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FIX POINTS OF LATTICE AND BOOLEAN TRANSFORMATION

BY

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A classical theorem due to Tarski [14] states that every isotone map $f: L \to L$ of a complete lattice into itself has a fix point. The proof is very easy: if a denotes the greatest lower bound (g.l.b) of the set $A = \{x \in L \mid f(x) \leqslant x\}$, it follows that $f(a)$ is a lower bound of A, hence $f(a) \leqslant a$, which implies $f(a) \in A$, therefore $a \leqslant f(a)$. As a matter of fact, a is the least element both of the set A and of the subset $F = \{x \in L \mid f(x) = x\}$ of fix points. Conversely, if L is a lattice such that every isotone map $f: L \to L$ has a fix point, then L is complete [4]. For various generalizations of this theorem see e.g. the papers quoted in [6], page 55.

A somewhat similar argument yields the Kleene fix point theorem [7]. Replace the completeness of L by the weaker condition of *ω-completeness*, which requires that every increasing sequence of elements from L have a least upper bound (l.u.b.), and replace the isotony of f by the stronger condition of *ω-continuity*, which requires that f commute with the l.u.b. of every increasing sequence. Then for every $x \in L$ such that $x \leqslant f(x)$, the l.u.b. of the increasing sequence

$$x \leqslant f(x) \leqslant f^2(x) \leqslant \ldots \leqslant f^n(x) \leqslant \ldots$$

(where f^n means iterated composition of functions) is a fix point of f. In particular the l.u.b. of the sequence $(f^n(0))_{n \in N}$ is the least fix point of f. This and several related theorems have important applications in informatics. We only mention here a recent generalization within the framework of category theory [3].

Our aim is to begin a study of fix points for maps defined over a distributive lattice or a Boolean algebra, say L. It might be natural to confine our attention to *lattice polynomials*, i.e., to maps $f: L^n \to L$ built up from variables by superpositions of the basic operations $\vee$ (disjunction) and $\wedge$ (conjunction) and also $'$ (negation) in the case of a Boolean algebra. More generally, by a *lattice function* (*Boolean function*, if L is a Boolean algebra) we will understand a map $f: L^n \to L$ obtained from a lattice polynomial $g: L^{n+p} \to L$ by substituting p fixed elements of L for some of the variables of g. These concepts are the specializations to lattices and Boolean algebras of Grätzer's universal-algebraic concepts of polynomial and algebraic function, respectively [5]. As a matter of fact, we will consider the slightly more general case of *lattice transformations*. By this term we mean a map $F = (f_1, \dots, f_n) : L^n \to L^n$, where each $f_i : L^n \to L$ is a lattice function and which is defined componentwise; the order relation of L^n is also defined componentwise. In particular if L is a Boolean algebra, we call F a *Boolean transformation*; for various properties of Boolean transformations see [9], [10], [11].

As might be expected, lattice transformations have strong properties. Thus if L is ω-complete and the functions of the form $f(x) = a \wedge x$ are ω-continuous, then every lattice transformation is ω-complete (Theorem 1), hence it has fix points (Corollary 1). In particular if B is a σ-Boolean algebra, then every isotone Boolean transformation $F : B^n \to B^n$ is ω-complete (Corollary 2). The subsequent results refer to arbitrary Boolean algebras (or distributive lattices with 0, in the case of Proposition 1). Using the tool which consists in solving Boolean equations, we give a necessary and sufficient condition for a Boolean transformation to have a fix point and we characterize the fix points as solutions of a Boolean equation (Theorem 2). Using this result and a combinatorial argument, we prove that every isotone Boolean transformation has fix points (Theorem 3).

For $n = 1$ we obtain the following stronger properties: every lattice function f has the least fix point $f(0)$ (Proposition 1) and, as was first shown by Scognamiglio [13], isotony is the necessary and sufficient condition for a Boolean function $f : B \to B$ to have fix points (Proposition 2). However for $n > 1$ there exist non-isotone Boolean transformations having fix points which cannot be reached by the Kleene iterative procedure.

The starting point of our research is the structure of lattice functions.

Lemma 1. *Let L be a distributive lattice with 0 and let $n \in N$. A map $f : L^n \to L$ is a lattice function if and only if it can be written in the form*

$$f(x_1, \dots, x_n) = \bigvee_{k=0}^{2^n-1} m_k(x_1, \dots, x_n), \tag{1}$$

where $\bigvee$ means iterated disjunction and for each $k \in \{0, 1, \dots, 2^n - 1\}$, $m_k(x_1, \dots, x_n)$ is either $x_{i_1} \wedge \dots \wedge x_{i_r}$ or $a_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r}$ with $a_k \in L$, the indices $i_1, \dots, i_r \in \{1, \dots, n\}$ being determined by the condition $2^{i_1} + \dots + 2^{i_r} = k$; in particular $m_0(x_1, \dots, x_n) = a_0 \in L$.

Proof. It is well known (see e.g. [6], ch. II, § 2, Theorem 1) that a lattice polynomial $f : L^n \to L$ over a distributive lattice L can be written as a disjunction of meet polynomials $x_{i_1} \wedge \dots \wedge x_{i_r}$ and since in our case the missing terms can be introduced as $0 \wedge x_{j_1} \wedge \dots \wedge x_{j_s}$, we obtain a representation of the form (1). A lattice function can be obtained from a lattice polynomial by giving fixed values in L to certain variables and after a suitable rearrangement of terms we obtain again a representation of the form (1).

Theorem 1. *Suppose L is an ω-complete distributive lattice with 0 and for every $a \in L$, the function $f_a : L \to L$ such that*

$$f_a(x) = a \wedge x \qquad (\forall x \in L) \tag{2}$$

is ω-continuous. Then every lattice transformation $F : L^n \to L^n$ is ω-continuous.

Proof. According to the hypothesis,

$$a \wedge \bigvee_{n \in N} x_n = \bigvee_{n \in N} (a \wedge x_n), \tag{3}$$

for every increasing sequence $(x_n)_{n \in N}$. This implies that

$$\left(\bigvee_{n \in N} x_n\right) \wedge \left(\bigvee_{n \in N} y_n\right) = \bigvee_{n \in N} (x_n \wedge y_n), \tag{4}$$

provided the sequences $(x_n)_{n \in N}$ and $(y_n)_{n \in N}$ are increasing. For in view of (3) we get

$$\left(\bigvee_{n \in N} x_n\right) \wedge \left(\bigvee_{n \in N} y_n\right) = \bigvee_{n \in N} \left[\left(\bigvee_{m \in N} x_m\right) \wedge y_n\right] = \bigvee_{n \in N} \bigvee_{m \in N} (x_m \wedge y_n) =$$
$$= \bigvee_{n \in N} (x_n \wedge y_n),$$

because for every $m, n \in N$ we have $x_m \wedge y_n \leqslant x_p \wedge y_p$, where $p = \max(m, n)$.

Now let $(x_{1p})_{p \in N}, \dots, (x_{np})_{p \in N}$ be increasing sequences. It follows immediately from (3) and (4) that

$$m_k\left(\bigvee_{p \in N} x_{1p}, \dots, \bigvee_{p \in N} x_{np}\right) = \bigvee_{p \in N} m_k(x_{1p}, \dots, x_{np}), \tag{5}$$

where the notation m_k is explained in Lemma 1. We claim that

$$f\left(\bigvee_{p \in N} x_{1p}, \dots, \bigvee_{p \in N} x_{np}\right) = \bigvee_{p \in N} f(x_{1p}, \dots, x_{np}) \tag{6}$$

for every lattice function $f : L^n \to L$. For Lemma 1, property (5) and the obvious fact that

$$\bigvee_{h=1}^{q} \bigvee_{p \in N} z_{hp} = \bigvee_{p \in N} \bigvee_{h=1}^{q} z_{hp} \tag{7}$$

for every q increasing sequences $(z_{1p})_{p \in N}, \dots, (z_{qp})_{p \in N}$, imply that

$$f\left(\bigvee_{p \in N} x_{1p}, \dots, \bigvee_{p \in N} x_{np}\right) = \bigvee_{k=0}^{2^n-1} m_k\left(\bigvee_{p \in N} x_{1p}, \dots, \bigvee_{p \in N} x_{np}\right) =$$
$$= \bigvee_{k=0}^{2^n-1} \bigvee_{p \in N} m_k(x_{1p}, \dots, x_{np}) = \bigvee_{p \in N} \bigvee_{k=0}^{2^n-1} m_k(x_{1p}, \dots, x_{np}) = \bigvee_{p \in N} f(x_{1p}, \dots, x_{np}).$$

Finally let $F = (f_1, \dots, f_n) : L^n \to L^n$ be a lattice transformation. Then it follows from (6) that

$$F\left(\bigvee_{p \in N} x_{1p}, \dots, \bigvee_{p \in N} x_{np}\right) = \left(f_i\left(\bigvee_{p \in N} x_{1p}, \dots, \bigvee_{p \in N} x_{np}\right)\right)_{i=1, \dots, n} =$$
$$= \left(\bigvee_{p \in N} (f_i(x_{1p}, \dots, x_{np})\right)_{i=1, \dots, n} = \bigvee_{p \in N} (f_i(x_{1p}, \dots, x_{np}))_{i=1, \dots, n},$$

that is

$$F\left(\bigvee_{p \in N} x_{1p}, \dots, \bigvee_{p \in N} x_{np}\right) = \bigvee_{p \in N} F(x_{1p}, \dots, x_{np}), \tag{8}$$

as desired.

Corollary 1. *Under the conditions of Theorem 1, every lattice transformation $F : L^n \to L^n$ has fix points.*

Proof. From Theorem 1 and the Kleene fix point theorem.

Corollary 2. *Every isotone Boolean transformation $F : B^n \to B^n$ over a σ-Boolean algebra B is ω-continuous.*

Proof. Let B and $F = (f_1, \dots, f_n)$ be as stated above. Clearly every f_i is an isotone Boolean function, therefore it has the form indicated in

Lemma 1 (see e.g. [9], Theorem 11.3). Consequently $F : B^n \to B^2$ is a lattice transformation over the distributive lattice B. On the other hand a σ-Boolean algebra is known to fulfil the countable distributivity law (3), so that we can apply Theroem 1.

Corollary 2 implies further that every isotone Boolean transformation has fix points, but we will prove this in Theorem 3 without any supplementary hypothesis on the Boolean algebra B.

Theorem 2. *Let $F = (f_1, \dots, f_n) : B^n \to B^n$ be a Boolean transformation over an arbitrary Boolean algebra B. Then:*

(i) *F has fix points if and only if*

$$\bigwedge_{\alpha_1, \dots, \alpha_n \in \{0,1\}} \bigvee_{i=1}^{n} f_i^{\alpha_i+1}(\alpha_1, \dots, \alpha_n) = 0, \tag{9}$$

where $\bigwedge$ means iterated conjunction, $+$ stands for the ring sum and the exponents 1, 0 denote identity and negation, respectively.

(ii) *The set of fix points coincides with the set of solutions to equation*

$$\bigvee_{\alpha_1, \dots, \alpha_n \in \{0,1\}} \left[\bigvee_{i=1}^{n} f_i^{\alpha_i+1}(\alpha_1, \dots, \alpha_n) \right] \wedge x_1^{\alpha_1} \wedge \dots \wedge x_n^{\alpha_n} = 0. \tag{10}$$

Comment. To facilitate the understanding of formulas (9) and (10), we transcribe them explicitly for $n = 2$. If we set

$$a_{00} = f_1(0, 0) \vee f_2(0, 0), \tag{11.1}$$

$$a_{01} = f_1(0, 1) \vee f_2'(0, 1), \tag{11.2}$$

$$a_{10} = f_1'(1, 0) \vee f_2(1, 0), \tag{11.3}$$

$$a_{11} = f_1'(1,1) \vee f_2'(1,1), \tag{11.4}$$

condition (9) becomes

$$a_{00} \wedge a_{01} \wedge a_{10} \wedge a_{11} = 0. \tag{12}$$

while equation (10) can be written

$$(a_{11} \wedge x_1 \wedge x_2) \vee (a_{10} \wedge x_1 \wedge x_2') \vee (a_{01} \wedge x_1' \wedge x_2) \vee (a_{00} \wedge x_1' \wedge x_2') = 0. \tag{13}$$

Proof. We use well-known properties of Boolean equations (see e.g. [9]). First we take an arbitrary $i \in \{1, \dots, n\}$ and write down the equation

$$f_i(x_1, \dots, x_n) = x_i$$

in the following equivalent forms:

$$[x_i' \wedge \bigvee_{\alpha_1, \dots, \alpha_n \in \{0,1\}} f_i(\alpha_1, \dots, \alpha_n) \wedge x_1^{\alpha_1} \wedge \dots \wedge x_n^{\alpha_n}] \vee$$

$$\vee [x_i \wedge \bigvee_{\alpha_1, \dots, \alpha_n \in \{0,1\}} f_i'(\alpha_1, \dots, \alpha_n) \wedge x_1^{\alpha_1} \wedge \dots \wedge x_n^{\alpha_n}] = 0,$$

then

$$\bigvee_{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n \in \{0,1\}} [f_i(\alpha_1, \dots, \alpha_n) \wedge x_1^{\alpha_1} \wedge \dots$$

$$\wedge x_i' \wedge \dots \wedge x_n^{\alpha_n} \vee f_i'(\alpha_1, \dots, \alpha_n) \wedge x_1^{\alpha_1} \wedge \dots \wedge x_i \wedge \dots \wedge x_n^{\alpha_n}] = 0.$$

or else

$$\bigvee_{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n \in \{0,1\}} \bigvee_{\alpha_i=0}^{1} [f_i^{\alpha_i+1}(\alpha_1, \dots, \alpha_n) \wedge x_1^{\alpha_1} \wedge \dots \wedge x_n^{\alpha_n}] = 0.$$

Therefore the condition

$$f_i(x_1, \dots, x_n) = x_i \qquad (i = 1, \dots, n)$$

for $(x_1, \dots, x_n)$ to be a fix point of f can be written in the form

$$\bigvee_{i=1}^{n} \bigvee_{\alpha_1, \dots, \alpha_n \in \{0,1\}} f_i^{\alpha_i+1}(\alpha_1, \dots, \alpha_n) \wedge x_1^{\alpha_1} \wedge \dots \wedge x_n^{\alpha_n} = 0,$$

which is equivalent to (10). The consistency condition for equation (10) is (9) (cf. [1], [12]; see e.g. [9], Theorem 2.3).

Theorem 3. *Every isotone Boolean transformation has fix points.*

Proof. Suppose B is an arbitrary Boolean algebra and $F = (f_1, \dots, f_n) : B^n \to B^n$ an isotone Boolean transformation. As we have already remarked, this implies that every f_i is isotone. We have to check condition (9) which, in view of distributivity, can be written

$$\bigvee_{\varphi} \bigwedge_{(\alpha_1, \dots, \alpha_n) \in \{0,1\}^n} f_{\varphi(\alpha)}^{\alpha_{\varphi(\alpha)}+1}(\alpha_1, \dots, \alpha_n) = 0, \tag{14}$$

where the disjunction is extended over all the maps $\varphi : \{0, 1\}^n \to \{1, \dots, n\}$ and $\varphi(\alpha)$ is an abbreviation for $\varphi(\alpha_1, \dots, \alpha_n)$. Therefore we take an arbitrary but fixed such φ and will prove that

$$\bigwedge_{\alpha_1, \dots, \alpha_n \in \{0,1\}} f_{\varphi(\alpha)}^{\alpha_{\varphi(\alpha)}+1}(\alpha_1, \dots, \alpha_n) = 0. \tag{15}$$

To prove (15) it suffices to find $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \{0, 1\}^n$ and $i \in \{1, \dots, n\}$ such that

$$\alpha < \beta \;\&\; \alpha_i = 0 \;\&\; \beta_i = 1 \;\&\; \varphi(\alpha) = \varphi(\beta) = i; \tag{16}$$

for (16) implies that $f_i(\alpha) < f_i(\beta)$ and that the left side of (15) is included in

$$f_i^{\alpha_i+1}(\alpha) \wedge f_i^{\beta_i+1}(\beta) = f_i(\alpha) \wedge f_i'(\beta) = 0.$$

To find α and β which fulfil (16), it is convenient to replace the poset $(\{0, 1\}^n, \leqslant)$ by its image $(\mathcal{P}(\{1, \dots, n\}), \leqslant)$ under the well-known isomorphism defined by:

$$(\gamma_1, \dots, \gamma_n) \to \{i_1, \dots, i_q\},$$

if and only if

$$\gamma_{i_1} = \dots = \gamma_{i_q} = 1 \;\&\; \gamma_{i_{q+1}} = \dots = \gamma_{i_n} = 0;$$

then condition (16) becomes

$$\alpha \subset \beta \;\&\; i \in \beta - \alpha \;\&\; \varphi(\alpha) = \varphi(\beta) = i. \tag{17}$$

To prove (17), define a sequence $(i_0, i_1, \dots, i_n)$ as follows: $i_0 = \varphi(\varnothing)$ and $i_h = \varphi(\{i_0, \dots, i_{h-1}\})$ $(h = 1, 2, \dots, n)$. Since $i_0, i_1, \dots, i_n \in \{1, \dots, n\}$, it follows that $i_j = i_k$ for some $j \neq k$. Let $j \in \{0, 1, \dots, n\}$ be the least index for which there exists $k \in \{0, 1, \dots, n\}$ such that $j < k$ and $i_j = i_k$. Take $\alpha = \{i_0, \dots, i_{j-1}\}$ or $\alpha = \varnothing$ as $j > 0$ or $j = 0$; take also $\beta = \{i_0, \dots, i_{k-1}\}$ and $i = i_j = i_k$. Then α, β and i fulfil (17) and this concludes the proof.

The last two propositions of this paper deal with the case $n = 1$.

Proposition 1. *Suppose L is a distributive lattice with 0. Then every lattice function $f : L \to L$ has the least fix point $f(0)$.*

Proof. In view of Lemma 1 the function f is of the form $f(x) = x \vee a_0$ or of the form $f(x) = (a_1 \wedge x) \vee a_0$. In both cases $f(0) = a_0$ and $f(a_0) = a_0$. On the other hand if p is an arbitrary fix point then $f(0) \leqq f(p) = p$.

Proposition 1 can be rephrased to the effect that the least fix point is reached in one step by the Kleene iterative procedure although L need not be ω-complete; take e.g. the increasing sequence $(\{1, \ldots, n\})_{n \in N}$ in the lattice of all finite subsets of N (or in the Boolean algebra of all finite and cofinite subsets of R).

Proposition 2. (Scognamiglio [13]). *Suppose B is an arbitrary Boolean algebra. Then a Boolean function $f : B \to B$ has fix points if and only if it is isotone, in which case the set of fix points is the range $[f(0), f(1)]$.*

Comment. For other properties equivalent to the isotony of the function f see [8], Theorem 11.2.

Proof. Condition (9) in Theorem 2 reduced to $f(0) \wedge f'(1) = 0$, that is to $f(0) \leqslant f(1)$, which is equivalent to isotony (cf. Mc Kinsey [8]; see also [9], Theorem 2.13). The fix points are determined by equation (10), which reduces to

$$(f(0) \wedge x') \vee (f'(1) \wedge x) = 0,$$

and is equivalent to $f(0) \leqslant x \leqslant f(1)$. But the interval

$$[f(0), f(1)] = [f(0) \wedge f(1), f(0) \vee f(1)]$$

is also the range of the function f (cf. Schröder [12]; see also [9] Theorem 2.4).

Finally we note that for $n > 1$ there exist non-isotone Boolean transformations whose fix points cannot be reached by the Kleene iterative procedure. Take e.g.

$$F(x, y) = (y', (x \wedge y) \vee (x' \wedge y')).$$

In this case formulas (11) yield $a_{00} = 1$, $a_{01} = 1$, $a_{10} = 0$, $a_{11} = 1$, so that equation (13) becomes $(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y) = 0$ or simply $x' \vee y = 0$, which shows that the unique fix point is $(1, 0)$. However $F(0, 0) = (1, 1)$, $F(1, 1) = (0, 1)$ and $F(0, 1) = (0, 0)$.

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