Une dernière remarque. L'existence d'une distribution de Killing sur une variété (M, g), ayant la propriété de locale symétrie, indique l'existence d'un groupe de mouvements qui agit sur M. Ainsi on arrive à un nouvel aspect d'un vieux problème de la classification des variétés (R) selon leur groupe de mouvements. La possibilité d'évaluer les invariants orthogonaux d'une distribution (K) permet d'entrevoir des nouveaux moyens d'utiliser en ce but [1].

Les généralisations faites aux vecteurs et distributions (K) et surtout le cas de quasi-conformité, indiquent des nouvelles perspectives d'étude. Les extensions sur les variétés $(\mathfrak{L})$ peuvent être utiles dans la théorie unitaire des champs physiques [3, 18].

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BICOFIBRATIONS

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In this Note we consider the notion of bicofibration which is a generalization of the topological sum of two spaces; it includes the join of complexes and it is dual to the notion of bifibration defined in [1]. The last result is used to construct some bifibrations and some simple bifibrations.

1. We recall from [1] that a bifibration is a pair $(X, B_i, p_i)_{i=1,2}$ of Hurewicz fibrations having the same total space X and such that for these fibrations there exist two lifting functions λ_1, λ_2 satisfying the following conditions:

$$(1)\qquad \begin{aligned} p_1\lambda_2(x_2, \omega_2) &= p_1(x_2) \text{ and} \\ p_2\lambda_1(x_1, \omega_1) &= p_2(x_1), \end{aligned}$$

for every $(x_i, \omega_i) \in \Omega_{p_i} = \{(x_i, \omega_i) \in X \times B_i^I / \omega_i(0) = p_i(x_i)\}$, $i = 1, 2$.

If the fibres of the fibrations and the total space X are connected, locally path connected and the base spaces B_i are locally path connected and semi-locally 1-connected, then the bifibration is called connected. A connected bifibration is called a simple bifibration if $p_1^{-1}(b_1) \cap p_2^{-1}(b_2)$ is a discrete subspace of X, for every points $b_1 \in B_1$, $b_2 \in B_2$.

Remark 1. Using arbitrary homotopies, the notion of bifibration can be given as follows: A pair of continuous maps (X, B_i, p_i) $(i=1,2)$ is a bifibration if for any topological space Y, any continuous map $f: Y \to X$ and any homotopies $F_i: Y \times I \to B_i$, $i = 1, 2$, such that $F_i(y, 0) = p_i f(y)$, there exist two homotopies $G_i: Y \times I \to X$, $i = 1, 2$, such that:

$$(2)\qquad G_i(y, 0) = f(y),\ \forall\, y \in Y,\ i = 1, 2,$$

$$(3)\qquad p_i G_i = F_i,\ i = 1, 2,$$

$$(4)\qquad \begin{aligned} p_2 G_1(y, t) &= F_2(y, 0) = p_2 f(y), \\ p_1 G_2(y, t) &= F_1(y, 0) = p_1 f(y), \end{aligned}$$

for every $(y, t) \in Y \times I$.

Example 1. Let $(X, B_i, p_i)_{i=1,2}$ be a pair of Hurewicz fibrations. We consider the continuous maps $p_i': X_1 \times X_2 \to B_i$, $i = 1, 2$, defined by $p_i'(x_1, x_2) = p_i(x_i)$, for $(x_1, x_2) \in X_1 \times X_2$. Then the pair $(X_1 \times X_2, B_i, p_i)((=1,2)$ is a bifibration. If p_i are covering projections, with X_i, $i = 1, 2$ connected, locally path connected spaces and with B_i, $i = 1, 2$ locally path connected and semi-locally 1-connected spaces, then the above bifibration is simple.

Let now $p_i : E \to B_i$, $i = 1, 2$, be two continuous maps. We suppose that the spaces B_i are with base points $*_i$, $i = 1, 2$. Let us denote $E_{*_i} = p_i^{-1}(*_i)$.

Definition 1. *A pair $(E, B_i, p_i, E_{*_i})_{i=1,2}$ as above, is a locally trivial bibundle if it consists of two locally trivial bundles having two trivialization atlases $(U^i, \varphi^i)_{i=1,2}$, $\varphi^i : U^i \times E_{*_i} \to p_i^{-1}(U^i)$ satisfying the following conditions:*

$$(5) \qquad \begin{aligned} p_2\varphi^1(x^1, h^1) &= p_2(h^1), \text{ for every } (x^1, h^1) \in U^1 \times E^1_{*_1}, \\ p_1\varphi^2(x^2, h^2) &= p_1(h^2), \text{ for every } (x^2, h^2) \in U^2 \times E^2_{*_2}. \end{aligned}$$

Example 2. If in the Example 1 the spaces B_i are with base points and if (X_1, B_1, p_1), (X_2, B_2, p_2) are locally trivial bundles, then the pair $(X_1 \times X_2, B_i, p_i')$ $i = 1, 2$ is a locally trivial bibundle.

In analogy with the Corollary 2.8.14 of [2, p. 136] one can prove the following theorem:

Theorem 1. *Let $(E, B_i, p_i, E_{*_i})(i=1,2)$ be a locally trivial bibundle for which the bases B_i are Hausdorff paracompact spaces. Then the pair $(E, B_i, p_i)(i=1,2)$ is a bifibration.*

2. Let $f_i : X_i \to X$, $i = 1, 2$ be two continuous maps.

Definition 2. *We say that the pair $(X_i, X, f_i)_{i=1,2}$ is a bicofibration, if for any topological space Y, any continuous map $g : X \to Y$ and any homotopies $G_i : X_i \times I \to Y$, satisfying*

$$(6) \qquad G_i(x_i, 0) = gf_i(x_i), \quad \forall x_i \in X_i, \ i = 1, 2,$$

there exist two homotopies $F_i : X \times I \to Y$, $i = 1, 2$, which satisfy the following conditions:

$$(7) \qquad F_i(x, 0) = g(x), \quad \forall x \in X, \ i = 1, 2,$$

$$(8) \qquad F_i(f_i(x_i), t) = G_i(x_i, t), \quad \forall x_i \in X_i, \ i = 1, 2, \ \forall t \in I,$$

$$(9) \qquad \begin{aligned} F_1(f_2(x_2), t) &= G_2(x_2, 0) = gf_2(x_2), \quad \forall x_2 \in X_2, \ \forall t \in I, \text{ and} \\ F_2(f_1(x_1), t) &= G_1(x_1, 0) = gf_1(x_1), \quad \forall x_1 \in X_1, \ \forall t \in I. \end{aligned}$$

Proposition 1. *If the pair $(X_i, X, f_i)_{i=1,2}$ is a bicofibration, then the maps $f_i : X_i \to X$, $i = 1, 2$, are cofibrations.*

Proof. Let Y be a topological space and let $g : X \to Y$, $G : X_1 \times I \to Y$ be continuous maps such that $G(x_1, 0) = gf_1(x)$. We take in Definition 2 $G_1 = G$ and $G_2(x_2, t) = gf_2(x_2)$, $\forall (x_2, t) \in X_2 \times I$. Then, there exists $F_1 : X \times I \to Y$ such that $F_1(x, 0) = g(x)$ and $F_1(f_1 \times 1) = G_1$, that is f_1 is a cofibration.

Proposition 2. *Let $(X_i, X, f_i)(i=1,2)$ be a bicofibration and let X_i be path connected spaces. Then $f_1(X_1) \cap f_2(X_2) = \varnothing$.*

Proof. Suppose there exists $x \in X$ such that $x = f_1(x_1) = f_2(x_2)$, $x_i \in X_i$, $i = 1, 2$. Let Y be an arbitrary path connected space and let $G_i : X_i \times I \to Y$ be two homotopies satisfying (6). From (7), (8), (9) we obtain $G_1(x_1, t) = F_1(f_1(x_1), t) = F_1(f_2(x_2), t) = gf_2(x_2) = gf_1(x_1)$, which is impossible since G_1 is an arbitrary homotopy.

Let $(X_i, X, f_i)_{i=1,2}$ be two cofibrations. Donote by $\bar{X}_i$ the spaces obtained from the sum $(X_i \times I) \vee (X \times 0)$ identifying $(x_i, 0) \in X_i \times I$ with $(f_i(x_i), 0) \in X \times 0$, $i = 1, 2$.

Let $r_i : X \times I \to \bar{X}_i$ be retract functions for f_i, $i = 1, 2$, that is r_i are continuous maps with conditions:

$$(10) \qquad \begin{aligned} r_i(x, 0) &= [x, 0]_i, \\ r_i(f_i(x_i), t) &= [x_i, t]_i, \quad i = 1, 2, \end{aligned}$$

where $[\ ]_i$ denotes an element of the quotient space $\bar{X}_i$.

Theorem 2. *A pair $(X_i, X, f_i)(i=1,2)$ of cofibrations is a bicofibration if and only if the maps f_i have two retract functions r_i, $(i=1,2)$ which satisfy the following conditions:*

$$(11) \qquad \begin{aligned} r_1(f_2(x_2), t) &= [f_2(x_2), 0]_1 = r_1(f_2(x_2), 0), \quad \forall x_2 \in X_2, \ \forall t \in I \text{ and} \\ r_2(f_1(x_1), t) &= [f_1(x_1), 0]_2 = r_2(f_1(x_1), 0), \quad \forall x_1 \in X_1, \ \forall t \in I. \end{aligned}$$

Proof. Let (X_i, X, f_i), $i = 1, 2$, be the pair of continuous maps with the retract functions $r_i : X \times I \to \bar{X}_i$, $i = 1, 2$, satisfying the conditions (10), (11).

Let $g : X \to Y$, $G_i : X_i \times I \to Y$, $i = 1, 2$, be continuous maps satisfying (6). We define the maps $\bar{G}_i : \bar{X}_i \to Y$ by $\bar{G}_i([x_i, t]_i) = G_i(x_i, t)$ and $\bar{G}_i([x, 0]_i) = g(x)$. Then, the maps $F_i = \bar{G}_i \circ r_i$, $i = 1, 2$, verify the conditions (7), (8), (9), that is the above pair is a bicofibration.

Reciprocally, suppose that the pair $(X_i, X, f_i)_{i=1,2}$ is a bicofibration. We define the maps $g_1 : X \to \bar{X}_1$, $G_{11} : X_1 \times I \to \bar{X}_1$, $G_{21} : X_2 \times I \to \bar{X}_1$ by $g_1(x) = [x, 0]_1$, $G_{11}(x_1, t) = [x_1, t]_1$, $G_{21}(x_2, t) = g_1f_2(x_2) = [f_2(x_2), 0]_1$. By Definition 2 it follows that there exist two continuous maps $r_1 : X \times I \to \bar{X}_1$ and $r_1' : X \times I \to \bar{X}_1$ such that $r_1(x, 0) = g_1(x) = [x, 0]_1$, $r_1(f_1(x_1), t) = G_{11}(x_1, t) = [x_1, t]_1$ and $r_1(f_2(x_2), t) = G_{21}(x_2, t) = [f_2(x_2), 0]_1$. Analogously, by taking the maps $g_2 : X \to \bar{X}_2$, $G_{12} : X_1 \times I \to \bar{X}_2$, $G_{22} : X_2 \times I \to \bar{X}_2$ defined by $g_2(x) = [x, 0]_2$, $G_{12}(x_1, t) = [f_1(x_1), 0]_2$, $G_{22}(x_2, t) = [x_2, t]_2$, it follows that there exists a map $r_2 : X \times I \to \bar{X}_2$, such that $r_2(x, 0) = [x, 0]_2$, $r_2(f_2(x_2), t) = [x_2, t]_2$ and $r_2(f_1(x_1), t) = [f_1(x_1), 0]_2$.

Proposition 3. *Let X_1, X_2 be two topological spaces and let $X_1 \vee X_2$ be their topological sum. Consider the inclusion maps $f_i : X_i \to X_1 \vee X_2$, $i = 1, 2$. Then, the pair $(X_i, X_1 \vee X_2, f_i)_{i=1,2}$ is a bicofibration.*

Proof. One sees easily that $\bar{X}_1$ can be identified with $(X_1 \times I) \vee X_2$ and $\bar{X}_2$ with $(X_2 \times I) \vee X_1$. Then we define:

$$r_1 : (X_1 \vee X_2) \times I \to (X_1 \times I) \vee X_2$$

$$r_2 : (X_1 \vee X_2) \times I \to (X_2 \times I) \vee X_1,$$

by $r_1(x_1, t) = (x_1, t)$, $r_1(x_2, t) = x_2$ and respectively $r_2(x_1, t) = x_1$, $r_2(x_2, t) = (x_2, t)$.

These functions satisfy Theorem 2.

More generally we have:

Proposition 4. *Let $f_i : X_i \to X'_i$ $i = 1,2$ be two cofibrations. Consider the maps $f'_i : X_i \to X'_1 \vee X'_2$ defined by $f'_i(x_i) = f_i(x_i)$, for $x_i \in X_i$. Then the pair $(X_i, X'_1 \vee X'_2, f'_i)_{i=1,2}$ is a bicofibration.*

The proof follows by Theorem 2.

Proposition 5. *Let K_1, K_2 be two simplicial complexes. Consider the join:*

$$K_1 * K_2 = K_1 \vee K_2 \cup \{s_1 \vee s_2 / s_1 \in K_1,\ s_2 \in K_2\}.$$

Let $f_i : |K_i| \to |K_1 * K_2|$, $i = 1, 2$, be the inclusion maps. Then, the pair $(|K_i|, |K_1 * K_2|, f_i)$ $(i=1, 2)$ is a bicofibration.

Proof. By the proof of Corollary 3.2.4 of [2, p. 154] it follows that there exist two retractions

$$r_i : |K_1 * K_2| \times I \to |K_1 * K_2| \times \{0\} \cup |K_i| \times I,$$

such that $r_i(x_j, t) = (x_j, 0)$, for $i, j = 1, 2$, $j \neq i$ and $\forall x_j \in K_j$.

Now, let the commutative diagram:

$$\begin{tikzcd} {|K_1|} \arrow[r, "f_1"] \arrow[d] & {|K_1 * K_2|} \arrow[d, "g"] & {|K_2|} \arrow[l, "f_2"'] \arrow[d] \\ {|K_1| \times I} \arrow[r, "G_1"] & Y & {|K_2| \times I} \arrow[l, "G_2"'] \end{tikzcd}$$

We define the maps $\bar{G}_i : |K_1 * K_2| \times \{0\} \cup |K_i| \times I \to Y$ by $\bar{G}_i(z, 0) = g(z)$, $z \in |K_1 * K_2|$ and $\bar{G}_i(x_i, t) = G_i(x_i, t)$, for $(x_i, t) \in |K_i| \times I$, $i = 1, 2$. Then, the maps $F_i = r_i \circ \bar{G}_i$ satisfy the condition (7), (8), (9) of Definition 2.

Now we give an example of bicofibration which is not contained in the Propositions 3, 4, 5.

Example 3. Let $X = I = [0,1]$, $X_1 = \{0,1\}$, $X_2 = \{1/3, 2/3\}$ and let $f_i : X_i \to X$, $i = 1, 2$ be the inclusion maps. Then, the pair (X_i, X, f_i), $i = 1, 2$, is a bicofibration.

In fact, let the commutative diagram:

$$\begin{tikzcd} \{0,1\} \arrow[r, "f_1"] \arrow[d] & I \arrow[d, "g"] & \{1/3,\ 2/3\} \arrow[l, "f_2"'] \arrow[d] \\ \{0,1\} \times I \arrow[r, "G_1"] & Y & \{1/3,\ 2/3\} \times I \arrow[l, "G_2"'] \end{tikzcd}$$

There exists a retraction $r_1 : I \times I \to I \times 0 \cup \{0, 1\} \times I$, such that $r_1(1/3, t') = (1/3, 0)$, $r_1(2/3, t') = (2/3, 0)$, $\forall t' \in I$. Then we define $F_1 = h_1 \circ r_1 : I \times I \to Y$, where $h_1 : I \times 0 \cup \{0,1\} \times I \to Y$ is defined by $h_1(t, 0) = g(t)$, $h_1(0, t') = G_1(0, t')$, $h_1(1, t') = G_1(1, t')$. We have $F_1(t, 0) = g(t)$,

$F_1|\{0,1\} \times I = G_1$ and $F_1(f_2 \times 1)(1/3, t') = F_1(1/3, t') = h_1 r_1(1/3, t') = h_1(1/3, 0) = g(1/3)$, $\forall t' \in I$, and $F_1(f_2 \times 1)(2/3, t') = g(2/3)$, $\forall t' \in I$.

Analogously one constructs $F_2 : I \times I \to Y$.

3. **Theorem 3.** *Let $(X_i, X, f_i)_{i=1,2}$ be a bicofibration with X_i, $i = 1, 2$ and X Hausdorff locally compact spaces. Let Y be an arbitrary topological space. Then, the pair of fibration (Y^X, Y^{X_i}, p_i) $(i=1,2)$ with $p_i : Y^X \to Y^{X_i}$ defined by $p_i(g) = g \circ f_i$, for $g \in Y^X$,* [2, p. 130], is a bifibration.

Proof. Let $r_i : X \times I \to X_i$, $i = 1, 2$, be retract functions for the considered bicofibration, satisfying the conditions (10), (11). Then, for the maps p_i we define the following lifting functions:

$$\lambda_i : \Omega_{p_i} \to (Y^X)^I, \text{ where}$$

$$\Omega_{p_i} = \{(g_i, G_i) \in Y^X \times (Y^{X_i})^I / G_i(0) = g_i \circ f_i\},$$

by

(12) $$\lambda_i(g_i, G_i)(t)(x) = (u_i \circ r_i)(x, t), \text{ where}$$

$$u_i : X_i \to Y,\ i=1, 2, \text{ are defined by}$$

(13) $$u_i([x_i, t]_i) = G_i(t)(x_i) \text{ and}$$
$$u_i([x_i, 0]_i) = g_i(x_i).$$

Then we have:

$p_2 \lambda_1(g_1, G_1)(t) = \lambda_1(g_1, G_1)(t) \circ f_2$, which implies: $(p_2 \lambda_1(g_1, G_1)(t))(x_2) = u_1 r_1(f_2(x_2), t) = u_1([f_2(x_2), 0]_1) = g_1 f_2(x_2)$, that is $p_2 \lambda_1(g_1, G_1)(t) = g_1 \circ f_2 = p_2(g_1)$, for every $(g_1, G_1) \in \Omega_{p_1}$ and any $t \in I$. Analogously one proves the equality $p_1 \lambda_2(g_2, G_2)(t) = p_1(g_2)$, for every $(g_2, G_2) \in \Omega_{p_2}$ and any $t \in I$.

Corollary 1. *Let X be a topological space and let $p_0 : X^I \to X$, $p_1 : X^I \to X$ be the maps $p_0(\omega) = \omega(0)$, $p_1(\omega) = \omega(1)$, for a path $\omega \in X^I$. The pair (X^I, X, p_i) $(i=0, 1)$ is a bifibration.*

Proof. Let $f_0 : \{0\} \to I$, $f_1 : \{1\} \to I$ be the inclusion maps. By the Proposition 5, the pair of these cofibrations is a bicofibration. Using Theorem 3, for an arbitrary topological space X, it follows that the pair of fibrations

$$X \equiv X^{\{0\}} \xleftarrow{p_0} X^I \xrightarrow{p_1} X^{\{1\}} \equiv X,$$

with $p_i(\omega) = (\omega \circ f_i)(1) = \omega(i)$, $i = 0, 1$, is a bifibration.

Remark 2. The bifibration of Corollary 1 (which can be called the bifibration of Serre) has the lifting functions:

(14) $$\lambda_0(\omega, \theta)(t) = (\omega^{-1} * \theta)^{-1}_{(t+1)/2}, \text{ for } \omega, \theta \in X^I, \text{ with } \omega(0) = \theta(0),$$
$$\lambda_1(\omega, \theta)(t) = (\omega * \theta)_{(t+1)/2}, \text{ for } \omega, \theta \in X^I, \text{ with } \omega(1) = \theta(0),$$

where for a path $\mu : I \to X$, we denote by μ_t the path $\mu_t(t') = \mu(tt')$ and μ^{-1} is the path $\mu^{-1}(t') = \mu(1-t')$. The continuity and the fact that these maps are lifting functions are well-known. Moreover, we have:

$$\tilde{p}_1 \lambda_0(\omega, \theta)(t) = (\omega^{-1} * \theta)^{-1}_{(t+1)/2}(1) = (\omega^{-1} * \theta)_{(t+1)/2}(0) = \omega^{-1}(0) = \omega(1) = p_1(\omega) \text{ and}$$

$$p_0 \lambda_1(\omega, \theta)(t) = (\omega * \theta)_{(t+1)/2}(0) = \omega(0) = p_0(\omega).$$

Using the bifibration of Serre for a suitable topological space, we can obtain some simple bifibrations by an analogous construction to the construction of covering spaces.

For this, let X be a connected, locally path connected, semilocally 1-connected and 1-simple space. Let H be a subgroup of the fundamental group $\Pi_1(X)$. Denote by $\tilde{X}$ the quotient space obtained from X^I with respect to equivalence relation: $\omega \sim \omega'$ iff $\omega(0) = \omega'(0)$, $\omega(1) = \omega'(1)$ and $[\omega * \omega'^{-1}] \in H$. Denote by $<\omega>$ the equivalence class of a path $\omega \in X^I$ and let $\tilde{p}_0 : \tilde{X} \to X$, $\tilde{p}_1 : \tilde{X} \to X$ be the maps defined by $\tilde{p}_0(<\omega>) = \omega(0)$ and $\tilde{p}_1(<\omega>) = \omega(1)$ respectively.

Theorem 4. *The pair $(\tilde{X}, \tilde{p}_i, X)(i=0, 1)$ is a simple bifibration.*

Proof. Let

$$\Omega_{\tilde{p}_0} = \{(<\omega>, \theta) \in \tilde{X} \times X^I/\omega(0) = \theta(0)\} \text{ and}$$

$$\Omega_{\tilde{p}_1} = \{(<\omega>, \theta) \in \tilde{X} \times X^I/\theta(0) = \omega(1)\}.$$

We consider $\tilde{\lambda}_i : \Omega_{\tilde{p}_i} \to \tilde{X}^I$, $i = 0,1$ by

$$\tilde{\lambda}_0(<\omega>, \theta)(t) = <(\omega^{-1} * \theta)^{-1}_{(t+1)/2}>, \text{ for } (<\omega>, \theta) \in \Omega_{\tilde{p}_0} \text{ and}$$
$$\tilde{\lambda}_1(<\omega>, \theta)(t) = <(\omega * \theta)_{(t+1)/2}>, \text{ for } (<\omega>, \theta) \in \Omega_{\tilde{p}_1}. \tag{15}$$

From [3] the maps $\tilde{\lambda}_i$, $i = 0,1$ are well defined and continuous. By the Remark 2 these maps are lifting functions for $\tilde{p}_i$ and satisfy the conditions:

$$\tilde{p}_1 \tilde{\lambda}_0(<\omega>, \theta)(t) = \tilde{p}_1(<\omega>), \text{ for } (<\omega>, \theta) \in \Omega_{\tilde{p}_0} \text{ and}$$

$$\tilde{p}_0 \tilde{\lambda}_1(<\omega>, \theta)(t) = \tilde{p}_0(<\omega>), \text{ for } (<\omega>, \theta) \in \Omega_{\tilde{p}_1}.$$

Thus, the considered pair is a bifibration. Moreover, by the construction of covering spaces, Theorem 2.5.13 of [2, p. 109], it follows that $\tilde{p}_0^{-1}(x_0) \cap \tilde{p}_1^{-1}(x_1) = \{<\omega>/\omega(0) = x_0, \omega(1) = x_1\}$, for $x_0, x_1 \in X$, is a discrete subspace of $\tilde{X}$.

Remark 3. Theorem 4 can be stated in more general conditions, replacing the subgroup H by a pair (H, U), where U is an open covering of X such that $\Pi_1(U) \subseteq H$.

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NORMAL CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD

BY

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The study of differential geometry of CR-submanifolds of a Kaehler manifold has been initiated in [1]. Fundamental results have been obtained in [2], [3], [4], [9].

The purpose of the present paper is to study normal CR-submanifolds of a Kaehler manifold.

In §1 we derive some fundamental formulas for a CR-submanifold immersed in a Kaehler manifold. In §2 we study normal CR-submanifolds and obtain necessary and sufficient conditions for a CR-submanifold in order to be normal. Finally, the last section is devoted to the study of cosymplectic CR-submanifolds in a Kaehler manifold.

1§. **CR-structure on a submanifold in a Kaehler manifold.** Let $\tilde{M}$ be a Kaehler manifold of complex dimension n. Denote by (J, g) the Kaehlerian structure of $\tilde{M}$, that is, J is the almost complex structure on $\tilde{M}$ and g is the Kaehlerian metric on $\tilde{M}$.

Let M be a Riemannian submanifold immersed in $\tilde{M}$ of real dimension m. The Riemannian metric on M will be denoted by the same symbol g as the Kaehlerian metric on $\tilde{M}$.

The submanifold M is called a CR-submanifold if the following conditions are satisfied [1].

1. *There exists a differentiable distribution D on M which is invariant by J, that is, $J(D_x) = D_x$ for all $x \in M$.*

2. *The orthogonal complementary distribution $D^\perp$ is anti-invariant, that is, $J(D^\perp_x) \subset (T_xM)^\perp$, where $(T_xM)^\perp$ is the normal space to M at x.*

The distribution D (resp. $D^\perp$) is called the *holomorphic distribution* (resp. *totally real distribution*). We suppose the real dimension of D_x (resp. $D^\perp_x$) be $2p$ (resp. q). The projectors which define the distributions D and $D^\perp$ are denoted respectively by P and Q.

As particular cases of CR-submanifolds we have:

1. *complex submanifolds*, when $q = 0$, [6],
2. *totally real submanifolds*, when $p = 0$, [5],
3. *generic submanifolds*, when $q = 2n - m$, [9].

The submanifold M is said to have an f-structure if there exists a non-null tensor field F on M of type (1,1) such that $F^3 + F = 0$ [8].

A proper CR-submanifold is a CR-submanifold which is neither a complex submanifold nor a totally real submanifold.