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VT-AUTOMATA AND METAALFEBRAIC FAMILIES

BY

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Introduction. In [1], S. Eilenberg and J. Wright have introduced the notions of T-algebra, T-automaton, relational T-automaton, polynomial and algebraic set. In [4] the algebra automata are used for dynamic analysis. In the recent papers [2, 3] the author has defined the notions of VT-algebra, relational VT-algebra, VT-automaton and relational VT-automaton over a free theory.

In this paper the notions of ω-polynomial over a free theory and metaalgebraic family are given. It is shown that the class of metaalgebraic families of order 1 coincides with the class of behaviors of restricted VT-automata over $S_0[\Omega]$.

1. ω-polynomials. For notations and definitions not explained in the following, see [2] and [3]. Let $T = S_0[\Omega]$ be a free theory.

Definition 1.1. *A ω-polynomial is the system $P_{n,p} = (P_1^0, \ldots, P_n^0, P_1^1, \ldots, P_n^1, \ldots, P_1^i, \ldots P_n^i, \ldots)$, where P_j^i are finite subsets of* $\mathrm{Morph}_{S_0[\Omega]}$ *$(1, |p|)$, $j = \overline{1, n}$, $i = 0, 1, 2, \ldots$ If there exists a natural number k, such that for every $\Phi \in P_j^i$, $d\,\Phi \leqslant k$, $j = \overline{1, n}$, $i = 0, 1, 2, \ldots$ ($d\,\Phi$ is the degree of Φ [2]) and there exist a pair of natural numbers i, j and a morphism $\Phi \in P_j^i$, such that $d\,\Phi = k$, then k is called the degree of $P_{n,p}$. If there exists a natural number m_0, such that for every $i = 0, 1, 2, \ldots, j = \overline{1, n}$, $|P_j^i| \leqslant \leqslant m_0$, then $P_{n,p}$ is called bounded ($|P_j^i|$ is the power of P_j^i).*

Let A be a relational VT-algebra over $S_0[\Omega]$ and $\hat{A}$ be the VT-algebra corresponding to A ([3], Remark 2.1). Let us consider $\tilde{A}^n = (\mathcal{S}(A_0)^n, \mathcal{S}(A_1)^n, \ldots)$. Let be $X^i = (X_1^i, \ldots, X_p^i)$, where $X_j^i \in A_i$, $j = 1, 2, \ldots, p$, $i = 0, 1, 2, \ldots, \ldots$, We denote

$$X^i P_j^i = \bigcup_{\Phi \in P_j^i} (X_1^i, \ldots, X_p^i)(\Phi\, \psi_{A,i}^{p,1}),\quad j = 1, 2, \ldots, n,$$

$$X^i P^i = (X^i P_1^i, \ldots, X^i P_n^i).$$

The polynomial $P_{n,p}$ defines a function $\mathfrak{L} : \tilde{A}^p \to \tilde{A}^n$ in the following way: if $X = (X^0, X^1 \ldots) \in \tilde{A}^p$ and $Z = (Z^0, Z^1 \ldots) \in \tilde{A}^n$, then $X\mathfrak{L} = Z$, where $Z^{i+1} = X^i P^i$, $i = 0, 1, 2, \ldots, Z^0 = X^0$ for $p = n$, $Z^0 = (X_1^0, \ldots, X_p^0, \varnothing, \ldots, \varnothing)$ for $p < n$ and $Z^0 = (X_1^0, \ldots, X_n^0)$ else. $\mathfrak{L}$ depends on A and $P_{n,p}$, will be denoted by $\mathfrak{L}_A$.

Proposition 1.1. *The function $\mathfrak{T}_A$ corresponding to a bounded ω-polynomial $P_{n,p}$ is monotone.*

Proof. We must show that

(1.1) $X\mathfrak{T}_A \subseteq Y\mathfrak{T}_A$ whenever $X \subseteq Y$ i.e. $X^i \subseteq Y^i$, $i = 0, 1, 2, \ldots$

This will be done by induction on $d\Phi$. We first show:

(1.2) for every $\Phi : I \to [p]$ — morphism in $S_0[\Omega]$, if

$$X^i \subseteq Y^i \text{ and } P^i_j = \{\Phi\}, \text{ then } X^i P^i_j \subseteq Y^i P^i_j.$$

Indeed, if Φ is of degree 0, then Φ is in S_0, therefore $X^i P^i_j = (X^i_1, \ldots, X^i_p)(\Phi\,\psi^{p,1}_{A,i}) = X^i_{1\Phi}\,\beta_i$, $Y^i P^i_j = Y^i_{1\Phi}\beta_i$; From $X^i \subseteq Y^i$, it follows $X^i_{1\Phi} \subseteq Y^i_{1\Phi}$ and $X^i_{1\Phi}\beta_i \subseteq X^i_{1\Phi}\beta_i$, so that (1.2) holds. Assume that (1.2) holds for all morphisms Φ with $d\,\Phi \leqslant l$. Consider $\Phi : I \to [p]$, $d\,\Phi = l + 1$. There exists a natural number k, a morphism $\theta : [k] \to [p]$ — in $S_0[\Omega]$, and $\omega \in \Omega_k$ ([1]), such that $\Phi = \omega\theta$, $d\,\Phi = 1 + d\theta$. It results $d\theta = l$. We have $\theta = \langle \theta_1, \ldots, \theta_k \rangle$, $\theta_j = j_k\theta$, $j = 1, 2, \ldots, k$, $d\,\theta = d\,\theta_1 + \ldots + d\,\theta_k$, $\theta_q : I \to [p]$, $d\,\theta_q \leqslant l$, $1 \leqslant q \leqslant k$. The relation (1.2) holds for $P^i_j = \{\theta_q\}$, $q = 1, 2, \ldots k$, i.e.

(1.3) $$X^i(\theta_q\,\psi^{p,1}_{A,i}) \subseteq Y^i(\theta_q\,\psi^{p,1}_{A,i}),\ q = 1, 2, \ldots, k.$$

We have:

(1.4) $$X^i P^i_j = (X^i_1 \ldots X^i_p)(\Phi\,\psi^{p,1}_{1,i}) = (X^i_1 \ldots X^i_p)(\theta\,\psi^{p,k}_{A,i})\tilde{\alpha}_i(\omega\,\psi^{k,1}_{A,i}) = \\ = ((X^i_1 \ldots X^i_p)(\theta_1\,\psi^{p,1}_{A,i}), \ldots, (X^i_1, \ldots X^i_p)(\theta_k\,\psi^{p,1}_{A,i}))\tilde{\alpha}_i(\omega\,\psi^{k,1}_{A,i}).$$

The relations (1.3), (1.4) and (1.4) for Y^i instead of X give (1.2). We now show

(1.5) If $P_{n,p} = P^1_{n,p} \cup P^2_{n,p}$, then $\mathfrak{T}_A = \mathfrak{T}^1_A \cup \mathfrak{T}^2_A$.

where the union for $P_{n,p}$ and $\mathfrak{T}_A$ is made coordinate by coordinate and $\mathfrak{T}^i_A$ is the function corresponding to $P^i_{n,p}$, $i = 1, 2$.

Let k be the degree of $P_{n,p}$ and $X \subseteq Y$. We represent $P_{n,p}$ in the form

(1.6) $$P_{n,p} = \bigcup_{s=0}^{k} P^s_{n,p},$$

where $P^s_{n,p} = (P^{0,s}_1, \ldots, P^{0,s}_n, P^{1,s}_1, \ldots P^{1,s}_n, \ldots)$ and the elements of $P^s_{n,p}$ have the degree s $(0 \leqslant s \leqslant k)$, whether $P^s_{n,p} \neq \varnothing$.

It results (1.7) $\mathfrak{T}_A = \mathfrak{T}^0_A \cup \ldots \cup \mathfrak{T}^k_A$, where $\mathfrak{T}^s_A$ is the function corresponding to $P^s_{n,p}$, $s = 0, 1, \ldots, k$ From (1.2) we have $X^i P^{i,s}_j \subseteq Y^i P^{i,s}_j$, $s = 0, 1, 2, \ldots, k$, $1 \leqslant j \leqslant n$, $i = 0, 1, \ldots$, hence $X^i P^{i,s} \subseteq Y^i P^{i,s}$, $s = 0, 1, 2, \ldots, k$, $i = 0, 1, 2, \ldots$, i.e. $X\mathfrak{T}^s_A \subseteq Y\mathfrak{T}^s_A$, $s = 0, 1, \ldots, k$. It follows from (1.7) that $X\,\mathfrak{T}_A \subseteq Y\,\mathfrak{T}_A$, as required.

If we consider $P_{n,n}$, then the function $\mathfrak{T}_A : \tilde{A}^n \to \tilde{A}^n$ may be iterated, giving $\mathfrak{T}^k_A : \tilde{A}^n \to \tilde{A}^n$. Let us denote $\varnothing = (\overline{\varnothing}_n, \overline{\varnothing}_n, \ldots)$, where $\overline{\varnothing}_n = (\underbrace{\varnothing, \ldots, \varnothing}_{n\text{-fold}})$. Because $\mathfrak{T}^k_A$ is monotone, it results:

(1.8) $$\varnothing \subseteq \varnothing\,\mathfrak{T}^1_A \subseteq \varnothing\,\mathfrak{T}^2_A \subseteq \ldots$$

Consider $\overline{\mathfrak{T}}_A = \bigcup_{k=1}^{\infty} \varnothing\,\mathfrak{T}^k_A$

Proposition 1.2. *If $\underline{\varnothing}\,P^k_A = (Z^{k,0}, Z^{k,1}, \ldots)$, $Z^{k,i} \in \mathcal{S}(A_i)^n$, $i = 0, 1, 2, \ldots$, $k = 1, 2, \ldots$, then*

(1.9) $Z^{l+1,h} = Z^{l,h}$ *for every* $k = 0, 1, 2, \ldots$, $h = 0, 1, 2, \ldots, k$, $l \geqslant k$.

Proof. For $k = 0$, $Z^{l,0} = (\varnothing, \ldots, \varnothing)$, $l = 0, 1, 2, \ldots$, for $k = 1$, $Z^{l,1} = (\varnothing, \ldots, \varnothing)\,P^0$, $l = 1, 2, \ldots$, therefore (1.9) holds. Assume that (1.9) holds for any $k \geqslant 1$. Then we have

(1.10) $$Z^{l+1,i} = Z^{l,i-1}\,P^{i-1},\ i = 1, 2, \ldots, l = 0, 1, 2, \ldots;$$

(1.11) $$Z^{l+1,k+1} = Z^{l,k}\,P^k = Z^{l-1,k}\,P^k = Z^{l,k+1},\ l \geqslant k + 1.$$

The relations (1.10) and (1.11) are (1.9) for $k + 1$.

Remark. *Relation* (1.9) *implies*

(1.12) $$Z^{k,i} = Z^{i,i},\ i = 0, 1, 2, \ldots, k \geqslant i.$$

Proposition 1.3. *If $\overline{\mathfrak{T}}_A = (\mathfrak{T}^0, \mathfrak{T}^1, \ldots)$, where $\mathfrak{T}^i \in \mathcal{S}(A_i)^n$, $i = 0, 1, 2, \ldots$, then we have*

(1.13) $$\overline{\mathfrak{T}}^{i+1} = \overline{\varnothing}\,P^0\,P^1 \ldots P^i,\ i = 0, 1, 2, \ldots, \overline{\mathfrak{T}}^0 = \overline{\varnothing}.$$

Proof. We have $\overline{\mathfrak{T}}^0 = \overline{\varnothing}$, because $\overline{\varnothing}$ is the first coordinate of $\overline{\varnothing}\,\mathfrak{T}^k$, for each k. Assume that (1.13) holds for any $i \geqslant 0$. Then we have $\overline{\mathfrak{T}}^i = Z^{0,i} \cup Z^{1,i} \cup \ldots \cup Z^{i,i}$, $\overline{\mathfrak{T}}^{i+2} = Z^{0,i+2} \cup Z^{1,i+2} \cup \ldots \cup Z^{i+2,i+2} = Z^{0,i+2} \cup Z^{0,i+1}\,P^{i+1} \cup \ldots Z^{i+1,i+1}\,P^{i+1} = \overline{\mathfrak{T}}^{i+1}\,P^{i+1} = \overline{\varnothing}\,P^0\,P^1 \ldots P^{i+1}$.

Proposition 1.4. *$\overline{\mathfrak{T}}_A$ is the least solution of the equation:*

(1.14) $$X\,\overline{\mathfrak{T}}_A = X$$

Proof. (1.13) is equivalent with $\overline{\mathfrak{T}}^i P^i = \overline{\mathfrak{T}}^{i+1}$, $i = 0, 1, 2, \ldots$, $\overline{\mathfrak{T}}^0 = \overline{\varnothing}$, i.e. $\overline{\mathfrak{T}}_A\,\mathfrak{T}_A = \overline{\mathfrak{T}}_A$. If Z is a solution for (1.14) we have $Z\,\mathfrak{T}_A = Z$. From $\varnothing \subseteq Z$ one obtains $\varnothing\,\mathfrak{T}^k_A \subseteq Z\,\mathfrak{T}^k_A = Z$ for each k, therefore $\bigcup_{k=1}^{\infty} \varnothing\,\mathfrak{T}^k_A \subseteq Z$, i.e. $\overline{\mathfrak{T}}_A \subseteq Z$. Similarly, we have;

Proposition 1.5. *$\overline{\mathfrak{T}}_A$ is the least solution of the inequality:*

(1.15) $$X\,\mathfrak{T}_A \subseteq X.$$

Proposition 1.6. *The following diagram*

(1.16) $$\begin{tikzcd} \tilde{C}^{0n} \arrow[r, "\mathfrak{T}_{\hat{C}^0}"] \arrow[d, "\tilde{\zeta}_A"'] & \tilde{C}^{0n} \arrow[d, "\tilde{\zeta}_A"] \\ \tilde{A}^n \arrow[r, "\mathfrak{T}_A"] & \tilde{A}^n \end{tikzcd}$$

where $\tilde{\zeta}_A = (\tilde{\zeta}_{A0}, \tilde{\zeta}_{A1}, \ldots)$ commutes. The proof is obvious.

2. Metaalgebraic families. Let C^0 be the relational VT-algebra, which is the initial object in the category C^{VTR} over the theory $T = S_0[\Omega]$

[3] and $\hat{C}^0$ be a VT-algebra corresponding to C^0. Let $P_{n,n}$ be a ω-polynomial over $S_0[\Omega]$ and

$$\mathfrak{T}_{\hat{C}^0} = (\overline{P}^0_1, \ldots, \overline{P}^0_n, \overline{P}^1_1, \ldots, \overline{P}^1_n, \ldots, \overline{P}^i_1, \ldots, \overline{P}^i_n, \ldots).$$

Definition 2.1. *The family $X = (X_i)$, $i = 0, 1, 2, \ldots$, $X_i \subseteq C_i$, is called metaalgebraic with respect to $\hat{C}^0$ if there exist a natural number n and a ω-polynomial $P_{n,n}$, such that $X_i = \overline{P}^i_1$, $i = 0, 1, 2, \ldots$*

Proposition 2.2. *Let A be a relational VT-algebra over $S_0[\Omega]$, $P_{n,n}$ be a ω-polynomial over $S_0[\Omega]$, k an integer, $1 \leqslant k \leqslant n$ and $\mathfrak{T}_A$ the function corresponding to $P_{n,n}$ and $\hat{A}$. If $\mathfrak{T}_A = (\overline{P}^0, \overline{P}^1 \ldots)$, $\overline{P}^i = (P^i_1, \ldots, P^i_n)$, $i = 0, 1, 2, \ldots$, then there exist a ω-polynomial over $S_0[\Omega]$, $Q_{n+1,n+1}$, such that $\overline{Q}_A = (\overline{Q}^0, \overline{Q}^1 \ldots)$, $\overline{Q}_i = (\overline{Q}^i_1, \ldots, \overline{Q}^i_{n+1})$, $i = 0, 1, 2, \ldots$, $\overline{Q}^i_1 = \bigcup_{j=1}^{k} \overline{P}^i_j$*

$$\overline{Q}^i_h = \overline{P}^i_{h-1},\ h = 2, 3, \ldots, n+1,\ i = 0, 1, 2, \ldots \tag{2.1}$$

Proof. We define ω-polynomial $Q_{n+1,n+1} = (Q^0_1, \ldots, Q^0_{n+1} \ldots)$ such that $Q^i_1 = \{\Phi\alpha \mid \Phi \in \bigcup_{j=1}^{k} P^i_j\}$, $Q^i_h = \{\Phi\alpha \mid \Phi \in P^i_{h-1}$, $2 \leqslant h \leqslant n+1$, $i = 0, 1, 2, \ldots$, where α is the morphism in S_0, $\alpha : [n] \to [n+1]$, defined by $t\alpha = t + 1$, $t = 1, 2, \ldots, n$. We have

$$\begin{gathered} \overline{\varnothing}_{n+1} Q^0 Q^1 \ldots Q^i = (\overline{\varnothing}_n P^0 P^1 \ldots P^{i-1} (P^i_1 \cup P^i_2 \cup \ldots \cup P^i_k), \\ \overline{\varnothing}_k P^0 P^1 \ldots P^i) = (P^i_1 \cup \ldots \cup P^i_k, \overline{\varnothing}_n P^0 P^1 \ldots P^i),\ i = 0, 1, 2, \ldots \end{gathered} \tag{2.2}$$

By the Proposition 1.3, it results (2.1).

Definition 2.3. *A relational VT-algebra A is called restricted if the functions $(\Phi\psi^{p,n}_i)$ are given only for $\Phi : [n] \to [p]$, $\Phi \in Z_i \subseteq S_0[\Omega]$, $i = 0, 1, 2, \ldots$, where $Z_i = S_0[\Omega]$, $i \geqslant 1$, $Z_0 = \{\Phi O_n$, $\Phi \in \Omega_0 \neq \varnothing$, $O_n : \varnothing \to [n]$, $n = 1, 2, \ldots$, O_n are the morphisms in $S_0\}$*

Proposition 2.4. *Let $S_0[\Omega]$ be a theory with $\Omega_0 \neq \varnothing$. For each ω-polynomial $P_{n,n}$, such that $P^0_j \subseteq Z_0$ and every element of P^i_j has the degree one, $1 \leqslant j \leqslant n$, $i = 1, 2, \ldots$, there exists a relational VT-algebra structure on $A = (A_i)$, $A_i = [n]$, $\alpha_i = \beta_i = 1_{[n]}$; $i = 0, 1, 2, \ldots$*

Conversely, for each relational VT-algebra structure on $A = (A_i)$, $A_i = [n]$, $\alpha_i = \beta_i = 1_{[n]}$, $i = 0, 1, 2, \ldots$, there exists a ω-polynomial $P_{n,n}$ such that $P^0_j \subseteq Z_0$ and every element of P_j has the degree one, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \ldots$.

Proof. Let $P_{n,n}$ be a ω-polynomial over $S_0[\Omega]$ with $P^0_j \subseteq Z_0$ and for every $\Phi \in P^i_j$, $d\Phi = 1$. Every morphism from P^0_j has also the degree one. Consider $\Phi : I \to [n]$, there exist the natural number p, ω, x, the morphisms in $S_0[\Omega]$, $\omega \in \Omega_p$, $x : [p] \to [n]$, such that $\Phi = \omega x$, $d\Phi = 1$, $d\Phi = 1 + dx$. It follows $dx = 0$, i.e. x is a morphism in S_0. If $\Phi \in P^i_{j_1} \cap P^i_{j_2} \cap \ldots \cap P^i_{j_k}(\{j_1, j_2, \ldots, j_k\} \subseteq [n])$ and $\Phi \notin P^i_l$, $l \neq j_s$, $s = 1, 2, \ldots, k$, we define

$$(I(1_p x), \ldots, I(p_p x)(\omega\psi^{p,1}_{A,i}) = \{j_1, j_2, \ldots, j_k\},\ i = 1, 2, \ldots \tag{2.3}$$

$(Z_1, Z_2, \ldots, Z_p)(\omega\psi^{p,1}_{A,i}) = \varnothing$, when $< y_1, \ldots, y_p > \neq x$, where $Iy_j = Z_j$, $1 \leqslant j \leqslant p$, y_j are the morphisms in S_0, $y_j : I \to [n]$.

If $\Phi 0_n \in P^i_{j_1} \cap P^i_{j_2} \cap \ldots \cap P^i_{j_k}$ and $\Phi \notin P^i_l$, $l \neq j_s$, $s = 1, 2, \ldots, k$, then

$$(\quad)(\Phi\psi^{0,1}_{A,i}) = \{j_1, j_2, \ldots, j_k\},\ i = 0, 1, 2, \ldots \tag{2.4}$$

It results a restricted relational VT-algebra structure on $A = (A_i)$. Conversely, let $A = (A_i)$ be a restricted relational VT-algebra.
For $\Phi \in S_0[\Omega]$ with $d\Phi = 1$, we have $\Phi = \omega x$, $\omega : I \to [p]$, $x : [p] \to [n]$, $x = < 1_p x, \ldots, p_p x >$. $I(j_p x) \in [n]$, $j = 1, 2, \ldots, n$. We define $P_{n,n}$ such that

$$\begin{gathered} \Phi = \omega x \in P^i_j \text{ iff } j \in (I(1_p x), \ldots, I(p_p x))(\omega\psi^{p,1}_{A,i}),\ 1 \leqslant j \leqslant n,\ i = \\ = 1, 2, \ldots,\ \Phi' = \Phi 0_n \in P^0_j \text{ iff } j \in (\quad)(\Phi\psi^{0,1}_{A,0}),\ \Phi \in \Omega_0,\ 1 \leqslant j \leqslant n. \end{gathered} \tag{2.5}$$

Remark 2.5. For VT-algebra $\hat{A}$ corresponding to relational VT-algebra A from the Proposition 2.4., we have $(Y_1, \ldots, Y_n)(\omega\psi^{n,1}_{\hat{A},i}) = \varnothing$, whenever there exists

$$j \in [1, n],\ \textit{such that}\ Y_j = \varnothing.$$

Theorem 2.6. *Let $P_{n,n}$ be a ω-polynomial over $S_0[\Omega]$. There exists a ω-polynomial over $S_0[\Omega]$, $T_{n,n}$ such that:*
(2.6) *The elements of T^i_j are the elements of P^i_j of degree > 0, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \ldots$*
(2.7) *$\mathfrak{T}_A = \mathfrak{F}_{\hat{A}}$ for each VT-algbra $\hat{A}$ corresponding to any relational VT-algebra A over $S_0[\Omega]$.*

Proof. If $P_{n,n} = (P^0_1, \ldots, P^0_n, P^1_1, \ldots, P^1_n, \ldots)$, consider $P^i_j = R^i_j \cup M^i_j$ and $P_{n,n} = R_{n,n} \cup M_{n,n}$, where M^i_j consists of all the elements of P^i_j of degree 0 and R^i_j consists of the elements of P^i_j of degree > 0. Let $S_0(I, [n]) = \{1_n, 2_n, \ldots, n_n\}$
We have

$$X^i M^i_j = \bigcup_{\Phi \in M^i_j} (X^i_1, \ldots, X^i_n)(\Phi\psi^{n,1}_{\hat{A},i}),\ X^i M^i = (X^i M^i_1, \ldots, X^i M^i_n)$$
$$1 \leqslant j \leqslant n,\ i = 0, 1, 2, \ldots$$

To each M^i we consider a matrix α^i, such that:

$$\alpha^i_{hk} = \begin{cases} 1 \text{ if } h_n : I \to [n] \text{ is in } M^i_k & 1 \leq h, k \leq n \\ 0 \text{ elsewhere} \end{cases}$$

Let us define

$$X^i\alpha^i = (Z_1, \ldots, Z_n),\ \text{where}\ Z_k = \bigcup_{\substack{h=1 \\ \alpha^i_{hk}=1}}^{n} X^i_h,\ k = 1, 2, \ldots, n,\ i = 0, 1, 2, \ldots$$

It is easy to verify that

$$X^i M^i = X^i\alpha^i,\quad i = 0, 1, 2, \ldots \tag{2.8}$$

Consider the lattice $L = \{0, 1\}$, 0 is the least element and 1 is the greatest element, with the binary compositions a $\wedge$ b and a $\vee$ b. We define the union of the matrices α, β, denoted by $\alpha \cup \beta$ and their product, denoted by $\alpha\beta$ thus

$$\alpha \cup \beta = (\alpha_{ij} \vee \beta_{ij}),\ \gamma = \alpha\beta,\ \gamma_{ij} = \bigvee_{k=1}^{n} (\alpha_{ik} \wedge \beta_{kj}),\ \alpha = (\alpha_{ij}),\ \beta = (\beta_{ij}),\ 1 \leqslant i, j \leqslant n.$$ The relation (2.8) implies:

(2.9) $X^i M^i M^{i+1} \dots M^{i+t} = X^i(\alpha^i \alpha^{i+1} \dots \alpha^{i+t})$, $t = 1, 2, \dots, i = 0, 1, 2, \dots$

Let B be the matrix

$$B = \begin{pmatrix} E & \alpha^0 & O & O & \dots \\ O & O & \alpha^2 & O & \\ O & O & O & \alpha^3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where E is the matrix with 1 on the diagonal and 0 elsewhere and O has all elements equal to 0. E and O are of the type $n \times n$. We define $T_{n,n} = (T^0, T^1, \dots)$ thus

(2.10) $T^i = R^i \cup R^0 \alpha^0 \alpha^1 \dots \alpha^{i-1} \cup R^1 \alpha^1 \dots \alpha^{i-1} \cup \dots \cup R^{i-1} \alpha^{i-1}$, $i = 1, 2, \dots$

$T^0 = R^0$

or shortly,

(2.11) $T_{n,n} = R_{n,n} (E' \cup \bigcup_{k=1}^{\infty} B^k)$, where E' is the matrix with 1 on the diagonal and 0 elsewhere.

Every element from T_j is of the degree > 0.

To shown that (2.12) $\overline{\mathfrak{L}}_A = \overline{\mathfrak{T}}_A$. For each $Z \in \tilde{A}^n$, we have

$$Z(\bigcup_{k=1}^{\infty} B^k) = \bigcup_{k=1}^{\infty} Z\, B^k. \tag{2.13}$$

Let $X = (X^0, X^1, \dots,)$ be from $\tilde{A}^n$, such that $X\, \mathfrak{L}_A \subseteq X$. From $P_{n,n} = R_{n,n} \cup M_{n,n}$ it results $X\, \mathfrak{R}_A \subseteq X$ and $X\, \mathfrak{M}_A \subseteq X$, where $\mathfrak{R}_A$ and $\mathfrak{M}_A$ are the functions corresponding to $R_{n,n}$ and $M_{n,n}$ respectively. But $X\, \mathfrak{M}_A = X\, B$, hence $X\, B \subseteq X$. Because $\mathfrak{M}_A$ is monotone, we have $X\, \mathfrak{M}_A^k \subseteq X$, for every $k = 1, 2, \dots$. It obtains $X\, B^k \subseteq X$, $X\, \mathfrak{T}_A = X\, \mathfrak{R}_A (E' \cup \bigcup_{k=1}^{\infty} B^k) \subseteq$ $\subseteq X (E' \cup \bigcup_{k=1}^{\infty} B^k) = X\, E' \cup \bigcup_{k=1}^{\infty} (X\, B^k) \subseteq X$. Because $\overline{\mathfrak{L}}_A$ is a solution for $X\, \mathfrak{L}_A \subseteq X$ and $\overline{\mathfrak{T}}_A$ is the least solution for $X\, \mathfrak{T}_A \subseteq X$, it results

$$\overline{\mathfrak{T}}_A \subseteq \overline{\mathfrak{L}}_A. \tag{2.14}$$

Let $X \in A^n$ be a solution for (2.15) $X\, \mathfrak{T}_A^{\wedge} = X$. From (2.11) we have $X\, \mathfrak{R}_A \subseteq X\, \mathfrak{T}_A = X$. On the other hand, it results $(E' \cup \bigcup_{k=1}^{\infty} B^k)\, B \subseteq E'\, B \cup \bigcup_{k=1}^{\infty} B^{k+1} = \bigcup_{k=1}^{\infty} B^k \subseteq E' \cup \bigcup_{k=1}^{\infty} B^k$, $X\, \mathfrak{T}_A\, \mathfrak{M}_A = X\, \mathfrak{R}_A^{\wedge}\, B = X\, \mathfrak{R}_A (E' \cup \bigcup_{k=1}^{\infty} B^k)$ $B \subseteq X\, \mathfrak{R}_A (E' \cup \bigcup_{k=1}^{\infty} B^k) = X\, \mathfrak{T}_A$.

$X\, \mathfrak{L}_A = X\, \mathfrak{R}_A \cup X\, \mathfrak{M}_A = X\, \mathfrak{R}_A \cup X\, \mathfrak{T}_A\, \mathfrak{M}_A \subseteq X\, \mathfrak{R}_A \cup X\, \mathfrak{T}_A \subseteq X$.

It obtains $\overline{\mathfrak{L}}_A \subseteq \overline{\mathfrak{T}}_A$, hence (2.12).

Theorem 2.7. *Let $P_{n,n}$ be a ω-polynomial over $S_0[\Omega]$, with Ω finite. There exists a ω-polynomial $T_{m,m}$ over $S_0[\Omega]$, $n \leqslant m$, ssuch that*

(i) *If $P_{n,n} = (P_1^0, \dots, P_n^0, \dots)$, $T_{m,m} = (T_1^0, \dots, T_m^0, T_1^1, \dots, T_m^1, \dots)$, then $\{\Phi / \Phi \in P_j^i$, Φ is morphism in $S_0\} = \{\Phi / \Phi \in T_j^i$, Φ is morphism in $S_0\}$, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \dots$*

(ii) *The elements from T_j^i are of degree 0 or 1, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \dots$*

(iii) *$\{\Phi / \Phi \in T_j^i$, Φ is morphism in $S_0\} = \varnothing$, $n < j \leqslant m$, $i = 0, 1, 2, \dots$*

(iv) *$\overline{P}_j^i = \overline{T}_j^i$, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \dots$*

Proof. Let k be the degree of $P_{n,n}$. For $k = 0$ or $k = 1$, the conditions (i — iv) of the theorem are satisfied taking $m = n$ and $T_{m,m} = P_{n,n}$. Assume that $k > 1$. There exists Φ in $P_{n,n}$, $\Phi : I \to [n]$, $d\,\Phi = k$. We define $T_j^i = \{\Phi' h / \Phi' \in P_j^i$, $\Phi' \neq \Phi$ whether $\Phi \in P_j^i\}$, $h : [n] \to [n+1]$ is the morphism in S_0 defined by $ih = i$, $i \in [n]$. There exist p, $\omega \in \Omega_p$, $\theta : [p] \to [n]$, such that $\Phi = \omega\,\theta$. We have $d\theta = k-1$, $\theta = <\theta_1, \dots, \theta_p>$, $\theta_h = h_p \theta$, $1 \leqslant h \leqslant n$, $d\theta = d\theta_1 + \dots + d\theta_p$. There exists j, $1 \leqslant j \leqslant p$, such that $d\theta_j > 0$, this implies $d(\theta_j \theta') > 0$. Let $(\Phi_1, \dots, \Phi_n)$ be from $C_s^{0^n}$, $s \geqslant 1$. We obtain $(\Phi_1, \dots, \Phi_n)(\Phi\, \psi_{C^0,s}^{n,1}) = \Phi < \Phi_1, \dots, \Phi_n > = \omega\,\theta < \Phi_1, \dots, \Phi_n > = \omega < \theta_1, \dots, \theta_p > < \Phi_1, \dots, \Phi_n > = \omega < \theta_1 < \Phi_1, \dots, \Phi_n >, \dots, \theta_p < < \Phi_1, \dots, \Phi_n \gg = \omega\,\eta_j < \Phi_1, \dots, \Phi_n, \Phi_{n+1} >$, where $\Phi_{n+1} = \theta_j < \Phi_1, \dots, \Phi_n >$ and η_j is defined by $\eta_{ji} = \theta_i h$, $i \in [1, p] - \{j\}$, $\eta_{jj} = (n+1)_{n+1}$ and $\eta_j = < \eta_{j1}, \dots, \eta_{jp} >$, $d\,\eta_j < d\theta$, $\omega\,\eta_j : I \to [n+1]$. It results

$$(\Phi_1, \dots, \Phi_n)(\Phi\, \psi_{C^0,s}^{n,1}) = (\Phi_1, \dots, \Phi_{n+1})((\omega\,\eta_j)\, \psi_{C^0,s}^{n+1,1}),\ s = 1, 2, \dots \tag{2.16}$$

If $\Phi \in P_j^i$ we adjoin $\omega\,\eta_j$ to T_j^i. The relation (2.16) is verified for $(Y_1, \dots, Y_n)$ where $Y_j = \{\Phi_j\}$ or $Y_j = \varnothing$, $\Phi_j \in S_0[\Omega] (I, \varnothing)$ and there is $i \in [1, n]$, such that $Y_i = \varnothing$. Indeed, if we consider $Y_{n+1} = (Y_1, \dots, Y_n)(\theta_j\, \psi_{\hat{C}^0,s}^{n,1})$, then $(Y_1, \dots, Y_n, Y_{n+1})((\omega\,\eta_j)\, \psi_{\hat{C}^0,s}^{n+1,1}) = (Y_1, Y_2, \dots, Y_n)(\Phi\, \psi_{\hat{C}^0,s}^{n,1})$.

We now show that there exist T_{n+1}^i, $i = 0, 1, 2, \dots$, such that

$$\bigcup_{\xi \in T_{n+1}^{i-1}} (T_1^{i-1}, \dots, T_n^{i-1}, T_{n+1}^{i-1})(\xi\, \psi_{\hat{C}^0,i-1}^{n+1,1}) = (T_1^i, \dots, T_n^i)(\theta_j\, \psi_{\hat{C}^0,i}^{n,1}),\ i = 1, 2 \dots \tag{2.18}$$

Indeed, (2.18) is equivalent with

$$\bigcup_{\xi \in T_{n+1}^{i-1}} (\overline{T}_1^{i-1}, \dots, \overline{T}_n^{i-1}, \overline{T}_{n+1}^{i-1})(\xi\, \psi_{\hat{C}^0,i-1}^{n+1,1}) = \bigcup_{\alpha \in V^{i-1}} (\overline{T}_1^{i-1}, \dots, \overline{T}_{n+1}^{i-1})(\alpha\, \psi_{\hat{C}^0,i}^{n+1,1}), \tag{2.19}$$

$i = 1, 2, \dots$, where $V^{i-1} = \theta_j < \overline{T}_1^{i-1}, \dots, \overline{T}_n^{i-1} > = \{\theta_j < \alpha_1, \dots, \alpha_n >$

$$| \alpha_k \in \overline{T}_k^{i-1},\ 1 \leqslant k \leqslant n\}.$$

A solution for (2.19) is $T_{n+1}^{i-1} = V^{i-1}$, $i = 1, 2, \dots$

It is clear that all elements from T_{n+1}^{i-1} have degree > 0.

Let us show (iv). We have $\overline{T}_{n+1}^i = (\overline{T}_1^i, \dots, \overline{T}_n^i)(\theta_j\, \psi_{\hat{C}^0,i}^{n,1})$, $i = 0, 1, 2, \dots$ and $\overline{P}_j^0 = \overline{T}_j^0$, $1 \leqslant j \leqslant n$. We assume that $\overline{P}_j^{i-1} = \overline{T}_j^{i-1}$, $1 \leqslant j \leqslant n$, for any $i \geqslant 1$. The relation (2.17) is satisfied replacing $Y_1, \dots, Y_n$, Y_{n+1} by $\overline{T}_1^{i-1}, \dots, \overline{T}_n^{i-1}$, $\overline{T}_{n+1}^{i-1}$ respectively.

If $\Phi \notin P_j^i$, then we have $\overline{T}_j^i = \bigcup_{\xi \in T_j^{i-1}} (\overline{T}_1^{i-1}, \ldots, \overline{T}_n^{i-1}, \overline{T}_{n+1}^{i-1})(\xi\, \psi^{n+1,1}_{\hat{C}^\circ, i-1}) =$

$$= \bigcup_{\Phi' \in P_j^{i-1}} (\overline{T}_1^{i-1}, \ldots, \overline{T}_n^{i-1}, \overline{T}_{n+1}^{i-1})((\Phi' h)\, \psi^{n+1,1}_{\hat{C}^\circ, i-1}) = \bigcup_{\Phi' \in P_j^{i-1}} (\overline{T}_1^{i-1}, \ldots, \overline{T}_n^{i-1})(\Phi'\, \psi^{n,1}_{\hat{C}^\circ, i-1}) =$$

$$= \bigcup_{\Phi' \in P_j^{i-1}} (\overline{P}_1^{i-1}, \ldots, \overline{P}_n^{i-1})(\Phi'\, \psi^{n,1}_{\hat{C}^\circ, i-1}) = \overline{P}_j^i.$$

If $\Phi \in P_j^{i-1}$, then $\overline{T}_j^i = \bigcup_{\Phi' \in P_j^{i-1}, \Phi' \neq \Phi} (\overline{T}_1^{i-1}, \ldots, \overline{T}_n^{i-1}, \overline{T}_{n+1}^{i-1})((\Phi' h)\, \psi^{n+1,1}_{\hat{C}^\circ, i-1}) \cup$

$$\cup\, (\overline{T}_1^{i-1}, \ldots, \overline{T}_n^{i-1})(\Phi\, \psi^{n,1}_{\hat{C}^\circ, i-1}) = \overline{P}_j^i.$$

Because Ω is finite, the number of the elements of degree k is finite. We apply above procedure for all morphisms of degree k from $P_{n,n}$, then for all morphisms of degree k—1, and so on, obtaining in T_j^i only the morphisms of degree $\leqslant 1$, $1 \leqslant j \leqslant m$, $i = 0, 1, 2, \ldots$

Theorem 2.8. *If $A = (A_i, \alpha_i, \beta_i)$, is a relational restricted VT-algebra over $S_0[\Omega]$ with $\alpha_i = \beta_i = 1_{[n]}$, $A_i = [n]$, $i = 0, 1, 2, \ldots$ and $P_{n,n}$ the homogeneous ω-polynomial over $S_0[\Omega]$ corresponding to A, then $\overline{P}_j^i = j\, \zeta_{A,i}^{-1}$, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \ldots$*

Proof. Let us denote $R_j^i = \{\Phi \mid \Phi : I \to \varnothing,\ j \in \Phi \zeta_{A,i}\}$, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \ldots, R_{n,n} = (R_1^0, \ldots, R_n^0, \ldots)$. Then we must verify that

$$\overline{P}_j^i = R_j^i,\ 1 \leqslant j \leqslant n,\ i = 0, 1, 2, \ldots. \tag{2.20}$$

For $i = 0$, we have $\overline{P}_j^0 = \varnothing = R_j^0$, $1 \leqslant j \leqslant n$, because $C_0^\circ = \varnothing$ and $\zeta_{A,i} : C_i^0 \to A_i$. We first show

$$\overline{P}_j^i \subseteq R_j^i,\ 1 \leqslant j \leqslant n,\ i = 1, 2, \ldots, \text{ or equivalently} \tag{2.21}$$

$$\overline{P}^i \subseteq R^i,\ i = 1, 2, \ldots, \text{ where } \overline{P}^i = (\overline{P}_1^i, \ldots, \overline{P}_n^i),\ R^i = (R_1^i, \ldots, R_n^i) \tag{2.22}$$

Let us denote $\mathfrak{R} = (R^0, R^1, \ldots)$. The following relations

$$R^i P^i \subseteq R^{i+1},\ i = 0, 1, 2, \ldots \tag{2.23}$$

give $\mathfrak{R}\, \mathfrak{P}_{\hat{C}^\circ} \subseteq \mathfrak{R}$, therefore $\mathfrak{P}_{\hat{C}^\circ} \subseteq \mathfrak{R}$, i.e. (2.21)

The relations (2.23) are equivalent with

(2.24) for every $\Phi \in P_j^i$, $(R_1^i, \ldots, R_n^i)(\Phi\, \psi^{n,1}_{\hat{C}^\circ, i}) \subseteq R_j^{i+1}$, $1 \leqslant j \leqslant n$,

$$i = 0, 1, 2, \ldots$$

We have two cases:

I) $R_j^i \neq \varnothing$ for every $j \in [1, n]$. Let $\beta_k \in R_k^i$, $1 \leqslant k \leqslant n$, $\beta_k : I \to \varnothing$, then $k \in \beta_k \zeta_{A,i}$. There exist p, $0 : [p] \to [n]$ in $S_0[\Omega]$, such that $\Phi = \omega\, 0$. We have $d\Phi = 1$, $d\, 0 = 0$, $0 = \langle 0_1, \ldots, 0_p \rangle$, $0_h : I \to [n]$, $d\, 0_h = 0$, $h = 1, 2, \ldots, p$.

By the Proposition 2.4 it follows $j \in (I(1_p\, \theta), \ldots, I(p_p\, \theta))(\omega\, \psi^{p,1}_{A,i})$. Then $\gamma = (\beta_1, \ldots, \beta_n)(\Phi\, \psi^{n,1}_{\hat{C}^\circ, i}) = (\beta_1, \ldots, \beta_n)(\theta\, \psi^{n,1}_{\hat{C}^\circ, i})(\omega\, \psi^{p,1}_{\hat{C}^\circ, i}) = (\beta_{m_1}, \beta_{m_2}, \ldots, \beta_{m_p})(\omega\, \psi^{p,1}_{\hat{C}^\circ, i})$, where $m_k = I(h_p\, \theta)$, $h = 1, 2, \ldots, p$. $\gamma \zeta_{A,i+1} = (\beta_{m_1}, \beta_{m_2}, \ldots, \beta_{m_p})(\omega\, \psi^{p,1}_{\hat{C}^\circ, i})\, \zeta_{A,i+1} = (\beta_{m_1} \zeta_{A,i}, \ldots, \beta_{m_p} \zeta_{A,i})(\omega\, \psi^{p,1}_{A,i})$. From $k \in \beta_k \zeta_{A,i}$, $1 \leqslant k \leqslant n$, it obtains $m_h \in \beta_{m_h} \zeta_{A,i}$, $h = 1, 2, \ldots, p$. Hence $j \in (m_1, \ldots, m_p)(\omega\, \psi^{p,1}_{A,i}) \subseteq (\beta_{m_1} \zeta_{A,i}, \ldots, \beta_{m_p} \zeta_{A,i})(\omega\, \psi^{p,1}_{A,i}) = \gamma\, \zeta_{A,i+1}$, i.e. $\gamma \in R_j^{i+1}$ (2.24).

II. There exists $j \in [1, n]$, such that $R_j^i = \varnothing$. By the Proposition 2.4, the Remark 2.5, we have

$(R_1^i, \ldots, R_n^i)(\Phi\, \psi^{n,1}_{\hat{C}^\circ, i})\, \zeta'_{A,i+1} = (R_1^i \zeta'_{A,i}, \ldots, R_n^i \zeta_{A,i})(\Phi\, \psi^{n,1}_{A,i}) = \varnothing$

It results that $(R_1^i, \ldots, R_n^i)(\Phi\, \psi^{n,1}_{\hat{C}^\circ, i}) = \varnothing \subseteq R_j^{i+1}$, therefore (2.24)

Let us show by induction on $d\Phi$:

$$R^i \subseteq P^i,\ i = 1, 2, \ldots \text{ or equivalentl y,} \tag{2.25}$$

(2.26) $R_j^i \subseteq P_j^i$, $1 \leqslant j \leqslant n$, $i = 1, 2, \ldots$, *i.e.* for every $\Phi : I \to \varnothing$, if $j \in \Phi \zeta_{A,i}$, then $\Phi \in P_j^i$.

First, we consider the morphism Φ with $d\Phi = 1$. There exist $k \geqslant 0$, $\omega \in \Omega_k$, $x : [k] \to \varnothing$, such that $\Phi = \omega x$. Then $dx = 0$, $k = 0$, $\Phi = \omega$. We have $(z_1, \ldots, z_n)((\Phi\, O_n)\, \psi^{n,1}_{A,i-1}) = (z_1, \ldots, z_n)(O_n \psi^{n,0}_{A,i-1})(\Phi\, \psi^{0,1}_{A,i-1}) = (\quad)(\Phi\, \psi^{0,1}_{A,i-1}) = (\quad)(\Phi\, \psi^{0,1}_{C^\circ,i-1})\, \zeta_{A,i} = \Phi\, \zeta_{A,i}$ From $j \in \Phi\, \zeta_{A,i}$ it obtains $j \in (\quad)(\Phi\, \psi^{0,1}_{A,i-1})$, hence $\Phi\, O_n \in P_j^{i-1}$, then $(\overline{P}_1^{i-1}, \ldots, \overline{P}_n^{i-1})((\Phi\, O_n)\, (\psi^{n,1}_{\hat{C}^\circ, i-1}) = (\quad)(\Phi\, \psi^{0,1}_{\hat{C}^\circ, i-1}) = \Phi$.

From $(\overline{P}_1^{i-1}, \ldots, \overline{P}_n^{i-1})((\Phi\, O_n)\, \psi^{n,1}_{\hat{C}^\circ, i-1}) \subseteq P_j^i$, it results $\Phi \in P_j^i$. Let $\Phi : I \to \varnothing$ be a morphism with $d\Phi = h > 1$. There exist $p > 0$, $\omega \in \Omega_p$, $x : [p] \to \varnothing$, such that $\Phi = \omega x$. Let $x = \langle x_1, \ldots, x_p \rangle$, $x_k : I \to \varnothing$, $1 \leqslant k \leqslant p$, $dx < h$ implies $dx_i < h$, $1 \leqslant i \leqslant p$. We have $j \in \Phi\, \zeta_{A,i}$. Consider $i > 1$. It obtains

$$\Phi = (\quad)(\Phi\, \psi^{0,1}_{C^\circ, i-1}) =$$

$$= [(\quad)(x_1\, \psi^{0,1}_{C^\circ, i-1}), (\quad)(x_2\, \psi^{0,1}_{C^\circ, i-1}), \ldots, (\quad)(x_p\, \psi^{0,1}_{C^\circ, i-1})](\omega\, \psi^{p,1}_{C^\circ, i-1}) =$$

$$= (x_1, \ldots, x_p)(\omega\, \psi^{p,1}_{C^\circ, i-1}).$$

$$\Phi\, \zeta_{A,i} = (x_1 \zeta_{A,i-1}, x_2 \zeta_{A,i-1}, \ldots, x_p \zeta_{A,i-1})(\omega\, \psi^{p,1}_{A,i-1}), \text{ therefore}$$

$$j \in (x_1 \zeta_{A,i-1}, \ldots, x_p \zeta_{A,i-1})(\omega\, \psi^{p,1}_{A,i-1}).$$

Because $\varnothing \neq x_k \zeta_{A,i-1} \subseteq A_{i-1} = [n]$, $1 \leqslant k \leqslant p$, there exists $t_k \in x_k \zeta_{A,i-1}$ and $z_k : I \to [n]$, such that $I z_k = t_k$ and $j \in (t_1, \ldots, t_p)(\omega \psi^{p,1}_{A,i-1})$. $1 \leqslant k \leqslant p$.

From $t_k \in x_k \zeta_{A,i-1}$, $dx_k < h$ and the assumption of the induction it follows $x_k \in P_{t_k}^{i-1}$, $k = 1, 2, \ldots, p$. Then $(P_1^{i-1}, \ldots, P_n^{i-1})((\omega z)\, \psi^{n,1}_{\hat{C}^\circ, i-1}) = (P_1^{i-1}, \ldots, P_n^{i-1})(z\, \psi^{n,p}_{\hat{C}^\circ, i-1})(\omega\, \psi^{p,1}_{\hat{C}^\circ, i-1}) = (P_{t_1}^{i-1}, P_{t_2}^{i-1}, \ldots, \overline{P}_{t_p}^{i-1})(\omega\, \psi^{p,1}_{\hat{C}^\circ, i-1}) \ni (x_1, \ldots, x_p)(\omega\, \psi^{p,1}_{\hat{C}^\circ, i-1}) = \Phi$.

Because $(I(1_p\, z), \ldots, I(p_p\, z))(\omega\, \psi^{p,1}_{A,i-1}) = (I z_1, \ldots, I z_p)(\omega\, \psi^{p,1}_{A,i-1}) = (t_1, \ldots, t_p)(\omega\, \psi^{p,1}_{A,i-1})$ and $j \in (t_1, \ldots, t_p)(\omega\, \psi^{p,1}_{A,i-1})$; we have $\omega z \in P_j^{i-1}$, $z = \langle z_1, \ldots, z_p \rangle$ therefore $(P_1^{i-1}, \ldots, P_n^{i-1})((\omega z)\, \psi^{n,1}_{\hat{C}^\circ, i-1}) \subseteq P_j^i$, it results $\Phi \in P_j^i$, as required.

If $i = 1$ and $j \in \Phi \zeta_{A,1}$, then we have $\Phi \in \overline{P}^1_j$, because $\Phi \zeta_{A,1} = (\quad)(\Phi \psi^{0,1}_{A,0})$.

Definition 2.9. *The family* (X_i), $i = 0, 1, 2, \ldots, X_i \subseteq C^0_i$ *is called metaalgebraic of order* 1 *with respect to* $\hat{C}_0$ *if there exist* n *and a* ω*-polynomial* $P_{n,n}$ *of degree* 1, *such that* $X_i = \overline{P}^i_1$, $i = 0, 1, 2, \ldots$

Theorem 2.10. *A metaalgebraic family* $X = (X_i)$, $i = 0, 1, 2, \ldots$ *of order* 1 *with respect to* $\hat{C}_0$ *is the behaviour of any restricted* VT*-automaton over* $S_0[\Omega]$.

Proof. The VT-automaton (A, t) over $S_0[\Omega]$ is called restricted, if A is a restricted VT-algebra over $S_0[\Omega]$. Let $P_{n,n}$ be a polynomial of degree 1 such that $X_i = \overline{P}^i_1$, $i = 0, 1, 2, \ldots$ By the Theorem 2.6 there exists a ω-polynomial $T_{n,n}$, such that $\overline{\mathfrak{L}}_{\hat{C}_0} = \overline{\mathfrak{T}}_{\hat{C}_0}$, therefore $X_i = \overline{T}^i_1$, $i = 0, 1, 2, \ldots$ Consider the relational VT-algebra $A = (A_i, \alpha_i, \beta_i)$, $A_i = [n]$, $\alpha_i = \beta_i = 1_{[n]}$, $i = 0, 1, 2, \ldots$ corresponding to $T_{n,n}$. By the Theorem 2,8, we have $\overline{T}^i_j = j\, \zeta^{-1}_{A,i}$, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \ldots$, therefore $X_i = 1\, \zeta^{-1}_{A,i}$. If $t_i = \{1\} \subseteq [n]$, $i = 0, 1, 2, \ldots$, it results that the family $X = (X_i)$, $i = 0, 1, 2, \ldots$ is the behavior of (A, t).

Theorem 2.11. *The behavior of the restricted* VT*-automaton* (A, t) *over* $S_0[\Omega]$, $A_i =$ *constant*, $\alpha_i = \beta_i = 1_{A_i}$, $t = (t_i)$, $t_i =$ *constant*, $i = 0, 1, 2, \ldots$ *is a metaalgebraic family with respect to* $\hat{C}_0$.

Proof. We may assume that $A_i = [n]$, $t_i = [k]$, $k \leqslant n$.

By the Proposition 2.4 there exists a ω-polinomial $P_{n,n}$ of degree 1 corresponding to A. We have $\beta((A, t)) = (X_i)$, $i = 0, 1, 2, \ldots$ where $X_i = t_i\, \zeta^{-1}_{A,i} = \bigcup_{h \in t_i = [k]} h\, \zeta^{-1}_{A,i}$. By the Theorem 2.8 $h\, \zeta^{-1}_{A,i} = \overline{P}^i_h$, $1 \leqslant j \leqslant n$, $i = 0, 1, 2, \ldots$ The family $(\overline{P}^0_1, \overline{P}^1_1, \ldots)$ is metaalgebraic with respect to $\hat{C}_0$. By the Proposition 2.2 it results that $X = (X_i)$, $X_i = \bigcup_{h=1}^{k} \overline{P}^i_h$, $i = 0, 1, 2, \ldots$ is metaalgebraic with respect to $\hat{C}^0$.

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ON THE COMPLETNESS IN ABSTRACT COMPLEXITY CLASSES OF LANGUAGES

BY

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1. Introduction. The importance of polynomial complete languages in concrete computational complexity has been pointed out by Cook [2] and Karp [5]. Book [3] shows the relationships between complexity classes defined by time-(space-) bounded multitape Turing machines with the bounding functions n, n^k, 2^n, 2^{nk}.

The goal of this paper is to show that some results of Book [3], are true in a more general context. We consider an „abstract complexity class" of languages to be a class of languages accepted by (non) deterministic Turing machines which run in time (space) bounded by functions from a class $\mathfrak{F}$. We look for the conditions which must be satisfied by $\mathfrak{F}$ and by the reductibilities, to obtain the Book-analogous relationships between „abstract complexity classes".

2. Preliminaries. Let Σ be a finite nonempty set called alphabet. The set of all strings (words) of elements from Σ (including the nul string $\wedge$) will be denoted by Σ^* (the free semigroup generated by Σ). The legth of a string $w \in \Sigma^*$ will be $|w|$ (which means the number of symbols of w). If $\sigma \in \Sigma$, the string $\sigma\sigma \ldots \sigma$ of length m, will be designed by σ^m. A language is a subset L of Σ^*.

We assume that the reader is familiar with the concept of multitape (non) deterministic Turing machine ([1] pag. 25, [4] pag 80). For a Turing machine M with the input alphabet Σ, we denote by $L(M)$, $L(M) \subseteq \Sigma^*$, the language accepted by M.

Let N be the set of positive integers.

Definition 1. *A function* $f: N \to N$ *is self computable* ([3]) *if there exists a deterministic Turing machine* M_1, *which, upon input* w, *runs for precisely* $f(|w|)$ *steps and halts, and a deterministic Turing machine* M_2, *which, upon input* w, *marks precisely* $f(|w|)$ *consecutive tape squares and halts.*

Definition 2. *A function* $f: N \to N$ *is a bounding function if*:

i) f *is nondecreasing*;

ii) f *is self-computable.*

Let M be a multitape Turing machine and f a bounding function. We say that *M operates within time (space) bound f* if, for each input string w, if $w \in L(M)$ there exists an accepting computation of M on w which