $z(n_0) + \sum_{s=n_0}^{n-1} b(s)$ represents a solution of the linear difference equation $\Delta z(n) = b(n)$ for all $n \in N$, where Δ is the operator defined by $\Delta z(n) = z(n+1) - z(n)$. It is supposed that $\sum_{s=n_0}^{n_0-1} b(s) = 0$. The expression $z(n_0) \prod_{s=n_0}^{n-1} c(s)$ represents a solution of the linear difference equation $z(n+1) = c(n)\, z(n)$ for all $n \in N$. It is supposed that $\prod_{s=n_0}^{n_0-1} c(s) = 1$. In our subsequent discussion we also use the following notions of the operators

$$\Delta_x[z(x, y)] = \Delta z_x(x, y) = z(x+1, y) - z(x, y) \text{ and}$$
$$\Delta_y[z(x, y)] = \Delta z_y(x, y) = z(x, y+1) - z(x, y), \text{ and}$$
$$\Delta_y[\Delta u_x(x, y)] = \Delta^2 u_{xy}(x, y) = \Delta u_x(x, y+1) - \Delta u_x(x, y),$$

and so on. We frequently use the letters x and y to denote the two independent variables which are the members of N.

A useful discrete version of the Wendroff's inequality is embodied in the following theorem.

Theorem 1. *Let $u(x, y)$ and $b(x, y)$ be real-valued nonnegative functions defined for $x \geq 0$, $y \geq 0$, and let $a(x, y)$ be positive, nondecreasing in both the variables, and defined for $x \geq 0$, $y \geq 0$, for which the inequality*

$$u(x, y) \leqslant a(x, y) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} b(s, t) u(s, t), \tag{1}$$

holds for $x \geq 0$, $y \geq 0$. Then

$$u(x, y) \leqslant a(x, y) \prod_{s=0}^{x-1} \left[1 + \sum_{t=0}^{y-1} b(s, t) \right], \tag{2}$$

for all $x \geq 0$, $y \geq 0$.

Proof. Since $a(x, y)$ is positive, nondecreasing, we observe from (1) that

$$\frac{u(x, y)}{a(x, y)} \leqslant 1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} b(s, t) \frac{u(s, t)}{a(x, y)} \leqslant 1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} b(s, t) \frac{u(s, t)}{a(s, t)}. \tag{3}$$

Define a function $m(x, y)$ by

$$m(x, y) = 1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} b(s, t) \frac{u(s, t)}{a(s, t)}, \quad m(x, 0) = m(0, y) = 1,$$

then

$$m(x+1, y) - m(x, y) = \sum_{t=0}^{y-1} b(x, t) \frac{u(x, t)}{a(x, t)}, \tag{4}$$

and from (4) we have

$$m(x+1, y+1) - m(x, y+1) - m(x+1, y) + m(x, y) = b(x, y) \frac{u(x, y)}{a(x, y)},$$

which in view of (3) implies

$$\Delta m_x(x, y+1) - \Delta m_x(x, y) \leqslant b(x, y)\, m(x, y). \tag{5}$$

From the definition of $m(x, y)$ we observe that $m(x, y) \leqslant m(x, y+1)$, for $x \geq 0$, $y \geq 0$. Using this fact in (5) we have $\Delta m_x(x, y+1) - \Delta m_x(x, y) \leqslant b(x, y)\, m(x, y+1)$, i.e.

$$\frac{\Delta m_x(x, y+1)}{m(x, y+1)} - \frac{\Delta m_x(x, y)}{m(x, y+1)} \leqslant b(x, y). \tag{6}$$

From (6) we observe that

$$\frac{\Delta m_x(x, y+1)}{m(x, y+1)} - \frac{\Delta m_x(x, y)}{m(x, y)} \leqslant b(x, y), \tag{7}$$

Now keeping x fixed in (7) and substituting $y = 0, 1, 2, \ldots, y-1$, we obtain the bound of $\Delta m_x(x, y)/m(x, y)$ such that

$$\frac{\Delta m_x(x, y)}{m(x, y)} \leqslant \sum_{t=0}^{y-1} b(x, t). \tag{8}$$

From (8) we observe that

$$m(x+1, y) \leqslant m(x, y) \left[1 + \sum_{t=0}^{y-1} b(x, t) \right], \tag{9}$$

for $x \geq 0$, $y \geq 0$. Now keeping y fixed in (9) and substituting $x = 0, 1, 2, \ldots, x-1$, we obtain the estimation for $m(x, y)$ such that

$$m(x, y) \leqslant \prod_{s=0}^{x-1} \left[1 + \sum_{t=0}^{y-1} b(s, t) \right].$$

Substituting this bound on $m(x, y)$ in (3) we obtain the desired bound in(2).

We now apply Theorem 1 to establish the following interesting and useful discrete inequality in two independent variables.

Theorem 2. *Let $u(x, y)$, $b(x, y)$ and $c(x, y)$ be real-valued nonnegative functions defined for $x \geq 0$, $y \geq 0$; and let $W(u)$ be continuous, positive strictly increasing function on $I = [u_0, \infty)$, $u_0 > 0$; suppose further that the inequality*

$$u(x, y) \leqslant M + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} b(s, t)\, u(s, t) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} c(s, t)\, W(u(s, t)), \tag{10}$$

is satisfied for $x \geq 0$, $y \geq 0$, where $M > 0$ is a constant. Then for $0 \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$

$$u(x, y) \leqslant \Omega^{-1}\left[\Omega(M) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} c(s, t)\, W\left(\prod_{r=0}^{s-1} \left[1 + \sum_{n=0}^{t-1} b(r, n)\right]\right)\right] \cdot \prod_{s=0}^{x-1} \left[1 + \sum_{t=0}^{y-1} b(s, t)\right] \tag{11}$$

where

$$\Omega(r) = \int_{r_0}^{r} \frac{dr}{W(s)}, \quad r \geqslant r_0 > 0, \tag{12}$$

Ω^{-1} is the inverse function of Ω, and

$$\Omega(M) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} c(s, t)\, W\left(\prod_{r=0}^{s-1}\left[1 + \sum_{n=0}^{t-1} b(r, n)\right]\right) \in \mathrm{Dom}\,(\Omega^{-1}),$$

for all x, y lying in the subintervals $0 \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$ of N.

Proof. Define

$$a(x, y) = M + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} c(s, t)\, W(u(s, t)), \quad a(x, 0) = a(0, y) = M, \tag{13}$$

then (10) can be restated as $u(x, y) \leqslant a(x, y) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} b(s, t)\, u(s, t)$. Since $a(x, y)$ is positive, nondecreasing, we have from Theorem 1

$$u(x, y) \leqslant a(x, y) \prod_{s=0}^{x-1} \left[1 + \sum_{t=0}^{y-1} b(s, t) \right]. \tag{14}$$

Further, $W(u(x,y)) \leqslant W(a(x,y))\, W\left(\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1} b(s,t)\right]\right)$, since W is submultiplicative. Hence,

$$\frac{c(x,y)\, W(u(x,y))}{W(a(x,y))} \leqslant c(x,y)\, W\left(\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1} b(s,t)\right]\right).$$

Because of (13), this reduces to

$$\frac{\Delta a_x(x,y+1)-\Delta a_x(x,y)}{W(a(x,y))} \leqslant c(x,y)\, W\left(\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1} b(s,t)\right]\right). \tag{15}$$

From (15) we observe that

$$\frac{\Delta a_x(x,y+1)}{W(a(x,y+1))}-\frac{\Delta a_x(x,y)}{W(a(x,y))} \leqslant c(x,y)\, W\left(\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1} b(s,t)\right]\right). \tag{16}$$

Now keeping x fixed in (16) and substituting $y=0,1,2,\ldots,y-1$, we obtain the estimate for $\Delta a_x(x,y)/W(a(x,y))$ such that

$$\frac{\Delta a_x(x,y)}{W(a(x,y))} \leqslant \sum_{t=0}^{y-1} c(x,t)\, W\left(\prod_{s=0}^{x-1}\left[1+\sum_{n=0}^{t-1} b(s,n)\right]\right). \tag{17}$$

From (12) and (17) we have

$$\Omega(a(x+1,y))-\Omega(a(x,y))=\int_{a(x,y)}^{a(x+1,y)}\frac{ds}{W(s)} \leqslant \frac{\Delta a_x(x,y)}{W(a(x,y))} \leqslant \sum_{t=0}^{y-1} c(x,t)\, W\left(\prod_{s=0}^{x-1}\left[1+\sum_{n=0}^{t-1} b(s,n)\right]\right).$$

Now keeping y fixed and substituting $x=0,1,\ldots,x-1$ in the above inequality, we obtain

$$\Omega(a(x,y)) \leqslant \Omega(M)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} c(s,t)\, W\left(\prod_{r=0}^{s-1}\left[1+\sum_{n=0}^{t-1} b(r,n)\right]\right). \tag{18}$$

The desired bound in (11), now follows by substituting the bound on $a(x,y)$ from (18) in (14). The subintervals of N for x and y are obvious.

We next establish the following two independent variable generalization of the discrete inequality recently established by Pachpatte [8, Theorem 1].

Theorem 3. *Let $u(x,y)$, $b(x,y)$, and $c(x,y)$ be as defined in Theorem 2; and let $a(x,y)$ be as defined in Theorem 1, for which the inequality*

$$u(x,y) \leqslant a(x,y)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,t)\left[u(s,t)+\sum_{r=0}^{s-1}\sum_{n=0}^{t-1} c(r,n)\, u(r,n)\right], \tag{19}$$

holds for $x \geqslant 0$, $y \geqslant 0$. Then

$$u(x,y) \leqslant a(x,y)\left[1+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,t)\prod_{r=0}^{s-1}\left[1+\sum_{n=0}^{t-1}(b(r,n)+c(r,n))\right]\right], \tag{20}$$

for all $x \geqslant 0$, $y \geqslant 0$.

Proof. Since $a(x,y)$ is positive, nondecreasing, we observe from (19) that

$$\frac{u(x,y)}{a(x,y)} \leqslant 1+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,t)\left[\frac{u(s,t)}{a(s,t)}+\sum_{r=0}^{s-1}\sum_{n=0}^{t-1} c(r,n)\frac{u(r,n)}{a(r,n)}\right]. \tag{21}$$

Define a function $m(x,y)$ by

$$m(x,y)=1+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,t)\left[\frac{u(s,t)}{a(s,t)}+\sum_{r=0}^{s-1}\sum_{n=0}^{t-1} c(r,n)\frac{u(r,n)}{a(r,n)}\right],$$

$$m(x,0)=m(0,y)=1,$$

then

$$m(x+1,y)-m(x,y)=\sum_{t=0}^{y-1} b(x,t)\left[\frac{u(x,t)}{a(x,t)}+\sum_{r=0}^{x-1}\sum_{n=0}^{t-1} c(r,n)\frac{u(r,n)}{a(r,n)}\right] \tag{22}$$

and from (22) we have

$$m(x+1,y+1)-m(x,y+1)-m(x+1,y)+m(x,y)=$$

$$=b(x,y)\left[\frac{u(x,y)}{a(x,y)}+\sum_{r=0}^{x-1}\sum_{n=0}^{y-1} c(r,n)\frac{u(r,n)}{a(r,n)}\right],$$

which in view of (21) implies

$$\Delta m_x(x,y+1)-\Delta m_x(x,y) \leqslant b(x,y)\left[m(x,y)+\sum_{r=0}^{x-1}\sum_{n=0}^{y-1} c(r,n)\, m(r,n)\right]. \tag{23}$$

If we put

$$v(x,y)=m(x,y)+\sum_{r=0}^{x-1}\sum_{n=0}^{y-1} c(r,n)\, m(r,n),\quad v(x,0)=v(0,y)=1, \tag{24}$$

then

$$v(x+1,y)-v(x,y)=\Delta m_x(x,y)+\sum_{n=0}^{y-1} c(x,n)\, m(x,n), \tag{25}$$

and from (25) we have

$$\Delta v_x(x,y+1)-\Delta v_x(x,y)=\Delta m_x(x,y+1)-\Delta m_x(x,y)+c(x,y)\, m(x,y). \tag{26}$$

Using the facts that $\Delta m_x(x,y+1)-\Delta m_x(x,y) \leqslant b(x,y)\, v(x,y)$ from (23) and $m(x,y) \leqslant v(x,y)$ from (24) in (26), we see that the inequality

$$\Delta v_x(x,y+1)-\Delta v_x(x,y) \leqslant [b(x,y)+c(x,y)]\, v(x,y), \tag{27}$$

is satisfied for $x \geqslant 0$, $y \geqslant 0$. Now by following the similar argument as in the proof of Theorem 1, we obtain the estimate for $v(x,y)$, such that

$v(x, y) \leqslant \prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1}[b(s, t) + c(s, t)]\right]$. Substituting this bound on $v(x, y)$ in (23) we have

$$(28) \qquad \Delta m_x(x, y+1) - \Delta m_x(x, y) \leqslant b(x, y) \prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1} b(s,t) + c(s,t)]\right].$$

Now keeping x fixed in (28) and substituting $y = 0, 1, 2, \ldots, y-1$, we obtain the estimation for $\Delta m_x(x, y)$ such that

$$(29) \qquad \Delta m_x(x, y) \leqslant \sum_{t=0}^{y-1} b(x,t) \prod_{s=0}^{x-1}\left[1 + \sum_{n=0}^{t-1}[b(s, n) + c(s, n)]\right].$$

Keeping y fixed in (29) and substituting $x = 0, 1, 2, \ldots, x-1$, we obtain the estimate for $m(x, y)$ such that

$$m(x, y) \leqslant 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t) \prod_{r=0}^{s-1}\left[1 + \sum_{n=0}^{t-1}[b(r, n) + c(r, n)]\right].$$

Substituting this bound on $m(x, y)$ in (21), we obtain the desired bound in (20).

We now apply Theorem 3 to establish the following more general inequality which can be used in some applications.

Theorem 4. *Let $u(x, y)$, $b(x, y)$, $c(x, y)$, and $k(x, y)$ be real-valued nonnegative functions defined for $x \geqslant 0$, $y \geqslant 0$; let $W(u)$ be the same function as defined in Theorem 2, and suppose further that the inequality*

$$(30) \qquad \begin{aligned} u(x, y) \leqslant M + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t)\left[u(s, t) + \sum_{r=0}^{s-1}\sum_{n=0}^{t-1} c(r, n)\, u(r, n)\right] + \\ + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} k(s, t)\, W(u(s, t)) \end{aligned}$$

is satisfied for $x \geqslant 0$, $y \geqslant 0$, where $M > 0$ is a constant. Then for $0 \leqslant x \leqslant x_2$, $0 \leqslant y \leqslant y_2$,

$$(31) \qquad u(x, y) \leqslant \Omega^{-1}\left[\Omega(M) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} k(s, t)\, W(P(s, t))\right] P(x, y),$$

where Ω, Ω^{-1} are as defined in Theorem 2, and

$$(32) \qquad P(x, y) = 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t) \prod_{r=0}^{s-1}\left[1 + \sum_{n=0}^{t-1}[b(r, n) + c(r, n)]\right],$$

and

$$\Omega(M) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} k(s, t)\, W(P(s, t)) \in \mathrm{Dom}\ (\Omega^{-1}),$$

for all x, y lying in the subintervals $0 \leqslant x \leqslant x_0$, $0 \leqslant y \leqslant y_2$ of N.

The proof of this theorem follows by the similar argument as in the proof of Theorem 2, by making use of Theorem 3. We omit the details.

3. **Further Inequalities.** In this section we establish some new discrete inequalities of the Wendroff type, involving two nonlinear functions on the right hand side. For this purpose we require the class of functions S as defined in [12].

A function $\Phi : [0, \infty) \to [0, \infty)$ is said to belong to the class S, if

(i) $\Phi(u)$ is positive, nondecreasing and continuous for $u \geqslant 0$,

$$\text{(ii)} \qquad \frac{1}{v}\,\Phi(u) \leqslant \Phi\left(\frac{u}{v}\right), \quad u > 0, \ v \geqslant 1.$$

Before giving the main results in this section, we first establish the following Wendroff type discrete inequalities which are useful in our further discussion.

Theorem 5. *Let $u(x, y)$, $a(x, y)$, and $b(x, y)$ be as defined in Theorem 1; let $\Phi \in S$; and let $H(u)$ be a positive, monotonic, nondecreasing subadditive and submultiplicative function defined on I, and H^{-1} denotes the inverse function of H, for which the inequality*

$$(33) \qquad u(x, y) \leqslant a(x, y) + H^{-1}\left[\Phi\left(\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t)\, H(u(s, t))\right)\right],$$

holds for $x \geqslant 0$, $y \geqslant 0$. Then for $0 \leqslant x \leqslant x_3$, $0 \leqslant y \leqslant y_3$,

$$(34) \qquad u(x, y) \leqslant a(x, y)\, H^{-1}\left[1 + \Phi\left(G^{-1}\left[G(0) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t)\right]\right)\right],$$

where

$$(35) \qquad G(r) = \int_{r_0}^{r} \frac{ds}{1 + \Phi(s)}, \quad r \geqslant r_0 > 0,$$

G^{-1} is the inverse of G, and

$$G(0) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t) \in \mathrm{Dom}\ (G^{-1}),$$

for all x, y lying in the subintervals $0 \leqslant x \leqslant x_3$, $0 \leqslant y \leqslant y_3$ of N.

Proof. Since H is subadditive and submultiplicative, we have from (33)

$$H(u(x, y)) \leqslant H(a(x, y)) + \Phi\left(\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t)\ H(u(s, t))\right).$$

Since $H(a(x, y))$ is positive, monotonic nondecreasing and $\Phi \in S$, we have

$$(36) \qquad \frac{H(u(x, y))}{H(a(x, y))} \leqslant 1 + \Phi\left(\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t)\ \frac{H(u(s, t))}{H(a(s, t))}\right).$$

Define a function $v(x, y)$ such that

$$v(x, y) = \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t)\ \frac{H(u(s, t))}{H(a(s, t))}, \quad v(x, 0) = v(0, y) = 0,$$

then

$$v(x+1,\ y) - v(x,\ y) = \sum_{t=0}^{y-1} b(x,\ t)\,\frac{H(u(x,\ t))}{H(a(x,\ t))}, \tag{37}$$

and from (37) we have $v(x+1,\ y+1) - v(x,\ y+1) - v(x+1,\ y) + v(x,\ y) = b(x,\ y)\,H(u(x,\ y))/H(a(x,\ y))$, which in view of (36) implies
$\Delta v_x(x,\ y+1) - \Delta v_x(x,\ y \leqslant b(x,\ y)\,[1 + \Phi\,(v(x,\ y))\,]$.
Now by following the similar argument as in the proof of Theorem 2 we obtain the estimate for $v(x, y)$ such that

$$v(x,\ y) \leqslant G^{-1}\left[G(0) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,\ t)\right].$$

Substituting this bound on $v(x, y)$ in (36), taking $H(a(x,\ y))$ in (36) on the right side, and then applying H^{-1} to both sides, we obtain the desired bound in (34).

Theorem 6. *Let $u(x,\ y)$, $a(x,\ y)$ and $b(x,\ y)$ be as defined in Theorem 1; and $H \in S$; and suppose further that the inequality*

$$u(x, y) \leqslant a(x, y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,\ t)\left((u(s,\ t) + \sum_{r=0}^{s-1}\sum_{n=0}^{t-1} b(r,\ n)\,H(u(r,\ n))\right), \tag{38}$$

is satisfied for $x \geqslant 0$, $y \geqslant 0$. Then for $0 \leqslant x \leqslant x_4$, $0 \leqslant y \leqslant y_4$

$$u(x, y) \leqslant a(x, y)\left[1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t)\,E^{-1}\left[E(1) + \sum_{r=0}^{s-1}\sum_{n=0}^{t-1} b(r,\ n)\right]\right], \tag{39}$$

where

$$E(r) = \int_{r_0}^{r} \frac{ds}{s + H(s)}, \quad r \geqslant r_0 > 0, \tag{40}$$

E^{-1} is the inverse function of E, and $E(1) + \sum_{r=0}^{x-1}\sum_{n=0}^{y-1} b(r,\ n) \in$ Dom (E^{-1}), for all x, y lying in the subintervals $0 \leqslant x \leqslant x_4$, $0 \leqslant y \leqslant y_4$ of N.

The proof of this theorem follows by a similar argument as in the proofs of Theorem 3 and Theorem 5 with suitable modifications. We omit the details.

As an applications of Theorem 5 and 6 we establish the foollowing more general discrete inequalities in two independent variables which may be convenient in some applications.

Theorem 7. *Let $u(x,\ y)$, $b(x, y)$, and $c(x,\ y)$ be as defined in Theorem 2; let $\Phi \in S$; let H, H^{-1} be defined as in Theorem 5; let W be the same function as defined in Theoreme 2, and suppose further that the inequality*

$$u(x, y) \leqslant M + H^{-1}\left[\Phi\left(\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,t)\ H(u(s,t))\right)\right] + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} c(s,\ t)\,W(u(s,\ t)), \tag{41}$$

is satisfied for $x \geqslant 0$, $y \geqslant 0$, where $M > 0$ ia a constant. Then for $0 \leqslant x \leqslant x_5$, $0 \leqslant y \leqslant y_5$,

$$u(x,\ y) \leqslant \Omega^{-1}\left[\Omega\,(M) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} c(s,\ t)\ W(Q(s,\ t))\right].\ Q(x,\ y), \tag{42}$$

where

$$Q(x,\ y) = H^{-1}\left[1 + \Phi\left(G^{-1}\left[G(0) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,\ t)\right]\right)\right], \tag{43}$$

in which Ω, Ω^{-1} and G, G^{-1} are as defined in Theorem 2 and Theorem 5, and $G(0) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,\ t) \in$ Dom (G^{-1}), and $\Omega(M) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} c(s,\ t)\ W\,(Q(s,\ t)) \in$ Dom (Ω^{-1}), *for all x, y lying in the subintervals $0 \leqslant x \leqslant x_5$, $0 \leqslant y \leqslant y_5$ of N.*

The details of the proof of this theorem follows by an argument similar to that in the proof of Theorem 2, by making use of Theorem 5, We omit the details.

Theorem 8. *Let $u(x,\ y)$, $b(x,\ y)$, and $c(x,\ y)$ be as defined in Theorem 3; $H \in S$; and W is the same function as defined in Theorem 2, and suppose further that the inequality*

$$u(x,\ y) \leqslant M + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,\ t)\left(u(s,\ t) + \sum_{r=0}^{s-1}\sum_{n=0}^{t-1} b(r,n)\ H(u(r,n))\right) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} c(s,\ t)\ W\,(u(s,\ t)), \tag{44}$$

is satisfied for $x \geqslant 0$, $y \geqslant 0$, where $M > 0$ is a constant. Then for $0 \leqslant x \leqslant x_6$, $0 \leqslant y \leqslant y_6$,

$$u(x,\ y) \leqslant \Omega^{-1}\left[\Omega(M) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} c(s,\ t)\ W(R(s,\ t))\right].\ R(x,\ y), \tag{45}$$

where

$$R(x,\ y) = 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,\ t)\ E^{-1}\left[E(1) + \sum_{r=0}^{s-1}\sum_{n=0}^{t-1} b(r,\ n)\right], \tag{46}$$

in which Ω, Ω^{-1} and E, E^{-1} are as defined in Theorem 2 and Theorem 6, and $E(1) + \sum_{r=0}^{x-1}\sum_{n=0}^{y-1} b(r,\ n) \in$ Dom (E^{-1}), *and* $\Omega(M) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} c(s,t)\ W(R(s,t)) \in$ Dom (Ω^{-1}), *for all x, y lying in the subintervals $0 \leqslant x \leqslant x_6$, $0 \leqslant y \leqslant y_6$ of N.*

The proof of this theorem proceeds much as that of Theorem 2, by making use of Theorem 6, and we leave the details to the reader.

4. Some Applications. In this section we indicate some applications of our results to obtain the bounds on the solutions of some finite difference equations involving two independent variables. We believe that the discrete inequalities established in this paper may be used in the theory of finite difference equations involving two independent variables in essentially the same capacity as the inequalities of the Gronwall [3] and Bihari type [2] are used in the theory of ordinary differential and integral equations. To illustrate the application of our Theorem 4 we establish the bound on the solution of a summary difference equation of the form

$$\Delta^2 u_{xy} = f\,[x,\ y,\ u] + F\left[x,\ y,\ u,\ \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h(x,\ y,\ s,\ t,\ u)\right], \tag{47}$$

with the given boundary conditions $u(x, 0) = g(x)$, $u(0, y) = e(y)$, $u(0,0) = 0$, where all the functions are defined on their respective domains of definitions and

$$|f[x, y, u]| \le k(x, y)\, W(|u|), \tag{48}$$

$$|F[x, y, u, v]| \le b(x, y)\, [\, |u| + |v|\,], \tag{49}$$

$$|h(x, y, s, t, u)| \le c(s, t)\, |u|, \tag{50}$$

for $x \ge 0$, $y \ge 0$, where W, $b(x, y)$, $c(x, y)$, and $k(x, y)$ are as defined in Theorem 4. By using the given boundary conditions, equation (47) can be represented by equivalent summary difference equation

$$u(x, y) = g(x) + e(y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} f[s, t, u(s, t)] + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} F\left[s, t, u(s, t), \sum_{r=0}^{s-1}\sum_{n=0}^{t-1} h(s, t, r, n, u(r, n))\right]. \tag{51}$$

If $|g(x)| + |e(y)| \le M$ (where $M > 0$ is a constant) then using (48), (49) and (50) in (51) and then applying Theorem 4, we obtain the bound on the solution $u(x, y)$ of (47).

In concluding this paper we also note that the discrete inequalities established in Theorem 2 and Theorem 8 can be used to obtain the bounds on the solutions of equations (47), when $h \equiv 0$ and $|h(x, y, s, t, u)| \le H(|u|)$ (where H is as defined in Theorem 8) respectively, under the conditions (48) and (49) on the functions f and F involved in (47).

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AN EXTENSION OF A FEASIBLE DIRECTIONS ALGORITHM TO MINIMIZATION OF NONDIFFERENTIABLE CONVEX FUNCTIONALS

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Many problems of mathematical programming, optimal control, game theory etc. involve the optimization of convex, not everywhere differentiable functionals. The well-known feasible directions methods [10] have been introduced basically for optimization of differentiable functionals and they are essentially linearization methods. When nondifferentiable convex functionals are involved, some of these methods can be easily adapted by using the directional differentials instead of the gradients and their convergence can be proved under appropriate hypotheses. Apparently these algorithms are not any longer linearization methods as in the differentiable case, because the directional differentials are not linear with respect to the direction at which they are calculated. However, for quite a large class of convex nondiferentiable functionals, the minimization problem which has to be solved at each step of the algorithm is still linear. Such is the case of some extremal-value functionals considered by W. Hogan [4], E. S. Levitin [9]. We shall discuss some other examples in the last part of the paper.

In this work we consider an extension of an algorithm studied previously by G. Zoutendijk [11] and E. Polak [10] for constrained minimization of convex functionals, for which we prove a theorem of convergence and consider some applications. In a previous paper [7], we have studied another feasible directions algorithm for which the convergence was proved in a similar manner. However the present is practically more eficient, because the number of constraints of the problem which has to be solved at each iteration, is smaller.

Anoptimality criterion and the description of the method. Let $J_i : E \to R$, $i = 0, 1, \dots, m$, be proper convex, lower semicontinuous functionals defined on the reflexive Banach space E, such that there exists $u_0 \in U = \{v \in E / J_i(v) \le 0,\ i = 1, \dots, m\}$ for which the set $W(u_0) = \{v \in U / J_0(v) \le J_0(u_0)\}$ is nonempty and bounded. Then the problem

$$\min_{v \in U} J_0(v) = J_0(u) \tag{1}$$

has at least a solution $u \in U \cap W(u_0)$. We assume also the usual hypothesis