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SEMIADMISSIBLE CATEGORIES AND CONTEXT-SENSITIVE LANGUAGES

BY

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1. Introduction. We shall use in this paper the notations from [1]. For any set X, $\hat{X}$ will denote the power set of X. A relation f from X to Y denoted by $f: X \to Y$ is a function $\hat{f}: \hat{X} \to \hat{Y}$, which is completely additive, that is $\hat{f}(\bigcup\{A_j \mid j \in J\}) = \bigcup\{\hat{f}(A_j) \mid j \in J\}$, for any family $\{A_j \mid j \in J\} \subseteq \hat{X}$.

The graph γf of a relation $f: X \to Y$ is the subset of $X \times Y$ defined by $\gamma f = \{(x, y) \mid y \in f(x)\}$. The domain (dom f) and the codomain (codom f) of a relation f are given by dom $f = \{x \mid x \in X, f(x) \neq \emptyset\}$, codom $f = \{y \mid y \in Y, y \in f(x)\}$.

By $I^*(I^+)$ we shall denote the free monoid (semigroup) generated by the alphabet I. The null word will be denoted by λ; we have $I^* = I^+ \cup \{\lambda\}$.

If N is the set of natural numbers, $N^{\cdot}$ will be the family of sets $N^{\cdot} = \{N^m \mid m \geqslant 0\}$, where N^0 consists from the single element "|" and N^m is the m-th cartesian power of N.

If I is the alphabet $I = \{z_1, \ldots, z_h\}$ and L is a language, $L \subseteq \{z_1\}^* \ldots \{z_h\}^*$, *$L$ represents a relation* $f: N^m \to N^n$, with $m + n = h$ if $z_1^{p_1} \ldots z_h^{p_h} \in L$ *iff* $(p_1, \ldots, p_h) \in \gamma f$. $\mathcal{R}_L$ is the set of relations represented by the language L. The relation $f: N^h \to N^0$ from $\mathcal{R}_L$ will be denoted by ρL.

A grammar is a 4-uple $G = (I_N, I_T, \zeta_0, F)$, where I_N, I_T are the alphabets of nonterminals and terminals, respectively, $\zeta_0 \in I_N$, is the initial symbol and F is a set of rules, $F = \{(u_j, v_j) \mid u_j \in I^* I_N I^*, v_j \in I^*, 1 \leqslant j \leqslant m\}$. Here we have taken $I = I_N \cup I_T$. If $(u_j, v_j) \in F$ we shall denote this by $u_j \to v_j$.

The relation „$\Rightarrow$" and its reflexive and transitive closure „$\overset{*}{\Rightarrow}$" are introduced in the usual way (see [3] or [4]).

Definition. *The grammar G computes the graph of a relation f iff $f \in \mathcal{R}_{L(G)}$.*

Every class of grammars K generates a class of relations $\mathcal{R}_K = \bigcup\{\mathcal{R}_{L(G)} \mid G \in K\}$. The name of K is usually transferred to $\mathcal{R}_K$. For instance we shall speak about the class of „context-free relations". If K is a class of Chomsky's hierarchy we shall use the notation $\mathcal{R}_j$ ($j \in \{0, 1, 2, 3\}$) to designate relations computed by grammars from the j-th class.

If $\mathcal{C}$ is a category, the class of objects of $\mathcal{C}$ is Ob $\mathcal{C}$, the set of morphisms from X to Y, is denoted by $\mathcal{C}(X, Y)$, where $X, Y \in$ Ob $\mathcal{C}$. By $\mathcal{R}$ we shall denote the category having as objects all sets and as sets of morphisms the sets $\mathcal{R}(X, Y) = \{f \mid f : X \to Y\}$, the composition of morphisms being the composition of relatons. The class of all morphisms of $\mathcal{C}$ is Mor $\mathcal{C}$.

The full subcategory of $\mathcal{R}$ determined by the set $N^{\cdot}$ is denoted by $\mathcal{R}(N^{\cdot})$.

An admissible subcategory of $\mathcal{R}(N^{\cdot})$ is a subcategory $\mathcal{A}$ of $\mathcal{R}(N^{\cdot})$ for which:

i) Ob $\mathcal{A} = N^{\cdot}$;

ii) if $f : N^r \to N^s \in \mathcal{A}(N^r, N^s)$ then so does the relation $N \times \times f : N^{1+r} \to N^{1+s}$ defined by $(N \times f)(m, n) = (m, f(n))$, $\forall m \in N$, $n \in \in N$ (the cylindrification property);

iii) the functions $\Theta_k : N^k \to N^k$ $(k \geqslant 2)$ defined by $\Theta_k(m, n, p) = (n, m, p)$, $\forall m, n \in N$, $p \in N^{k-2}$, $\Delta : N \to N^2$ defined by $\Delta(n) = (n, n)$, $\forall n \in N$ and the projection $\Pi : N \to N^0$ (which is unique) belong to Mor $\mathcal{A}$.

We shall call a category $\mathcal{A}$ semiadmissible if only Θ_k and Δ belong to Mor $\mathcal{A}$.

Let $[n]$ be the set $\{1, \ldots, n\}$ for $n \in N$, where $[0] \in \emptyset$. The logical function ([1]) induced by the function $f : [p] \to [q]$ is the function $f^{\#} : N^q \to N^p$ given by $f^{\#}(n_1, \ldots, n_q) = (n_{f(1)}, \ldots, n_{f(p)})$, for every $(n_1, \ldots, n_q) \in N^q$.

If $\mathcal{A}$ is a semiadmissible category, the following facts are quite straightforward (the proofs for admissible categories can be founded in [1]):

i) if $f \in \mathcal{A}(N^r, N^s)$, $g \in \mathcal{A}(N^t, N^u)$ then so is $f \times g : N^{r+t} \to N^{s+u}$, defined by $(f \times g)(x, y) = (f(x), g(y))$, for every $x \in N^r$, $y \in N^t$;

ii) if $f \in \mathcal{A}(N^r, N^s)$ and $g \in \mathcal{A}(N^r, N^t)$ then so is $< f, g > : N^r \to N^{s+t}$, where $< f, g > (x) = (f(x), g(x))$;

iii) if $f : [p] \to [q]$ is a surjective mapping (hence $p \geqslant q$) then $f^{\#} \in \mathcal{A}(N^q, n^p)$.

When $\mathcal{A}$ is an admissible category, for every function $f : [p] \to [q]$, the logical function $f^{\#}$ belongs to $\mathcal{A}(N^q, N^p)$.

II. Context-sensitive Relations and Admisible Categories. Using Parikh's theorem [3] it follows that the class $\mathcal{R}_2$ of context-free relations is rather simple since, if $f \in \mathcal{R}_2$, γf has to be a semilinear subset of N^n. However, some important functions belong to $\mathcal{R}_2$.

Theorem 1. *The succesor function $\Sigma : N \to N$, $\Sigma(n) = n + 1$, $\forall n \in N$, the projections $\Pi^k_j : N^k \to N$, $\Pi^k_j(n_1, \ldots, n_k) = n_j$, $1 \leqslant j \leqslant k$ and the constant function $K_a : N^k \to N$, $K_a(n_1, \ldots, n_k) = a$, $\forall (n_1, \ldots, n_k) \in N^k$, are context-free relations.*

Proof. We have to prove that the language $L_\Sigma = \{x_1^n y_1^{n+1} \mid n \in N\}$, $L_{\Pi_j} = \{x_1^{n_1} \ldots x_j^{n_j} \ldots x_k^{n_k} y_1^{n_j} \mid (n_1, \ldots, n_k) \in N^k\}$, $L_{K_a} = \{x_1^{n_1} \ldots x^{n_k} y_1^a \mid (n_1, \ldots n_k) \in N^k\}$ are context-free. The construction of the corresponding context-free grammars is rather simple and it is left to the reader.

Remark. If $f, g \in \mathcal{R}_1$, their composition as relations does not belong, in general, to $\mathcal{R}_1$. Indeed, let $\{x^n \mid n \in B\}$ be a language from $\mathcal{L}_0 \setminus \mathcal{L}_1$. According to a result from [3], p. 89, there exists a type 1 language L_1 such

that L_1 consists of words of the form $x^n y y_2^{k(n)}$, $\forall n \in B$, where $k : B \to N$ is an appropiate function. This language represents the graph of a relation $f : N \to N^2$. Let us consider also the relation $g : N \to N^0$, whose domain is $\{(1, n) \mid n \in N\}$. It is clear that $\gamma(g \circ f) = \{n \mid n \in B\}$ and this set cannot be the graph of a relation from $\mathcal{R}_1(N, N^0)$. Therefore, it is not possible to obtain a subcategory of $\mathcal{R}$ by restricting the set of relations $\mathcal{R}(N^p, N^q)$ to the context-sensitive relations.

Definition. *An extending relation $f : N^p \to N^q$ is a relation having the property $\| x \| < M_f \| y \|$ for every $x \in$ dom f, $y \in f(x)$.*

Theorem 2. *The subcategory X of $\mathcal{R}(N^{\cdot})$, where $X(N^p, N^q)$ contains the extending relations from N^p to N^q, $\forall p, q \in N$ is a semiadmissible category.*

Proof. Let $f : N^p \to N^q$, $g : N^q \to N^r$ be two extending relations. Since $\| x \| \leqslant M_f \| y \|$ for every $x \in$ dom f, $y \in f(x)$ and $\| y \| \leqslant M_g \| z \|$ for every $y \in$ dom g and $z \in g(y)$, it follows that $\| x \| \leqslant M_f \| y \| \leqslant M_g M_f \| z \|$, $\forall x \in$ dom $f \circ g$, $z \in (f \circ g)(x)$. Hence $f \circ g$ is an extending relation. We conclude that X is indeed a subcategory $\mathcal{R}(N^{\cdot})$.

If $f : N \to N^q$ is an extending relation so is $N \times f$, hence X is closed with respect to cylindrification.

It is clear also that Δ and Θ_k, $K \geqslant 2$ are extending relations.

The next result is a more intricate one.

Theorem. 3. *The subcategory $\mathcal{C}$ of $\mathcal{R}(N^{\cdot})$, where $\mathcal{C}(N^p, N^q) = \mathcal{R}_1 \cap X(N^p, N^q)$ is semiadmissible.*

Proof. Suppose that $f \in \mathcal{C}(N^p, N^q)$ $g \in \mathcal{C}(N^q, N^r)$ are extending, context-sensitive relations. Since $f \circ g$ is extending, we have to prove that the language $L_{f \circ g}$ is context-sensitive, from the fact that L_f and L_g are context-sensitive languages. There exists the length increasing grammars G_f and G_g for which $L(G_f) = \{x_1^{n_1} \ldots x_p^{n_p} y_1^{m_1} \ldots y_q^{m_q} \mid (n_1, \ldots, n_p, m_1, \ldots, m_q) \in \gamma f\}$ and $L(G_g) = \{\overline{x}_1^{k_1} \ldots \overline{x}_q^{k_q} \overline{y}_1^{l_1} \ldots, \overline{y}_r^{l_r} \mid (k_1, \ldots, k_q, l_1, \ldots, l_r) \in \gamma g\}$. Let us consider the type-0 grammar $G = (I_{Nf} \cup I_{Ng} \cup \{\xi_0, \eta, \check{\xi}_1, \ldots, \check{\xi}_q, \xi_1, \ldots, \xi_q\}, I_{Tf} \cup I_{Tg}, \xi_0, F)$, where $G_f = (I_{Nf}, I_{Tf}, \xi_{0f}, F_f)$, $G_g = (I_{Ng}, I_{Tg}, \xi_{0g}, F_g)$ and F contains the following rules:

i) F'_f, which is obtained from F_f by replacing $y_1, \ldots, y_q$ by $\check{\xi}_1, \ldots, \check{\xi}_q$, respectively;

ii) F'_g, which is obtained from F_g by replacing $\overline{x}_1, \ldots, \overline{x}_q$ by $\xi_1, \ldots, \xi_q$, respectively;

iii) $\xi_0 \to \xi_{0f}\ \eta\ \xi_{0g}$;

iv) $\check{\xi}_m \check{\xi}_n \to \check{\xi}_n \check{\xi}_m$, with $1 \leqslant m < n \leqslant q$;

v) $\xi_m \xi_n \to \xi_n \xi_m$, with $1 \leqslant m < n \leqslant q$;

vi) $\check{\xi}_l\ \eta\ \xi_l \to \eta$, with $1 \leqslant l \leqslant q$;

vii) $\eta \to \lambda$,

where λ is the null word.

The unique way to start from ξ_0 and to derive a word from the terminal alphabet is the following:

$$\xi_0 \Rightarrow \xi_{0f}\ \eta\ \xi_{0g} \overset{\bullet}{\Rightarrow} x_1^{n_1} \ldots x_p^{n_p} \check{\xi}_1^{m_1} \ldots \check{\xi}_q^{m_q}\ \eta\ \xi_1^{k_1} \ldots \xi_q^{k_q} \overline{y}_1^{l_1} \ldots \overline{y}_r^{l_r} \overset{\bullet}{\Rightarrow} x_1^{n_1} \ldots x_p^{n_p} \check{\xi}_1^{m_1-1} \ldots$$
$$\check{\xi}_q^{m_q} \check{\xi}_1\ \eta\ \xi_1\ \xi_1^{k_1-1} \ldots \xi_q^{k_q} \overline{y}_1^{l_1} \ldots \overline{y}_r^{l_r} \Rightarrow x_1^{n_1} \ldots x_p^{n_p} \check{\xi}_1^{m_1-1} \ldots \check{\xi}_q^{m_q}\ \eta\ \xi_1^{k_1-1} \ldots \xi_q^{k_q} \overline{y}_1^{l_1} \ldots \overline{y}_r^{l_r}.$$

The necessity of elimination of non-terminals using the rules $\zeta_i\ \eta_j\ \xi_i \to \eta_j$ implies the equalities $m_j = k_j$, $1 \leqslant j \leqslant q$. Therefore, in these conditions we have $\xi_0 \overset{\bullet}{\Rightarrow} x_1^{n_1} \dots x_p^{n_p} \bar{y}_1^{l_1} \dots \bar{y}_r^{l_r}$, when $(n_1, \dots, n_p, l_1, \dots, l_r) \in \gamma(f \circ g)$.

If $WS_G(w)$ is the workspace of the word w in the grammar G we can write $WS_G\ (x_1^{n_1} \dots x_p^{n_p} \bar{y}_1^{l_1} \dots \bar{y}_r^{l_r}) = \| (n_1, \dots, n_p) \| + 2 \| (m_1, \dots, m_q) \| + 1 + \| (l_1, \dots, l_r) \|$, for $(m_1, \dots, m_q) \in f(n_1, \dots, n_p)$ and $(l_1, \dots, l_r) \in g(m_1, \dots, m_q)$ since G_f and G_g were supposed to be length increasing. Since f and g are extending relations we have:

$$WS_G\ (x_1^{n_1} \dots x_p^{n_p} \bar{y}_1^{l_1} \dots \bar{y}_r^{l_r}) \leqslant \| (n_1, \dots, n_p) \| + 1 + \| (l_1, \dots, l_r) \| (1 + 2M_g) \leqslant$$
$$\leqslant 2 \| (n_1, \dots, n_p) \| + \| (l_1, \dots, l_r) \| (2 + 2M_g) \leqslant (2 + 2M_g) \| (n_1, \dots, n_p, l_1, \dots, l_r) \| =$$
$$= (2 + 2M_g)\, l(x^{n_1} \dots x^{n_p} \bar{y}_1^{l_1} \dots \bar{y}_1^{l_r}),$$

for all nonemply words $x_1^{n_1} \dots x_p^{n_p} \bar{y}_1^{l_1} \dots \bar{y}_r^{l_r} \in L(G)$. Using the workspace theorem, it follows that $L(G)$ is context-sensitive.

Let us prove that $\mathcal{C}$ is closed with respect to cylindrification. Suppose that $f \in \mathcal{C}(N^r, N^s)$. Since $\| x \| \leqslant M_f \| y \|$, for every $y \in f(x)$, it follows that $\| (n, x) \| = n + \| x \| < n + M_f \| y \| < M_f \| (n, y) \|$ for every $(n, y) \in (N \times f)(n, x)$.

There exists a context-sensitive grammar G for which $L(G) = \{x_1^{n_1} \dots x_r^{n_r} y_1^{m_1} \dots y_s^{m_s} \mid (n_1, \dots, n_r, m_1, \dots, m_s) \in \gamma f\}$. We have to construct a context-sensitive grammar G' such that $L(G') = \{x^n x_1^{n_1}, \dots, x_r^{n_r} y^n y_1^{m_1} \dots y_1^{m_s} \mid n \in N, (n_1, \dots, n_r, m_1, \dots, m_s) \in \gamma f\}$.
To this end, let us consider the grammar $G' = (\{\mu_0, \mu, \mu', \mu'', \xi_1, \dots, \xi_r\} \cup I_N, \{x, y\} \cup I_T, \mu_0, F')$, where F' contains the following rules:

i) the rules of F in which the terminal symbols $x_1, \dots, x_r$ have been replaced by $\xi_1, \dots, \xi_r$, respectively:

ii) $\mu_0 \to \xi_0$, $\mu_0 \to \mu\xi_0$;
iii) $\mu \to \mu'\mu''$, $\mu \to \mu'\mu\mu''$;
iv) $\mu' \to x$;
v) $\mu''\zeta \to \zeta\mu''$, $\forall \zeta \in \{\xi_1, \dots, \xi_r\}$;
vi) $x\xi_j \to xx_j$, $1 \leqslant j \leqslant r$;
vii) $x_j \xi_h \to x_j x_h$, $1 \leqslant j, h \leqslant r$;

Using these rules it is possible to write $\mu_0 \Rightarrow \mu\xi_0 \overset{\bullet}{\Rightarrow} \mu'^n \mu''^n \xi_0 \overset{\bullet}{\Rightarrow} \mu'^n \mu''^n \xi_1^{n_1} \dots \xi_r^{n_r} y_1^{m_1} \dots y_s^{m_s} \overset{\bullet}{\Rightarrow} x^n \mu''^n \xi_1^{n_1} \dots \xi_r^{n_r} y_1^{m_1} \dots y_s^{m_s}$.
At this stage we have to move ξ_j, $1 \leqslant j \leqslant r$ over μ''^n in order to transform them into terminals using either vi) or vii). We have $x^n \mu''^n \xi_1^{n_1} \dots \xi_r^{n_r} y_1^{m_1} \dots y_s^{m_s} \overset{\bullet}{\Rightarrow} x^n x_1^{n_1} \dots x_r^{n_r} \mu''^n y_1^{m_1} \dots y_s^{m_s} \overset{\bullet}{\Rightarrow} x^n x_1^{n_1} \dots x^{n_r} y^n y_1^{m_1} \dots y_s^{m_s}$, hence G' computes indeed $N \times f$.

It is clear that Δ belongs to $\mathcal{C}(N, N^2)$, because the language $L_\Delta = \{x_1^n y_1^n y_2^n \mid n \in N\}$ is context sensitive and Δ is extending.

For Θ_k we have $L_{\Theta_k} = \{x_1^m x_2^n x_3^{p_1} \dots x_k^{p_{k-2}} y_1^n y_2^m y_3^{p_1} \dots y_k^{p_{k-2}}, m, n \in N, (p_1, \dots, p_{k-2}) \in N^{k-2}\}$. To prove that L_{Θ_k} is context-sensitive in a simple manner we shall give a scattered context grammar which generates L_Θ.

A scattered context-grammar (scg) is a 4-uple $G = (I_N, I_T, \xi_0, F)$, where I_N, I_T, ξ_0 have the same significances as in the definition of grammars and F is a set of rules having the form $(\xi_1, \dots, \xi_n) \to (w_1, \dots, w_n) \in [(I_N \cup I_T)^+]^n$ or $(\xi_0) \to (e)$ (when we suppose that ξ_0 does not occur in any right side of a rule). We shall write $u_1 \xi_1 u_2 \xi_2 \dots u_n \xi_n u_{n+1} \Rightarrow u_1 w_1 u_2 w_2 \dots u_n w_n u_{n+1}$, for every $u_1, \dots, u_n \in (I_N \cup I_T)^*$. The language generated by G is the language $L(G) = \{\alpha \mid \xi_0 \overset{\bullet}{\Rightarrow} \alpha\} \cap I_T^*$, where „$\overset{\bullet}{\Rightarrow}$" is, as usual, the reflexive and transitive closure of „$\Rightarrow$". Every language which can be generated by a scg is context-sensitive.

Let us consider the scg $G = (\{\xi_0, \zeta, \mu, \xi_2, \dots, \xi_k, \eta_2, \dots, \eta_k\}, \{x_1, \dots, x_k, y_1, \dots, y_k\}, \xi_0, F)$, where F contains the following rules:

i) $(\xi_0) \to (\zeta)$, $(\xi_0) \to (\zeta\ \eta)$;
ii) $(\zeta) \to (\mu)$, $(\zeta) \to (x_1\, \zeta\, y_2)$, $(\zeta) \to (x_1 y_2)$;
iii) $(\mu) \to (x_2\, \mu\, y_1)$, $(\mu) \to (x_2 y_1)$;
iv) $(\mu, \eta) \to (\xi_{j_1} \dots \xi_{j_l}, \eta_{j_1} \dots \eta_{j_l})$ with $3 \leqslant j_1 < j_2 < \dots < j_l \leqslant k$;
v) $(\xi_j, \eta_j) \to (\xi_j \xi_j, \eta_j \eta_j)$, $3 \leqslant j \leqslant k$;
vi) $(\xi_j, \eta_j) \to (x_j, y_j)$, $3 \leqslant j \leqslant k$;
vii) $(\xi_0) \to (e)$.

It is easy to see that a derivation in this grammar starts either with the rule $(\xi_0) \to (\zeta)$ or with $(\xi_0) \to (\zeta\ \eta)$.

In the first case, we can have one of the following three types of derivations:

a) $\xi_0 \Rightarrow \zeta \Rightarrow \mu \overset{\bullet}{\Rightarrow} x_2^n y_1^n$, for $n \geqslant 1$;

b) $\xi_0 \overset{\bullet}{\Rightarrow} \zeta \Rightarrow x_1^m \zeta y_2^m \Rightarrow x_1^m \mu y_2^m \overset{\bullet}{\Rightarrow} x_1^m x_2^n y_1^n y_2^m$, for $m, n \geqslant 1$;

c) $\xi_0 \Rightarrow \zeta \Rightarrow x_1^m y_2^m$, with $m \geqslant 1$.

In the second case the aspect of the initial part of the derivation is

$$\xi_0 \Rightarrow \zeta\, \eta \overset{\bullet}{\Rightarrow} x_1^m x_2^n \zeta y_1^n y_2^m \eta \Rightarrow x_1^m x_2^n \xi_{j_1} \dots \xi_{j_l} y_1^n y_2^m \eta_{j_1} \dots \eta_{j_l}.$$

Using the 5[th] group of rules we obtain the word $x_1^m x_2^n x_{j_1}^{p_1} \dots x_{j_l}^{p_l} y_1^n y_2^m y_{j_1}^{p_1} \dots y_{j_l}^{p_l}$. From the previous arguments it follows that $L_{\Theta_k} = L(G)$, hence $\Theta_k \in \mathcal{C}(N^k, N^k)$ and the proof is completed.

Using the closure properties of the class of morphisms of a semiadmissible category exposed in the introductory chapter it follows immediately the following

Corollary. a) *If $f : [p] \to [q]$ is a surjective function, then the language $L_f{*} = \{x_1^{n_1} \dots x_q^{n_q} y^{n_{f(1)}} \dots y_q^{n_{f(p)}} \mid (n_1, \dots, n_q) \in N^q\}$ is context-sensitive.*

b) *Suppose that $L_A = \{x_1^{n_1} \dots x_r^{n_r} y_1^{m_1} \dots y_s^{m_s} \mid (n_1, \dots, n_r, m_1, \dots, m_s) \in A\}$ and $L_B = \{x_1^{p_1} \dots x_t^{p_t} y_1^{q_1} \dots y_u^{q_u} \mid (p_1, \dots, p_t, q_1, \dots, q_u) \in B\}$ are two context sensitive languages for which $\| (n_1, \dots, n_r) \| \leqslant M_A \| (m_1, \dots, m_s)$ and $\| (p_1, \dots, p_t) \| \leqslant M_B \| (q_1, \dots, q_u) \|$, for every $(n_1, \dots, n_r, m_1, \dots, m_s) \in A$*

and $(p_1, \ldots, p_r, q_1, \ldots, q_u) \in B$. *Then the language* $L_{A \times B} = \{x_1^{n_1}, \ldots, x_r^{n_r} x_1'^{p_1} \ldots \ldots x_t'^{p_t} y_1^{m_1} \ldots y_s^{m_s} y_1'^{q_1} \ldots y_u'^{q_u} \mid (n_1, \ldots, n_r, m_1, \ldots, m_s) \in A, (p_1, \ldots, p_t, q_1, \ldots, q_u) \in B\}$ *is context-sensitive.*

Indeed, when L_A and L_B satisfy the previous conditions there exist two relations $f_A : N^r \to N^s$ and $f_B : N^t \to N^u$ such that $f_A, f_B \in \mathrm{Mor}\, \mathcal{C}$. Since $\mathrm{Mor}\, \mathcal{C}$ is closed with respect to combina ion it follows that $L_{A \times B}$ is context-sensitive.

c) If the languages $L_A = \{x_1^{n_1} \ldots x_r^{n_r} y_1^{m_1} \ldots y_s^{m_s} \mid (n_1, \ldots, n_r, m_1, \ldots, m_s) \in A\}$, $L_B = \{x_1^{n_1} \ldots x_r^{n_r} y_1^{p_1} \ldots y_t^{p_t} \mid (n_1, \ldots, n_r, p_1, \ldots, p_t) \in B\}$ are context sensitive and $\| (n_1, \ldots, n_r) \| \leqslant M_A \| (m_1, \ldots, m_s) \|$, $\|(n_1, \ldots, n_r) \| \leqslant M_B \| (p_1, \ldots, p_t) \|$ for all $(n_1, \ldots, n_r, m_1, \ldots, m_s) \in A$, $(n_1, \ldots, n_r, p_1, \ldots, p_t) \in B$ it follows that the language $L_{<A,B>} = \{x_1^{n_1} \ldots x_n^{n_r} y_1^{m_1} \ldots y_s^{m_s} y_1^{p_1} \ldots y_t^{p_t} \mid (n_1 \ldots n_r, m_1, \ldots, m_s) \in A, (n_1, \ldots, n_r, p_1, \ldots, p_t) \in B\}$ is also context-sensitive.

The proof uses essentially the same ideea as the previous one, taking into account that $\mathrm{Mor}\, \mathcal{C}$ is closed with respect to the operation $<.,.>$.

Regarding the composition of context-sensitive relations we are now able to present a stronger result.

Lemma. *The language* $\{y_1^{q_1} \ldots y_s^{q_s} y_{s+1} y_{s+2}^{q} y_{s+3}^{q_1} \ldots y_{2s+1}^{q_s} \mid (q, q_1, \ldots, q_s) \in N^{1+s}\}$ *is context-sensitive.*

Proof. Let $g : N^0 \to N^2$ be the relation containing the pairs $(1, q)$ for $q \in N$. Since g is extending and $L_G = y_{s+1} \{y_{s+2}\}^*$, it follows that $g \in \mathcal{C}(N^0, N^2)$. Taking into account that C is closed with respect to „$\times$" we have $h = 1_N \times \ldots \times 1_N \times g \in \mathcal{C}(N^s, N^{s+2})$, hence the language $L_h = \{y_1^{q_1} \ldots \ldots y_s^{q_s} y_{s+1} y_{s+2}^{q} \times_{s+3}^{q_1} \ldots y_{2s+2}^{q_s} \mid (q, q_1, \ldots, q_s) \in N^{1+s}\}$ is context-sensitive.

From this lemma it follows that the relation $k : N^{s+2} \to N^s$, with $k \in R_{L_h}$ is also context-sensitive.

Theorem 4. *The class* R_0 *consists of relations which can be expressed as compositions of context-sensitive relations.* $R_0 = \{f \mid f = f_1 \circ f_2, f_1, f_2 \in \mathcal{R}_1\}$

Proof. If $f : N^s \to N^s$ is a relation from $\mathcal{R}_0$ the language $L_f = \{x_1^{n_1} \ldots \ldots x_r^{n_r} y_1^{q_1} \ldots y_s^{q_s} \mid (n_1, \ldots, n_s, q_1, \ldots, q_s) \in \gamma f\}$ is a type-0 language. Therefore, there exists a mapping $\varphi : N^{r+s} \to N$ for which the language:

$$L_1 = \{x_1^{n_1} \ldots x_r^{n_r} y_1^{q_1} \ldots y_s^{q_s} y_{s+1} y_{s+2}^{\varphi(n_1, \ldots, n_r, q_1, \ldots, q_s)} \mid (n_1, \ldots, n_r, q_1, \ldots, q_s) \in \gamma f\}$$

is context-sensitive. Let f_1 be the relation $f_1 : N^r \to N^{s+2}$ from $\mathcal{R}_{L_1}$. If we take f_2 equal to k (which was introduced after the previous lemma), $f_2 : N^{s+2} \to N^s$ it follows that $f = f_1 \circ f_2$.

Indeed, let $(n_1, \ldots, n_r, q_1, \ldots, q_s) \in \gamma f$. In view of the definition of f_1 and f_2 it follows that $(n_1, \ldots, n_r, q_1, \ldots q_s, 1, \varphi(n_1, \ldots, n_r, q_1, \ldots, q_s)) \in \gamma f_1$ and $(q_1, \ldots, q_s, 1, \varphi(n_1, \ldots, n_r, q_1, \ldots, q_s), q_1, \ldots, q_s) \in \gamma f_2$, hence $(n_1, \ldots, n_r, q_1, \ldots, q_s) \in \gamma (f_1 \circ f_2)$. Conversely, if $(n_1, \ldots, n_r, q_1, \ldots, q_s) \in \gamma(f_1 \circ f_2)$ there exists an $(s+2)$-uple $(m_1, \ldots, m_s, m_{s+1}, m_{s+2})$, such that $(n_1, \ldots, n_r, m_1, \ldots, m_s, m_{s+1}, m_{s+2}) \in \gamma f_1$ and $(m_1, \ldots, m_s, m_{s+1}, m_{s+2}, q_1, \ldots, q_s) \in \gamma f_2$. Using the definition of f_1 and f_2 it follows that $(n_1, \ldots, n_r, m_1, \ldots, m_s) \in \gamma f$

and $m_j = q_j$, for $1 \leqslant j \leqslant s$. Therefore $(n_1, \ldots, n_r, q_1, \ldots, q_s) \in \gamma f$ and we have $\gamma f = \gamma(f_1 \circ f_2)$, hence $f = f_1 \circ f_2$. We have obtained the inclusion $\mathcal{R}_0 \subseteq \{f \mid f = f_1 \circ f_2, f_1, f_2 \in R_1\}$.

To prove the convers inclusion we can use the construction which we have used in the proof of Theorem 3 for proving that $\mathcal{C}$ is closed with respect to composition of relations. We have considered the type-0 grammar G, which is able to compute the graph of f. But, since we have not supposed here that f_1 and f_2 are extending relations, we can not infer that $L(G)$ is context-sensitive. Hence we have obtained also the converse inclusion.

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