$$p_1 \lambda_0(\omega, \theta)(t) = (\omega^{-1} * \theta)^{-1}_{(t+1)/2} (1) = (\omega^{-1} * \theta)_{(t+1)/2}(0) = \omega^{-1}(0) = \omega(1) = p_1(\omega) \text{ and}$$

$$p_0 \lambda_1 (\omega, \theta)(t) = (\omega * \theta)_{(t+1)/2} (0) = \omega(0) = p_0(\omega).$$

Using the bifibration of Serre for a suitable topological space, we can obtain some simple bifibrations by an analogous construction to the construction of covering spaces.

For this, let X be a connected, locally path connected, semilocally 1-connected and 1-simple space. Let H be a subgroup of the fundamental group $\Pi_1(X)$. Denote by $\tilde{X}$ the quotient space obtained from X^I with respect to equivalence relation: $\omega \sim \omega'$ iff $\omega(0) = \omega'(0)$, $\omega(1) = \omega'(1)$ and $[\omega * \omega'^{-1}] \in H$. Denote by $<\omega>$ the equivalence class of a path $\omega \in X^I$ and let $\tilde{p}_0 : \tilde{X} \to X$, $\tilde{p}_1 : \tilde{X} \to X$ be the maps defined by $\tilde{p}_0(<\omega>) = \omega(0)$ and $\tilde{p}_1 (<\omega>) = \omega(1)$ respectively.

Theorem 4. *The pair $(\tilde{X}, \tilde{p}_i, X)(i=0, 1)$ is a simple bifibration.*

Proof. Let

$$\Omega_{\tilde{p}_0} = \{(<\omega>, \theta) \in \tilde{X} \times X^I / \omega(0) = \theta (0)\} \text{ and}$$

$$\Omega_{\tilde{p}_1} = \{(<\omega>, \theta) \in \tilde{X} \times X^I / \theta (0) = \omega (1)\}.$$

We consider $\tilde{\lambda}_i : \Omega_{\tilde{p}_i} \to \tilde{X}^I$, $i = 0,1$ by

$$\text{(15)} \qquad \begin{aligned} &\tilde{\lambda}_0(<\omega>, \theta)(t) = <(\omega^{-1} * \theta)^{-1}_{(t+1)/2}>, \text{ for } (<\omega>, \theta) \in \Omega_{\tilde{p}_0} \text{ and} \\ &\tilde{\lambda}_1(<\omega>, \theta)(t) = <(\omega * \theta)_{(t+1)/2}>, \text{ for } (<\omega>, \theta) \in \Omega_{\tilde{p}_1}. \end{aligned}$$

From [3] the maps $\tilde{\lambda}_i$, $i = 0,1$ are well defined and continuous. By the Remark 2 these maps are lifting functions for $\tilde{p}_i$ and satisfy the conditions:

$$\tilde{p}_1 \tilde{\lambda}_0 (<\omega>, \theta)(t) = \tilde{p}_1(<\omega>), \text{ for } (<\omega>, \theta) \in \Omega_{\tilde{p}_0} \text{ and}$$

$$\tilde{p}_0 \tilde{\lambda}_1 (<\omega>, \theta)(t) = \tilde{p}_0(<\omega>), \text{ for } (<\omega>, \theta) \in \Omega_{\tilde{p}_1}.$$

Thus, the considered pair is a bifibration. Moreover, by the construction of covering spaces, Theorem 2.5.13 of [2, p. 109], it follows that $\tilde{p}_0^{-1}(x_0) \cap \tilde{p}_1^{-1} (x_1) = \{<\omega> / \omega(0) = x_0, \omega(1) = x_1\}$, for $x_0, x_1 \in X$, is a discrete subspace of $\tilde{X}$.

Remark 3. Theorem 4 can be stated in more general conditions, replacing the subgroup H by a pair (H, U), where U is an open covering of X such that $\Pi_1(U) \subseteq H$.

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NORMAL CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD

BY

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The study of differential geometry of CR-submanifolds of a Kaehler manifold has been initiated in [1]. Fundamental results have been obtained in [2], [3], [4], [9].

The purpose of the present paper is to study normal CR-submanifolds of a Kaehler manifold.

In §1 we derive some fundamental formulas for a CR-submanifold immersed in a Kaehler manifold. In §2 we study normal CR-submanifolds and obtain necessary and sufficient conditions for a CR-submanifold in order to be normal. Finally, the last section is devoted to the study of cosymplectic CR-submanifolds in a Kaehler manifold.

1§. **CR-structure on a submanifold in a Kaehler manifold.** Let $\tilde{M}$ be a Kaehler manifold of complex dimension n. Denote by (J, g) the Kaehlerian structure of $\tilde{M}$, that is, J is the almost complex structure on $\tilde{M}$ and g is the Kaehlerian metric on $\tilde{M}$.

Let M be a Riemannian submanifold immersed in $\tilde{M}$ of real dimension m. The Riemannian metric on M will be denoted by the same symbol g as the Kaehlerian metric on $\tilde{M}$.

The submanifold M is called a CR-submanifold if the following conditions are satisfied [1].

1. *There exists a differentiable distribution D on M which is invariant by J, that is, $J(D_x) = D_x$ for all $x \in M$.*
2. *The orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is, $J(D^{\perp}_x) \subset (T_xM)^{\perp}$, where $(T_xM)^{\perp}$ is the normal space to M at x.*

The distribution D (resp. $D^{\perp}$) is called the *holomorphic distribution* (resp. *totally real distribution*). We suppose the real dimension of D_x (resp. $D^{\perp}_x$) be $2p$ (resp. q). The projectors which define the distributions D and $D^{\perp}$ are denoted respectively by P and Q.

As particular cases of CR-submanifolds we have:

1. *complex submanifolds*, when $q = 0$, [6].
2. *totally real submanifolds*, when $p = 0$, [5].
3. *generic submanifolds*, when $q = 2n - m$, [9].

The submanifold M is said to have an f-structure if there exists a non-null tensor field F on M of type (1,1) such that $F^3 + F = 0$ [8]. *A proper CR-submanifold* is a CR-submanifold which is neither a complex submanifold nor a totally real submanifold.

Proposition 1.1. *Let M be a proper CR-submanifold of a Kaehler manifold $\widetilde{M}$. Then, there exist an f-structure F on M and a non-null normal bundle valued 1-form u on M, satisfying*

$$u \circ F = 0, \tag{1.1}$$

$$FX = 0 \text{ if and only if } X \in D^{\perp}, \tag{1.2}$$

$$uY = 0 \text{ if and only if } Y \in D. \tag{1.3}$$

Proof. We define F and u by

$$FX = JPX \text{ and respectively} \tag{1.4}$$

(1.5) $uX = JQX$ for all $X \in TM$. Now, the assertions of the proposition are simple verifications.

The pair (F, u) is called the CR-structure on the CR-submanifold M. The 1-form u is called the CR-form on M.

Lemma 1.1. *The CR-structure (F, u) satisfies*

$$g(FX, FY) = g(X, Y) - g(uX, uY), \tag{1.6}$$

for all $X, Y \in TM$.

The proof is a consequence of (1.4), (1.5) and of the fact that g is a Hermitian metric on $\widetilde{M}$.

We define on the CR-submanifold M the 2-form Φ by

$$\Phi(X, Y) = g(X, FY). \tag{1.7}$$

The skew-symmetry of Φ follows from (1.4). Also we have

$$\Phi(FX, FY) = \Phi(X, Y). \tag{1.8}$$

Denote by $\widetilde{\nabla}$ the Kaehlerian connection on $\widetilde{M}$ and by ∇ the Riemannian connection on M. The normal connection on M is denoted by $\nabla^{\perp}$. Then we can define the covariant derivative of F and u by

$$(\nabla_X F)Y = \nabla_X FY - F(\nabla_X Y), \tag{1.9}$$

$$(\nabla_X u)Y = \nabla_X^{\perp} uY - u(\nabla_X Y), \tag{1.10}$$

for all $X, Y \in TM$.

The exterior differential of u and Φ are given respectively by

$$du(X, Y) = \frac{1}{2}\{\nabla_X^{\perp} uY - \nabla_Y^{\perp} uX - u([X, Y])\} \text{ and} \tag{1.11}$$

$$\begin{aligned} d\Phi(X, Y, Z) = \frac{1}{3}\{&X\Phi(Y, Z) + Y\Phi(Z, X) + Z\Phi(X, Y) - \\ &- \Phi([X, Y], Z) - \Phi([Z, X], Y) - \Phi([Y, Z], X)\}. \end{aligned} \tag{1.12}$$

The Nijenhuis tensor of F is a tensor field N_F of type (1, 2) given by

$$N_F(X, Y) = F^2[X, Y] + [FX, FY] - F[FX, Y] - F[X, FY]. \tag{1.13}$$

Now, we can state

Lemma 1.2. *The covariant derivative of F is given by*

$$\begin{aligned} 2g((\nabla_X F)Y, Z) &= 3d\Phi(X, FY, FZ) - 3d\Phi(X, Y, Z) + \\ &+ g(N_F(Y, Z), FX) + 2g(du(FY, Z) - du(FZ, Y), uX) + \\ &+ 2g(du(FY, X), uZ) - 2g(du(FZ, X), uY), \end{aligned} \tag{1.14}$$

for all vector fields X, Y, Z tangent to M.

Proof. Since the Riemannian connection ∇ is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) \\ + g([Z, X], Y) - g([Y, Z], X), \end{aligned} \tag{1.15}$$

we have

$$\begin{aligned} 2g((\nabla_X F)Y, Z) &= 2g(\nabla_X FY, Z) + 2g(\nabla_X Y, FZ) = Xg(FY, Z) + \\ &+ FYg(X, Z) - Zg(X, FY) + g([X, FY], Z) + g([Z, X], FY) - \\ &- g([FY, Z], X) + Xg(Y, FZ) + Yg(X, FZ) - FZg(X, Y) + \\ &+ g([X, Y], FZ) + g([FZ, X], Y) - g([Y, FZ], X). \end{aligned} \tag{1.16}$$

Next, by means of (1.6) – (1.8) from (1.16) we obtain

$$\begin{aligned} 2g(\nabla_X F)Y, Z &= 3d\Phi(X, FY, FZ) - 3d\Phi(X, Y, Z) + g(N_F(Y, Z), \\ &FX) + g(\nabla_{FY}^{\perp} uZ - u([FY, Z]), uX) + g(\nabla_{FY}^{\perp} uX - u([FY, X]), uZ) \\ &+ g(uX, u([FZ, Y])) + g(uY, u([FZ, X])) - FZg(uX, uY). \end{aligned} \tag{1.17}$$

Finally, by using

$$du(FY, Z) = \frac{1}{2}\{\nabla_{FY}^{\perp} uZ - u([FY, Z])\}, \tag{1.18}$$

$$FZg(uX, uY) = g(\nabla_{FZ}^{\perp} uX, uY) + g(uX, \nabla_{FZ}^{\perp} uY), \tag{1.19}$$

from (1.17) we get (1.14). Thus, the proof is complete.

Now, if h is the second fundamental form of M, then the equations of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.20}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{1.21}$$

for all $X, Y \in TM$ and $N \in (TM)^{\perp}$. Here, A_N is the fundamental tensor of Weingarten, corresponding to the normal section N. We have

$$g(A_N X, Y) = g(h(X, Y), N). \tag{1.22}$$

The CR-submanifold M is called *mixed totally geodesic* if we have

$$h(X, Y) = 0 \text{ for all } X \in D \text{ and } Y \in D^{\perp}. \tag{1.23}$$

For each normal section N we put

$$JN = BN + CN, \tag{1.24}$$

where BX and CX are respectively the tangential part and the normal part of JX. Thus, C is an endomorphism of the normal bundle. Moreover, C is an f-structure on $(TM)^{\perp}$, [1].

Since $\tilde{\nabla}$ is a Kaehlerian connection on $\tilde{M}$ we have

$$P(\nabla_X FY) - P(A_{uY}X) = F(\nabla_X Y), \tag{1.25}$$

$$Q(\nabla_X FY) - Q(A_{uY}X) = Bh(X, Y), \tag{1.26}$$

$$h(X, FY) + \nabla^{\perp}_X uY = u(\nabla_X Y) + Ch(X, Y). \tag{1.27}$$

As a consequence of (1.25) — (1.27), for the covariant derivative of F and u we obtain

$$(\nabla_X F)\, Y = Bh(X, Y) + A_{uY}X, \tag{1.28}$$

$$(\nabla_X u)Y = Ch(X, Y) - h(X, FY), \tag{1.29}$$

for all X, Y tangent to M.

§2. Normal CR-submanifolds of a Kaehler manifold.

Let M be a CR-submanifold of a Kaehler manifold $\tilde{M}$. By means of the CR-structure (F, u) we define the tensor field S by

$$S(X, Y) = N_F(X, Y) - 2\, Bdu(X, Y), \tag{2.1}$$

for all $X, Y \in TM$. The tensor field S is called the torsion tensor of the CR-structure (F, u).

Substituting du by its expression from (1.11) and taking account that ∇ is a linear connection without torsion, the right hand side of (2.1) becomes

$$S(X, Y) = (\nabla_{FX}F)Y - (\nabla_{FY}F)X + F\{(\nabla_Y F)X - (\nabla_X F)Y\} - B\{(\nabla_X u)Y - (\nabla_Y u)X\}. \tag{2.2}$$

The CR-submanifold M is said to be *normal* if the torsion tensor S of the CR-structure (F, u) vanishes on M.

It is not difficult to see that when M is a real hypersurface of a Kaehler manifold $\tilde{M}$ we get the well known notion of normal real hypersurface of a Kaehler manifold [7].

Now, we can state

Theorem 2.1. *The CR-submanifold M is normal if and only if we have*

$$A_{uY}(FX) = F(A_{uY}X), \tag{2.3}$$

for all $X \in D$ and $Y \in D^{\perp}$.

Proof. By using (1.28) and (1.29) into (2.2) we obtain

$$S(X, Y) = (A_{uY}F - FA_{uY})X - (A_{uX}F - FA_{uX})Y, \tag{2.4}$$

for all $X, Y \in TM$. Suppose M is a normal CR-submanifold of $\tilde{M}$. Then, from (2.4) follows (2.3) since $A_{uX} = 0$ for all $X \in D$. Now, if (2.3) is satisfied we shall prove

$S = 0$ by using the decomposition $TM = D \oplus D^{\perp}$. First, we see from (2.4) that $S(X, Y) = 0$ for all $Y, X \in D$. Next, for $X \in D$ and $Y \in D^{\perp}$ we obtain

$$S(X, Y) = (A_{uY}F - FA_{uY})\, X = 0,$$

by using (2.3). Finally, for $X, Y \in D^{\perp}$, (2.4) becomes

$$S(X, Y) = F(A_{uX}Y - A_{uY}X), \tag{2.5}$$

since $FX = FY = 0$ in this case. By (1.25) and (1.26) we obtain

$$A_{uX}Y - A_{uY}X = F([X, Y]), \tag{2.6}$$

for all $X, Y \in D^{\perp}$. Since the totally real distribution is always involutive [4], from (2.5), (2.6) and (1.4) we obtain $S(X, Y) = 0$. Thus, the proof is complete.

Suppose now $\{E_1, \ldots, E_q\}$ be a local field of orthonormal frames for the totally real distribution. Denote by A_i the fundamental tensors corresponding to $N_i = JE_i \in (TM)^{\perp}$, for $i = 1, \ldots, q$. Then, from Theorem 2.1, we have

Corallary 2.1. *The CR-submanifold M is normal if and only if the fundamental tensors of Weingarten A_i commute with F on the holomorphic distribution, that is,*

$$A_i F = FA_i, \; i = 1, \ldots, q. \tag{2.7}$$

This corollary is a generalisation of a result obtained by M. Okumura [7] concerning normal real hypersurfaces of a Kaehler manifold.

Now, we need

Lemma 2.1. *A CR-submanifold N of a Kaehler manifold $\tilde{M}$ is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$ and $N \in (TM)^{\perp}$.*

By using this lemma we can prove

Proposition 2.1. *Each normal generic CR-submanifold of a Kaehler manifold $\tilde{M}$ is mixed totally geodesic.*

Proof. By using Corollary 2.1, we obtain $FA_iX = A_iFX$ for all $X \in D$. Thus, the assertion follows by means of (1.4) and Lemma 2.1.

From the equations of Gauss and Weingarten we have

$$\nabla_X E_i = FA_iX - B\,\nabla^{\perp}_X N_i \text{ and} \tag{2.8}$$

$$\nabla^{\perp}_X N_i = JQ\,\nabla_X E_i + Ch(X, E_i), \tag{2.9}$$

for all $X \in TM$ and $i = 1, \ldots, q$.

It is known that X is a Killing vector field if and only if we have

$$g(\nabla_Z X, Y) + g(Z, \nabla_Y X) = 0, \tag{2.10}$$

for all $Y, Z \in TM$. We introduce here a weaker condition for X. Thus we say that X is a D-Killing vector field on M if we have

$$g(P\nabla_Z X, PY) + g(PZ, P\nabla_Y X) = 0, \tag{2.11}$$

for all $Y, Z \in TM$. Then, we have

Theorem 2.2. *A necessary and sufficient condition for the CR-submanifold M to be normal is that $E_i (i = 1, \dots, q)$ be D-Killing vector fields.*

Proof. By using (2.8) we obtain

$$g(\nabla_Z E_i, Y) + g(\nabla_Y E_i, Z) = g((FA_i - A_iF)Z, Y) - g(B\nabla^\perp_Z N_i, Y) - g(B\nabla^\perp_Y N_i, Z), \tag{2.12}$$

for all $Y, Z \in TM$. From (2.9) we have

$$B\nabla^\perp_X N_i = -Q\nabla_X E_i, \tag{2.13}$$

for all $X \in TM$. Hence (2.12) becomes

$$g(P\nabla_Z E_i, PY) + g(P\nabla_Y E_i, PZ) = g((FA_i - A_iF)Z, Y). \tag{2.14}$$

Then, the assertion of our theorem follows from (2.14) and Corollary 2.1.

From (2.7) we see that each totally geodesic CR-submanifold of a Kaehler manifold is a normal CR-submanifold. We shall obtain a more general class of normal CR-submanifolds.

The CR-submanfold M is called pseudo-umbilical with respect to the normal sections $N_i = JE_i$ $(i = 1, \dots, q)$ if the fundamental tensors A_i are given by

$$A_iX = a_iX + b_ig(X, E_i)E_i, \tag{2.15}$$

where a_i, b_i are differentiable functions on M and X is an arbitrary vector field on M. Then, we have

Proposition 2.2. *Each CR-submanifold M of a Kaehler manifold $\widetilde{M}$ which is pseudo-umbilical with respect to the normal sections $N_i = JE_i$ is a normal CR-submanifold.*

The proof of proposition follows from (2.7) and (2.15).

From this proposition we have

Corollary 2.2. *Each totally umbilical CR-submanifold of a Kaehler manifold is a normal CR-submanifold.*

Normal CR-submanifolds can be characterized by another tensor field S^* on M defined by

$$S^*(X, Y) = (L_XF)Y, \tag{2.16}$$

where L_X denotes the operator of Lie derivation with respect to the vector field X.

Theorem 2.3. *Let M be a CR-submanifold of a Kaehler manifold $\widetilde{M}$, such that*

$$Q\nabla_XY = 0 \text{ for all } X \in D \text{ and } Y \in D^\perp. \tag{2.17}$$

Then, M is normal if and only if,

$$S^*(Y, X) = 0, \tag{2.18}$$

for all $Y \in D^\perp$ and $X \in D$.

Proof. We can see from the proof of Theorem 2.1. that M is normal if and only if $S(X, Y) = 0$ for all $X \in D$ and $Y \in D^\perp$. By means of (1.11) (1.29) and (2.1) we obtain

$$S(X, Y) = F(F[X, Y] - [JX, Y]) - Bh(JX, Y), \tag{2.19}$$

for all $X \in D$ and $Y \in D^\perp$. Next, from (1.27) we have

$$h(JX, Y) = JQ(\nabla_YX) + Ch(X, Y), \tag{2.20}$$

which implies

$$Bh(JX, Y) = -Q(\nabla_YX). \tag{2.21}$$

Hence (2.19) becomes

$$S(X, Y) = F(F[X, Y] - [JX, Y]) + Q(\nabla_YX). \tag{2.22}$$

On the other hand, from (2.16) follows

$$S^*(Y, X) = F[X, Y] - [JX, Y], \tag{2.23}$$

for all $X \in D$ and $Y \in D^\perp$.

Now, if M is normal, from (2.22) we obtain

$$PS^*(Y, X) = 0 \text{ and} \tag{2.24}$$

$$Q(\nabla_YX) = 0. \tag{2.25}$$

Also, from (2.23) we see that $QS^*(Y, X) = 0$ if and only if $Q([JX, Y]) = 0$. But the last equality follows from (2.17) and (2.25). Therefore, $S^*(Y, X) = 0$ for all $Y \in D^\perp$ and $X \in D$.

Conversely, $S^*(Y, X) = 0$ implies

$$F(F[X, Y] - [JX, Y]) = 0 \text{ and} \tag{2.26}$$

$$Q([X, Y]) = 0, \tag{2.27}$$

for all $X \in D$ and $Y \in D^\perp$. Thus, by using (2.17), (2.26) and (2.27) we obtain $S(X, Y) = 0$. This completes the proof.

§3. Cosymplectic CR-submanifolds of a Kaehler manifold. The CR-submanfold M of a Kaehler manifold $\widetilde{M}$ is called a *cosymplectic CR-submanifold* if it is normal and the differential forms u and Φ are closed, that is,

$$du = 0 \text{ and } d\Phi = 0. \tag{3.1}$$

In this paragraph we are dealing only with cosymplectic generic CR-submanifolds. In this case (1.28) and (1.29) become

$$(\nabla_XF)Y = Jh(X, Y) + A_{uY}X \text{ and respectively,} \tag{3.2}$$

$$(\nabla_Xu)Y = -h(X, FY), \tag{3.3}$$

for all vector fields X, Y tangent to M.

We call F (resp. u) parallel on M if we have $\nabla_XF = 0$ (resp. $\nabla_Xu = 0$) for each $X \in TM$.

Now, we can state

Theorem 3.2. *Let M be a generic CR-submanifold of a Kaehler manifold $\bar{M}$. Then, F is parallel if and only if u is parallel.*

Proof. Suppose F is parallel on M. Then, for all $Y \in D$ and $X \in TM$, from (3.2) follows $h(X, Y) = 0$. Hence in this case, by means of (3.3), we have $(\nabla_X u)Y = 0$. When $Y \in D^\perp$, (3.3) implies $(\nabla_X u)Y = 0$ since $FY = 0$.

Conversely, if u is parallel, then from (3.2) and (3.3) follows $(\nabla_X F)Y = 0$ for each $Y \in D$. Now, we take Y from $D^\perp$ and we have

$$g(PA_{JY}X, Z) = g(h(X, Z), JY) = 0, \tag{3.4}$$

for all $X \in TM$ and $Z \in D$. From (3.2) follows that $(\nabla_X F)Y = 0$ for all $X \in TM$ and $Y \in D^\perp$ if and only if we have

$$PA_{JY}X = 0 \text{ and} \tag{3.5}$$

$$Bh(X, Y) + QA_{JY}X = 0. \tag{3.6}$$

But, (3.5) follows from (3.4) and (3.6) is a consequence of (1.26). Thus, the proof is complete.

Theorem 3.2. *Let M be a generic CR-submanfold of a Kaehler manifold $\bar{M}$. Then, the following three assertions are equivalent to each other*

1. *The CR-submanifold M is cosymplectic;*
2. *The CR-form u is parallel;*
3. *The f-structure F is parallel.*

Proof. The equivalence of assertions 2 and 3 has been proved in Theorem 3.1. Suppose the condition 1 is accomplished. Then, since M is normal, and $du = 0$, form (2.1) we obtain $X_F = 0$. Hence (1.14) implies F is parallel on M. Now, if F is parallel then from (2.2) we see that M is a normal CR-submanifold. Of course $du = 0$, since $\nabla_X u = 0$ for each $X \in TM$. Thus we have only to prove $d\Phi = 0$. First, from (1.14) we have

$$d\Phi(X, Y, Z) = d\Phi(X, FY, FZ), \tag{3.7}$$

for all $X, Y, Z \in TM$. If we put $Y = X$ into (3.7) we obtain $d\Phi(X, FY, FZ) = 0$ which implies

$$d\Phi(X, FY, FZ) + d\Phi(Y, FX, FZ) = 0. \tag{3.8}$$

Also,, from (3.7) we have

$$d\Phi(X, Y, Z) = 0, \tag{3.9}$$

whenever $Y \in D^\perp$ and X, Z are arbitrary vector fields on M.

Next, by using (1.4), (1.14) and (3.7) – (3.9) we obtain

$$\begin{aligned} d\Phi(X, FY, FZ) &= -d\Phi(Y, FX, FZ) = -d\Phi(FX, FZ, Y) = \\ &= -d\Phi(FX, F^2Z, FY) = d\Phi(FX, PZ, FY) = d\Phi(FX, Z, FY) = \\ &= -d\Phi(Z, X, Y) = -d\Phi(X, Y, Z). \end{aligned} \tag{3.10}$$

Thus, from (3.7) and (3.10) follows $d\Phi(X, Y, Z) = 0$. Therefore, assertion 2 implies assertion 1. The proof of the theorem is complete.

From (3.3) and Theorem 3.2, we have

Theorem 3.3. *Let M be a generic CR-submanifold of a Kaehler manifold $\bar{M}$. Then, M is a cosymplectic CR-submanifold, if and only if we have*

$$h(X, Y) = 0, \tag{3.11}$$

for all $X \in TM$ and $Y \in D$.

Now, we can state

Theorem 3.4. *A generic CR-submanifold M of a Kaehler manifold $\bar{M}$ is cosymplectic if and only if we have*

$$(\nabla_X F)X = 0, \tag{3.12}$$

for all vector fields X tangent to M.

Proof. Of course, if M is a cosymplectic CR-submanifold, then (3.12) is satisfied since F is parallel on M by Theorem 3.2.

Conversely, suppose (3.12) is satisfied on M. Then, substituting both X and Y from (3.2) by $X + Y$ and taking account of (3.12) we obtain

$$2Jh(X, Y) + A_{uY}X + A_{uX}Y = 0, \tag{3.13}$$

for all $X, Y \in TM$. If we take X and Y from D, then (3.13) implies

$$h(X, Y) = 0 \quad \forall X, Y \in D. \tag{3.14}$$

Next, for $X \in D^\perp$ and $Y \in D$, (3.13) becomes

$$2Jh(X, Y) + A_{JX}Y = 0. \tag{3.15}$$

On the other hand, from (1.26) we obtain

$$Jh(X, Y) + QA_{JX}Y = 0. \tag{3.16}$$

Thus, from (3.15) and (3.16), taking into account that $J = B$, we have

$$h(X, Y) = 0 \text{ for all } X \in D^\perp \text{ and } Y \in D. \tag{3.17}$$

Finally, we see that (3.14) and (3.17) imply (3.11). Thus the proof is complete.

From Proposition 4. of [3] and our Theorem 3.2, we obtain

Theorem 3.5. *Each cosymplectic generic CR-submanifold M of a Kaehler manifold $\bar{M}$ is locally a Riemannian direct product $M^T \times M^\perp$ where M^T is a totally geodesic complex submanifold of $\bar{M}$ of complex dimension $m - n$ and $M^\perp$ is a totally real submanifold of $\bar{M}$ of real dimension $2n - m$.*

In a forthcoming paper, we shall study normal CR-submanifolds and cosymplectic CR-submanifolds in a complex space form.

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CONGRUENCES DE CÔNES QUADRATIQUES TANGENTS À LA SURFACE SUPPORT

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0. Introduction. Dans l'espace projectif P_3, on considère une surface regulière Σ de classe $C^t (t \geqslant 1)$ et, à chaque point $M(u_1, u_2) \in \Sigma$, on attache un cône quadratique (Q), dont le sommet est au point M. On y distingue trois variétés biparamétriques de cônes quadratiques, selon que les droites suivant lesquelles le plan (π) — tangent à Σ en M, - coupe le cône (Q) sont : 1) réelles et distinctes, 2) imaginaires ou 3) réelles et coïncidentes. Nous notons ces variétés par $V_i(2 ; (1,1) ; P_3)$, où les valeurs 1, 2, 3 de l'indice i correspondent aux trois situations mentionées et, dans la paranthèse sont précisés : le nombre 2 des paramètres, l'espèce 1 et la signature 1 du cône générateur, l'espace projectif ambiant, P_3 [3]. Les variétés V_1, V_2 ont été étudiées dans [1] ; nous allons nous occuper des variétés V_3, que nous nommons *congruences de cônes quadratiques tangents à la surface support* Σ. L'image de V_3 dans l'espace S_9 [2] est une surface, que nous notons par $\overline{V}_3$. Les indices utilisés sont : $h = 1, 2$; $i = 1, 2, 3$; $\alpha = \overline{0,3}$; $s = \overline{1,7}$.

1. Les formules Frenet et le théorème fondamental. Nous choisissons les repères d'ordre zéro par les mêmes conditions que dans [2], de manière que l'équation du cône local, rapporté à ces repères, est

$$Q(u_1, u_2 ; t_s) \equiv (x^1)^2 + (x^2)^2 - (x^3)^2 = 0, \quad x^i = f^i(u_h ; t_s). \tag{1.1}$$

où t_s sont les paramètres secondaires. Avec les conditions de fixité d'un point dans P_3 (ou du point-image (Q) dans S_9 [2]), nous obtenons huit pfaffiens principaux d'ordre zéro : ω_0^i, $\omega_2^1 + \omega_1^2$, $\omega_3^1 - \omega_1^3$, $\omega_3^2 - \omega_2^3$, $\omega_1^1 - \omega_2^2$, $\omega_1^1 - \omega_3^3$. En excluant de l'étude les variétés pour lesquelles au moins un de ces pfaffiens est $\equiv 0$ ou $|\omega_0^1, \omega_0^2| = 0$ (variétés, dont les images $\overline{V}_3$ sont situées dans un espace projectif P_r $(r < 8)$, nous notons : $\omega_0^3 = \alpha_h \omega_0^h$, $\omega_2^1 + \omega_1^2 = \alpha_{2, h} \omega_0^h, \ldots, \omega_1^1 - \omega_3^3 = \alpha_{11, h} \omega_0^h$. Alors le plan (Π), déterminé par A_i et dA_0, a l'équation

$$(\Pi) \equiv \alpha_1 x^1 + \alpha_2 x^2 - x^3 = 0. \tag{1.2}$$

L'intersection du plan (Π) et du cône local (Q) est donnée par le système

$$\alpha_1 x^1 + \alpha_2 x^2 - x^3 = 0, \quad (1 - \alpha_1^2)(x^1)^2 - 2\alpha_1\alpha_2 x^1 x^2 + (1 - \alpha_2^2)(x^2)^2 = 0. \tag{1.3}$$