with the given boundary conditions $u(x, 0) = g(x)$, $u(0, y) = e(y)$, $u(0,0) = 0$, where all the functions are defined on their respective domains of definitions and

$$|f[x, y, u]| \leqslant k(x, y)\, W(|u|), \tag{48}$$

$$|F[x, y, u, v]| \leqslant b(x, y)\,[\,|u| + |v|\,], \tag{49}$$

$$|h(x, y, s, t, u)| \leqslant c(s, t)\,|u|, \tag{50}$$

for $x \geqslant 0$, $y \geqslant 0$, where W, $b(x, y)$, $c(x, y)$, and $k(x, y)$ are as defined in Theorem 4. By using the given boundary conditions, equation (47) can be represented by equivalent summary difference equation

$$u(x, y) = g(x) + e(y) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} f[s, t, u(s, t)] + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} F\left[s, t, u(s, t), \sum_{r=0}^{s-1}\sum_{n=0}^{t-1} h(s, t, r, n, u(r, n))\right]. \tag{51}$$

If $|g(x)| + |e(y)| \leqslant M$ (where $M > 0$ is a constant) then using (48), (49) and (50) in (51) and then applying Theorem 4, we obtain the bound on the solution $u(x, y)$ of (47).

In concluding this paper we also note that the discrete inequalities established in Theorem 2 and Theorem 8 can be used to obtain the bounds on the solutions of equations (47), when $h \equiv 0$ and $|h(x, y, s, t, u)| \leqslant H(|u|)$ (where H is as defined in Theorem 8) respectively, under the conditions (48) and (49) on the functions f and F involved in (47).

REFERENCES

1. Beckenbach E. F. and Bellman R. — *Inequalities*, Springer–Verlag, Berlin, 1961.
2. Bihari I. — *A generalization of lemma of Bellman and its applications to uniqueness problems of differential equations*, Acta. Math. Acad. Sci. Hung. 7 (1976), 81–94.
3. Gronwall T. H. — *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. Math. 20 (1919), 292–296.
4. Henrici P. — *Discrete variable method in ordinary differential equations*, Wiley, New York, 1962.
5. Hull T. E. and Luxemburg W. A. J. — *Numerical methods and existence theorems for ordinary differential equations*, Numer. Math. 2 (1960), 30–41.
6. Jones G. S. — *Fundamental inequalities for discrete and discontinuous functional equation*, J. Soc. Ind. Appl. Math. 12 (1964), 43–57.
7. Pachpatte B. G. — *On the discrete generalizations of Gronwall's inequality*, J. Indian Math. Soc. 37 (1973), 147–156.
8. Pachpatte B. G. — *Finite difference inequalities and their applications*, Proc. Nat. Acad. Sci. India 43 (A) (1973), 348–356.
9. Pachpatte B. G. — *On discrete inequalities related to Gronwall's inequality*, Proc. Indian Acad. Sci. 85 (A) (1977), 26–40.
10. Pachpatte B. G. — *A note on some fundamental discrete inequalities of the Gronwall-Bellman type*, Bull. Inst. Math. Acad. Sinica 5 (1977), 121–128.
11. Pachpatte B. G. — *On some nonlinear discrete inequalities of Gronwall type*. Bull. Inst. Math. Acad. Sinica 5 (1977), 305–315.
12. Pachpatte, B.G. — *On some integral inequalities similar to Bellman-Bihari inequalities*, J. Math. Anal. Appl. 49 (1975), 794–802.

Received 1.IV.1978

Department of Mathematics
Marathwada University
Aurangabad — 431004
(Maharashtra), India

Analele științifice ale Universității „Al. I. Cuza" Iași
Tomul XXVI, s. I a, 1980, fasc. 1

AN EXTENSION OF A FEASIBLE DIRECTIONS ALGORITHM TO MINIMIZATION OF NONDIFFERENTIABLE CONVEX FUNCTIONALS

BY

C. ILIOI

Many problems of mathematical programming, optimal control, game theory etc. involve the optimization of convex, not everywhere differentiable functionals. The well-known feasible directions methods [10] have been introduced basically for optimization of differentiable functionals and they are essentially linearization methods. When nondifferentiable convex functionals are involved, some of these methods can be easily adapted by using the directional differentials instead of the gradients and their convergence can be proved under appropriate hypotheses. Apparently these algorithms are not any longer linearization methods as in the differentiable case, because the directional differentials are not linear with respect to the direction at which they are calculated. However, for quite a large class of convex nondiferentiable functionals, the minimization problem which has to be solved at each step of the algorithm is still linear. Such is the case of some extremal-value functionals considered by W. Hogan [4], E. S. Levitin [9]. We shall discuss some other examples in the last part of the paper.

In this work we consider an extension of an algorithm studied previously by G. Zoutendijk [11] and E. Polak [10] for constrained minimization of convex functionals, for which we prove a theorem of convergence and consider some applications. In a previous paper [7], we have studied another feasible directions algorithm for which the convergence was proved in a similar manner. However the present is practically more eficient, because the number of constraints of the problem which has to be solved at each iteration, is smaller.

Anoptimality criterion and the description of the method. Let $J_i : E \to \bar{R}$, $i = 0, 1, \ldots, m$, be proper convex, lower semicontinuous functionals defined on the reflexive Banach space E, such that there exists $u_0 \in U = \{v \in E / J_i(v) \leqslant 0,\ i = 1, \ldots, m\}$ for which the set $W(u_0) = \{v \in U / J_0(v) \leqslant J_0(u_0)\}$ is nonempty and bounded. Then the problem

$$\min_{v \in U} J_0(v) = J_0(u) \tag{1}$$

has at least a solution $u \in U \cap W(u_0)$. We assume also the usual hypothesis

$$\exists z \in E,\ J_i(z) < 0,\ i = 1, 2, \ldots, m. \tag{2}$$

Now we can prove an optimality criterion for the problem (1) extending a similar result of Zoutendijk [11], which is essential for the construction of the algorithm.

Lemma 1. *Under the above hypotheses, J_0 achieves its infimum on the set U at $u \in U$, if and only if the relation*

$$\min_{h\in S}\ \max_{i\in I_0(u)} J'_{i+}(u, h) = 0, \quad \text{for } I_0(u) = \{0\} \cup \{i \neq 0,\ J_i(u) = 0\}, \tag{3}$$

holds for any set $S \subseteq E$ containing 0.

Proof. The relation (3) is equivalent to

$$\max\{J'_{i+}(u, h),\ i \in I_0(u)\} \geq 0,\ \forall h \in E, \tag{4}$$

because for $h = 0$ this maximum is obviously zero.

Let us suppose that $u \in U$ satisfies (4). Then, for any $v \in U$, $0 \geq J_i(v) \geq J_i(u) + J'_{i+}(u, v-u)$, $i = 1, 2, \ldots, m$. Therefore $J'_{i+}(u, v-u) \leq 0$ for each $i \in I_0(u)$, $i \neq 0$. Then, according to (4), we obtain $J'_{0+}(u, v-u) \geq 0$, $\forall v \in U$, which implies that u is a minimizing point for J_0 on the set U.

Conversely, let $u \in U$ be a minimizing point for J_0 on U and suppose that there exists $h \in E$ such that $J'_{i+}(u, h) < 0$, $\forall i \in I_0(u)$. Then, according to the definition of $J'_{i+}(u, h)$, for each $i \in I_0(u)$, $i \neq 0$, there exists some $\theta_i > 0$ such that for $0 < \varepsilon < -J'_{i+}(u, h)$, we have

$$0 < J_i(u + \theta_i h) - J_i(u) - \theta_i J'_{i+}(u, h) \leq \theta_i \varepsilon,$$

where from $J_i(u + \theta_i h) < 0$. For the other constraints J_i with $i \notin I_0(u)$, the continuity implies also the existence of $\theta_i > 0$, so that $J_i(u + \theta_i h) \leq 0$. Therefore, for $\theta = \min\{\theta_i,\ i = 1, 2, \ldots, m\}$, we have $u + \theta h \in U$. Since u is a minimizing point on U, the inequality $J'_{0+}(u, \theta h) \geq 0$ follows, contrary to the above assumption. Hence that supposition is not true, so that the relation (4) is proved.

Now we shall introduce the feasible directions algorithm of the form

$$u_{n+1} = u_n + \lambda_n h_n,\quad n = 0, 1, \ldots, \tag{5}$$

where for each n, $u_n \in U$, $\lambda_n > 0$ and $h_n \in E$, with $\|h_n\| = 1$ is a descent direction of J_0 at u_n. For the calculation of h_n, let us consider certain bounded convex closed set S. For any $v \in U$, let us denote

$$y_0(v) = \min_{x\in S}\ \max_{i\in I_0(v)} J'_{i+}(v, x) \tag{6}$$

and by $x_0(v) \in S$ a solution of this minimization problem. It is easy to see that $y_0(v) \leq 0$, and, according to Lemma 1, the equality holds only when v is a minimizing point of J_0 on U. Then, apparently, a natural choice for the directions h_n at each iteration would be $h_n = x_0(u_n)/\|x_0(u_n)\|$ for which $J'_{0+}(u_n, h_n) \leq y_0(u_n)/\|x_0(u_n)\| \leq 0$. But this alternative leads to an algorithm which may not converge even in the differentiable case [11]. In order to obtain convergent methods, we have to introduce an ε-perturbation technique.

Let $\varepsilon > 0$ and denote, for each $v \in U$,

$$I_\varepsilon(v) = \{0\} \cup \{i / J_i(v) + \varepsilon \geq 0\}, \tag{7}$$

$$y_\varepsilon(v) = \min_{x\in S}\ \max_{i\in I_\varepsilon(v)} J'_{i+}(v, x), \tag{8}$$

and by $x_\varepsilon(v) \in S$ a solution of (8).

Lemma 2. *If $v \in U$ is not a solution of the problem (1), then there exists $\varepsilon > 0$ such that*

$$y_\varepsilon(v) \leq -\varepsilon. \tag{9}$$

Proof. According to Lemma 1, if $v \in U$ is not a minimizing point of J_0 on U, then $y_0(v) < 0$. It is easy to verify that, with the notation (7), $I_0(v) \subseteq I_\rho(v)$, for any $\rho > 0$. Now, suppose that there exists $i \notin I_0(v)$. Then for any $0 < \rho < -\max\{J_i(v)/i \notin I_0(v)\} = \bar{\rho}$, we have $J_i(v) + \rho < 0$, for all $i \notin I_0(v)$. Therefore $I_\rho(v) = I_0(v)$ and by (8), $y_\rho(v) = y_0(v)$ for $0 < \rho < \bar{\rho}$. If our supposition is not true, then $I_0(v) = \{0, 1, \ldots, m\} = I_\rho(v)$ and $y_0(v) = y_\rho(v)$ for any $\rho > 0$. Anyway, for any $0 < \varepsilon < \min\{\bar{\rho}, -y_0(v)\}$, we have $y_\varepsilon(v) = y_0(v) < -\varepsilon$, that is the lemma is proved.

For our iterative method we shall take as the direction h_n at each u_n which is not a minimizing point of J_0 on U, a solution $x_\varepsilon(u_n)$ of the problem (8) for $v = u_n$ and for a positive ε for which $y_\varepsilon(u_n) \leq -\varepsilon$. Practical procedure to obtain such a ε is easy to produce. By starting with an $\varepsilon_0 > 0$, we have only to verify successively the inequality (9) for $\varepsilon = \varepsilon_0 \beta^j$, $j = 0, 1, \ldots$, with certain $\beta \in (0, 1)$, by solving the corresponding problems of the form (8). According to Lemma 2, after a finite number of steps, we get to a ε satisfying (9). If u_n is not a minimizing point for (1), $x_\varepsilon(u_n) \neq 0$, so that we can take

$$h_n = x_\varepsilon(u_n)/\|x_\varepsilon(u_n)\|.$$

For the steplength $\lambda_n > 0$ at each iteration, on the direction h_n, we shall consider a large class of procedures. Let us denote

$$\nu_n = \max\{\lambda \geq 0,\ u_n + \lambda h_n \in U\}. \tag{10}$$

We consider also a sequence $\{\mu_n\}$ of positive constants for which there exists a real positive function $\gamma(t)$ with the property that $\gamma(t)$ can converge to zero, only if t converges to zero (a „forcing function"), and satisfying the inequality

$$J_0(u_n) - J_0(u_n + \mu_n h_n) \geq \gamma(-J'_{0+}(u_n, h_n)). \tag{11}$$

In the previous paper [5] we have shown that under appropriate hypotheses, which are given in the next section, many practical procedures can be described for obtaining sequences $\{\mu_n\}$ satisfying (11): the maximal decreasing procedure, the approximate maximal decreasing, extensions of the Goldstein's and Armijo's algorithms. Then for each iteration u_n we shall take

$$\lambda_n = \min\{\nu_n, \mu_n\}. \tag{12}$$

providing that all iterations belong to the domain U of the constraints.

The convergence of the method. According to the previous section, we can consider now the following algorithm which we shall call the extended Zoutendijk-Polak method :

Step 0. Choose $u_0 \in U$, $\varepsilon_0 > 0$, $\beta \in (0, 1/2]$, $\varepsilon_1 = \varepsilon_0 \beta^k$, and put $n = 0$.

Step 1. Put $v = u_n$, $\varepsilon = \varepsilon_0$.

Step 2. Solve the problem (8) to obtain $x_\varepsilon(v)$, $y_\varepsilon(v)$.

Step 3. If $y_\varepsilon(v) \leqslant -\varepsilon$, take $\varepsilon(v) = \varepsilon$, put $h_n = x_\varepsilon(v)/\|x_\varepsilon(v)\|$ and go to Step 6 ; otherwise go to Step 4.

Step 4. If $\varepsilon = \varepsilon_1$ solve the problem (6) to determine $y_0(v)$ and go to Step 5 ; otherwise put $\varepsilon = \beta\varepsilon$ and go to Step 2.

Step 5. If $y_0(v) = 0$, stop ; otherwise put $\varepsilon = \beta\,\varepsilon$ and go to Step 2.

Step 6. Calculate λ_n by (12), put $u_{n+1} = u_n + \lambda_n h_n$, $n = n + 1$ and go to Step 1.

The positiv integer k is taken so that $\varepsilon_1 = \varepsilon_0 \beta^k$ be quite small in such a way that we expect to have practically $y_0(u_n) = 0$ when $-\varepsilon_1 < y_{\varepsilon_1}(u_n) \leqslant 0$. In that case u_n is a solution of the problem (1) and we reach the stop at Step 5. Except that case the algorithm is infinite, so that we have to prove the convergence for obtaining an implementable algorithm by stopping the calculation after a finite number of iterations.

Theorem. *If, in addition to the previous assumption, for each $i = 0, 1, \ldots, m$, we suppose also the hypothesis :*

$$\forall\, \varepsilon > 0,\ \exists\, \delta_i(\varepsilon) > 0,\ 0 \leqslant \lambda < \delta_i(\varepsilon) \Rightarrow 0 \leqslant J'_{i+}(v + \lambda h, h) - \\ - J'_{i+}(v, h) < \varepsilon,\ \forall\, v \in W(u_0),\ \forall\, h \in E,\ \|h\| = 1. \tag{13}$$

then, for the iterations u_n given by the extended Zoutendijk-Polak algorithm, one of the following two alternatives holds :

(i) *for certain n, u_n is a solution of the problem* (1) *and we reach the stop at Step* 5 ;

(ii) *every weak accumulation point of the sequence u_n is a solution of* (1).

Proof. Since u_n belong to the bounded set $U \cap W(u_0)$, there exist weak accumulation points of this sequence. Let $u_{n'} \rightharpoonup u^*$ be a weakly convergent subsequence. Since J_0 is convex lower semicontinuous, $\liminf\limits_{n' \to \infty} J_0(u_{n'}) \geqslant J_0(u^*) \geqslant J_0(u)$ for any solution u of the problem (1). The construction of u_n provides that the sequence $\{J_0(u_n)\}$ is decreasing.

Let us assume that u^* is not a minimizing point of J_0 on U. There exists then $l \in R$ such that

$$J_0(u_{n'}) \geqslant J_0(u^*) > l > J_0(u),\ J_0(z) > l. \tag{14}$$

Let $\theta \in (0, 1)$ so that, at $w = \theta z + (1-\theta)u$, $J_0(w) = l$. Since J_i are convex and because of the hypothesis (2), $0 > J_i(w) \geqslant J_i(u_{n'}) + J'_{i+}(u_{n'}, w - u_{n'})$, for $i = 1, 2, \ldots, m$. Also we have $0 > l - J_0(u^*) \geqslant J_0(w) - J_0(u_{n'}) \geqslant J'_{0+}(u_{n'}, w - u_{n'})$.

For $\max\{l - J_0(u^*), J_1(w), J_2(w), \ldots, J_m(w)\} = -\eta < 0$, and because $\beta \in (0, 1/2]$, we have, with $\tau = \beta\,\eta$, $J'_{0+}(u_{n'}, w - u_{n'}) \leqslant -\tau$. Also, for any $i \in I_\tau(u_{n'})$, $i \neq 0$, $J'_{i+}(u_{n'}, w - u_{n'}) \leqslant J_i(w) - J_i(u_{n'}) \leqslant -\eta + \tau \leqslant$

$\leqslant -2\,\tau + \tau = -\tau$. But we always can choose S large enough providing that $w - u_{n'} \in S$, so that, according to (8), we have $y_\tau(u_{n'}) \leqslant -\tau$, that is, by virtue of the algorithm, $\varepsilon(u_{n'}) \geqslant \tau > 0$. Therefore

$$y_{\varepsilon(u_{n'})}(u_{n'}) \leqslant -\varepsilon(u_{n'}) \leqslant -\tau < 0. \tag{15}$$

Hence

$$J'_{0+}(u_{n'}, x_{\varepsilon(u_{n'})}(u_{n'})) \leqslant -\tau < 0 \tag{16}$$

and

$$J'_{i+}(u_{n'}, x_{\varepsilon(u_{n'})}(u_{n'})) \leqslant -\tau,\ \forall\, i \in I_{\varepsilon(u_{n'})}(u_{n'}),\ i \neq 0. \tag{17}$$

Now, let $i \in I_{\varepsilon(u_{n'})}(u_{n'})$, $i \neq 0$. According to the hypothesis (13), for $0 < \rho < \delta_i(\tau/r) = \rho_i$ (where r is the radius of the smallest sphere containing S), we have

$$J_i(u_{n'} + \rho\, h_{n'}) \leqslant J_i(u_{n'}) + \rho\, J'_{i+}(u_{n'} + \rho h_{n'}, h_{n'}) \leqslant J_i(u_{n'}) + \\ + \rho\left(J'_{i+}(u_{n'}, h_{n'}) + \frac{\tau}{r}\right).$$

Then, since $\|x_{\varepsilon(u_{n'})}(u_{n'})\| \leqslant r$ and by using the inequality (17), we obtain

$$J_i(u_{n'} + \rho h_{n'}) \leqslant J_i(u_{n'}) + \rho\,\tau\,\frac{\|x_{\varepsilon(u_{n'})}(u_{n'})\| - r}{r\,\|x_{\varepsilon(u_{n'})}(u_{n'})\|} \leqslant 0. \tag{18}$$

If now, $i \notin I_{\varepsilon(u_{n'})}(u_{n'})$, i.e. $J_i(u_{n'}) < \varepsilon(u_{n'}) \leqslant -\tau$, because of the continuity of J_i, there exists $\rho_i > 0$ such that $0 < \rho < \rho_i$ implies $J_i(u_{n'} + \rho\, h_{n'}) \leqslant J_i(u_{n'}) + \tau/2 \leqslant -\tau + \tau/2 = -\tau/2 < 0$. Therefore, together with (18), for any $0 < \rho < \min\{\rho_i,\ i = 1, 2, \ldots, m\} = \rho_0$, $J_i(u_{n'} + \rho\, h_{n'}) \leqslant \leqslant 0$, for $i = 1, 2, \ldots, m$. As a consequence

$$\nu_{n'} \geqslant \rho_0 > 0. \tag{19}$$

In the last part of our proof we show that

$$\lim_{n' \to \infty} J'_{0+}(u_{n'}, x_{\varepsilon(u_{n'})}(u_{n'})) = 0. \tag{20}$$

If $\lambda_{n'} = \mu_{n'} \leqslant \nu_{n'}$, the convergence of the decreasing sequence $\{J_0(u_{n'})\}$ and the inequality (11) imply $\lim\limits_{n' \to \infty} J'_{0+}(u_{n'}, h_{n'}) = 0$, where from (20) follows easily, because $\|x_{\varepsilon(u_{n'})}(u_{n'})\| \leqslant 2\,r$.

For $\lambda_{n'} = \nu_{n'} < \mu_{n'}$, the convexity of J_0 implies

$$J_0(u_{n'}) - J_0(u_{n'} + \nu_{n'}\, h_{n'}) \geqslant \frac{\nu_{n'}}{\mu_{n'}}\,(J_0(u_{n'}) - J_0(u_{n'} + \mu_{n'}\, h_{n'})).$$

By virtue of the inequalities (11) and (19) we obtain $\lim\limits_{n' \to \infty} \gamma(-J'_{0+}(u_{n'}, h_{n'})) = 0$, which leads to (20) as before. Therefore the limit (20) is proved in any possible case. But this is contrary to the inequality (16). Hence our supposition (14) is not true and that implies the conclusion of the theorem.

For the norm-convergence of the sequence u_n, additional hypotheses must be supposed, like the compactity of U or the uniform convexity of J_0.

Applications. The algorithm can be applied practically only to convex functionals satisfying the following two features:

(i) the directional differentials satisfy the uniformity hypotheses (13),

(ii) the auxiliary problem is relatively easy to solve at each iteration.

Both these conditions depend on the properties and simplicity of calculation of the directional differentials of the functionals which are involved. The hypothesis (13) is quite difficult to be verified in the original form, but can be replaced by the slightly stronger and more convenient condition

$$\begin{aligned}(21)\qquad & \forall\, \varepsilon > 0,\ \exists\, \delta_i(\varepsilon) > 0,\ 0 \leqslant \lambda < \delta_i(\varepsilon) \Rightarrow 0 \leqslant J'_{i+}(v + \lambda h,\ h) - \\ & - J'_{i+}(v,\ h) < \varepsilon,\ \forall\, v \in W(u_0),\ \forall\, h \in E,\ \|h\| = 1,\end{aligned}$$

or by the uniform directional differentiability of the functionals J_i (see [5], Part II).

The auxiliary problem (8) is equivalent to

$$(22)\qquad \min\,\{y / J'_{i+}(v,\ x) \leqslant y,\ i \in I_\varepsilon(v),\ x \in S\} = y_\varepsilon(v),$$

which is a problem of minimization of a linear functional on $E \times R$, subject to some constraints. The convex bounded set S is arbitrary, so we can always choose it such that, the belonging to S involves only linear constraints. For example, when $E = R^n$ we can take $S = \{s = (s^1,\ s^2, \ldots, s^n),\ |s^i| \leqslant \alpha = \text{const.}\}$. The other constraints from (22) are generally sublinear. But for many convex functionals, the computation of the directional differentials is a linear optimization problem. In fact, in view of the relation

$$J'_{i+}(v,\ h) = \max\,\{<v^*\ h>,\ \forall\, v^* \in \partial J_i(v)\},$$

this is true whenever the subdifferentials $\partial J_i(v)$ are linear polytops.

We consider now some special cases.

1) Let $J_i : E \to \bar{R}$ be convex, Gâteaux-differentiable functional. In this case the hypotheses (13) are equivalent to the uniform continuity of the gradients of J_i on $W(u_0)$. The constraints of (22) concerning the directional differentials of J_i, are linear, so that the auxiliary problem (22) is linear.

2) Let E be a Banach space, S a compact topological space and $f : E \times S \to R$ convex, uniformly directionally differentiable with respect to $s \in S$, for each $x \in E$. Let us denote

$$(23)\qquad J(x) = \inf\,\{f(x,\ s),\ s \in S\}$$

and $S_0 = \{s \in S / f(x,\ s) = J(x)\}$. Then $J : E \to R$ defined by (23) is uniformly directionally differentiable at every $x \in E$ and

$$(24)\qquad J'_+(x,\ h) = \inf\,\{f'_{x+}(x,\ s\,;\ h),\ s \in S_0\}$$

(see [8], Theorem 3, Chap. 4). The most interesting case is that when the function $f(x,\ s)$ is continuously differentiable with respect to both variables. Then the directional differential of J is computed by the optimization problem

$$J'_+(x,\ h) = \min\,\{<\operatorname{grad}_x f(x,\ s),\ h> / \operatorname{grad}_s f(x,\ s) = 0\}.$$

If the constraints $\operatorname{grad}_s f(x,\ s) = 0$ are linear with respect to s, we obtain a linear optimization problem.

3) A more sophisticated case is that when in the previous example the compact set S depends on x, especially via a collection of inequalities $S(x) = \{x \in S / g(x,\ s) \leqslant 0\}$, where S is a compact linear topological space and $g : E \times S \to R^m$. We consider

$$(25)\qquad J(x) = \inf\,\{f(x,\ s) / s \in S,\ g(x,\ s) \leqslant 0\}.$$

Such a functional was considered by W. Hogan [4] in a finite dimensional setting. If f, g are convex, lower semicontinuous with respect so both variables, then J is also convex, lower semicontinuous [6]. Also the directional differential has the form

$$(26)\qquad J'_+(x\,;\ h) = \inf\,\{f'_+(x,\ s\,:\ h,\ t) / t \in H(x,\ s),\ s \in S_0(x)\},$$

where $S_0(x) = \{s \in S / g(x,\ s) \leqslant 0,\ f(x,s) = J(x)\}$ and $H(x,\ s) = \{t \in S / g(x + \lambda h,\ s + \lambda t) \leqslant 0,\ s + \lambda t \in S$, for some $0 < \lambda < \bar{\lambda}\}$. In [6] we have proved also that the hypothesis (13) is satisfied for the functional J, whenever the function f is uniformly directionally differentiable on every compact set. If we assume f and g Gâteaux-differentiable with uniformly continuous gradients on $W(u_0)$, we obtain for $J'_+(x,\ h)$ a linear optimization problem with linear constraints (see [4]) and the hypothesis (13) is satisfied, too. The minimization of such functionals arises for example in connection with some economical problems, like the coordination of two-level organizations with multiple objectives [2].

4) Often the functional to be minimized or even the contraints are sublinear functionals. Especially when a Banach space norm $N(x) = \|x\|$ is involved, as it is well known

$$N'_+(x,\ h) = \sup\,\{<h,\ x^*> / x^* \in E^*,\ <x,\ x^*> = \|x\|,\ \|x^*\|_* = 1\},$$

where E^* is the dual space and $\|\cdot\|_*$ is the norm of E^*. This is a maximizing problem of a linear functional subject to constraints which are also linear, except the last one. But, in many cases, the constraint $\|x^*\|_* = 1$ is also linear; for example, when $E = L^1(S)$, $E^* = L^\infty(S)$ it is equivalent to $|x(s)| \leqslant 1$, a.e. on S. Therefore our algorithm could be applied to some friction problems considered in [3]. Of course, for numerical treatment of such a problem, we have first to apply a discretization procedure in order to obtain a finite dimensional approximate problem.

Other examples of nondifferentiable convex functionals for which our method can be used, were considered by E. S. Levitin [9] who has developed also many applications to optimal control, minimax and nonlinear programming.

REFERENCES

1. Céa J. *Optimisation. Théorie et algorithmes*, Dunod, Paris, 1971.
2. Geoffrion, A. M., Hogan W. — *Coordination of two-level organizations with multiple objectives*, in „Techniuqes of Optimization" Academic Press, New York, 1972.
3. Glowinski R., Lions, J. L., Trémolières R. — *Analyse numérique des inéquations variationelles*, Dunod, Paris, 1976.
4. Hogan W. — *Directional derivatives for extremal-value functions with applications to the completely convex case*, Oper. Res. 21, 1973, 188—209.
5. Ilioi C. — *A unified approach of the descent methods for minimization of convex functionals.* Part I : *General theory*, An. şt. Univ. Matematica, Iaşi, Tom XXIV, 1978, 307—319 ; *Part II : Steplength algorithms*, Bull. Soc. Math. R.S.R. 23, 1979, 143—156
6. Ilioi C. — *Constrained minimization of nonsmooth functionals by an extension of the Frank-Wolfe algorithm*, An. şt. Univ. Matematica, Iaşi, Tom XXV, 1979, 332—336
7. Ilioi C. — *On the convergence of a feasible directions algorithm for minimization of convex nonsmooth constraints*, to appear in Mathematica, Cluj.
8. Ioffe, A. D., Tihomirov, V. M. — *The theory of the extremal problems*, (in Russian), Nauka, Moskow, 1974.
9. Levitin E. S. — *A general method of minimization for extremal nonsmooth problems* (in Russian), J. Vicisl. Mat. I Mat. Fiz. 9,1969, 783—806.
10. Zoutendijk G. — *Methods of feasible directions*, Elsevier, 1960.
11. Polak E. — *Computational methods in optimization : a unified approach*, Academic Press, New York, 1971.

Received 11.X.1979

Faculty of Mathematics
University of Iaşi
Iaşi, R.S. Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" Iaşi
Tomul XXVI, s. I a, 1980, fasc. 1

SUR LES VECTEURS ET LES DISTRIBUTIONS DE KILLING ET LEURS EXTENSIONS

PAR

GH. GHEORGHIEV

1. Introduction. En examinant beaucoup de modèles géométriques) utilisée dans les théories unitaires des champs physiques (voir la bibliographie) on arrive à la conclusion, qu'on doit admettre l'existence d'un ou de plusieurs champs de vecteur de Killing orthogonaux sur les variétés pseudoriemanni- nes, ce qui nous a conduit à considérer les distributions de Killing (§6). Après une exposition succinte de la méthode des coordonées non-holonomes et orthogonales sur les variétés de Riemann et de Lorentz (§2), de même que celle du repère mobile (§3), on passe à l'étude d'un champ de Killing généralisé et d'autres extensions (§4), apparus de même dans la théorie de la relativité sous la forme du mouvement rigide de Rosen-Rayner [9] ; ici sont inclus les vecteurs de Killing, le cas conforme, ainsi que celui de quasi-conformité. Dans le §5 on détérmine tous les invariants ortho- gonaux d'un champ de Killing générique et on établit la liaison avec la courbure Frenet de ses lignes vectorielles et avec la torsion de non-holonomie de l'hypersurface normale. Ensuite, on étudie une p-distribution de Killing sur une variété de Riemann et on démontre quelques propriétés, comme par exemple : la distribution de Killing est localement symétrique tandis que sa distribution complémentaire est totalement géodésique, en étant aussi localement plate (§6).

Enfin, en suivant le procédé appliqué au vecteur de Killing, on es- quisse quelques extensions d'une distribution de Killing qui conservent aussi les inclusions effectives (18) établies au §4.

2. Sur les coordonnées non-holonomes et orthogonales. La méthode des coordonées non-holonomes a été initiée par E. Cartan, T. Levi-Civita et Gr. Ricci au comencement du siècle.

Soient, M — une variété C^∞-différentiable, $T(M)$ et $T^*(M)$ — les fibrés tangent et cotangent de M, et, encore, $P(M)$ — le fibré principal des repères de M. En chaque point $x \in M$ on considère les repères linéai- res duaux, (X_i), (ω^j), $i, j = 1, 2, \ldots, n = \dim M$, dans les fibres locales $T_x(M)$ et $T_x^*(M)$. Dans le cas d'un repère naturel, qui conduit au coordonées courvilignes ou holonomes sur M, on a : $X_i = \partial/\partial x^i$, $\omega^j = dx^j$, où (x^i) sont les coordonées de x, dans un charte locale autour de ce point. En général, dans le cas non-holonome, les crochets des vecteurs d'une base sont par donnés