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ON THE MAXIMUM EDGES OF A HEREDITARY HYPERGRAPH

BY

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1. Introduction. A *hypergraph* $H = (E, \mathcal{F})$ is given by a finite set of *vertices* $E = \{x_j ; j \in J\}$ and a finite family $\mathcal{F} = (E_i ; i \in I)$ of subsets of E such that $\bigcup_{i \in I} E_i = E$, whose elements are the *edges* of H.

Our aim is to find a characterization for maximum cardinality edges of a hypergraph H. On the other hand, the paper is devoted to give an extension of the „alternating chains" technics, to a general class of hypergraphs, hereditary hypergraphs. This technics introduced by Berge [1] and Norman and Rabin [2] for matching problems, was extended by D. K. Ray-Chaudhury [3] for general b-matching problems. However, A.A. Zykov [4] gives a counter example to the Chaudhury's work. Our concept of alternating chain gives the correct approach for the b-matching problems as a special case.

2. Preliminaries. In the following, $|A|$ denotes the cardinality of the set A and $A \Delta B$ is $(A - B) \cup (B - A)$. For any edge F of a hypergraph H, we denote

$$\mathcal{G}_F = \{G ; G \subseteq E, F \Delta G \in \mathcal{F} \text{ and } |G| > 2 |F \cap G| \}.$$

The family $\mathcal{G}_F$ can be used to obtain a characterization for maximum cardinality edges (shortly, maximum edges) of the hypergraph H.

Lemma 1. *F is a maximum edge if and only if $\mathcal{G}_F = \varnothing$.*

Proof. If F is a maximum egde, then, for any $G \subseteq E$ such that $F \Delta G \in \mathcal{F}$ we have $|F \Delta G| \leqslant |F|$, $|F| + |G| - 2 |F \cap G| \leqslant |F|$, $|G| \leqslant 2 |F \cap G|$, that is $\mathcal{G}_F = \varnothing$.

Conversely, let F be any edge such that $\mathcal{G}_F = \varnothing$, and let F_0 be a maximum edge. Let $G = F \Delta F_0$. Obviously $F \Delta G = F_0 \in \mathcal{F}$ and therefore $|G| \leqslant 2 |F \cap G|$ that means $|F - F_0| + |F_0 - F| \leqslant 2 |F - F_0|$, $|F_0 - F| \leqslant |F - F_0|$, $|F_0| \leqslant |F|$; it follows $|F| = |F_0|$ and therefore F is a maximum edge.

The result of this lemma has the disadvantage that the family $\mathcal{G}_F$ can be very large and therefore difficult to be constructed. An interesting problem for the characterization of the maximum edges of a hypergraph is to obtain a result as in lemma 1, but using only a subfamily of $\mathcal{G}_F$. This is deduced in the next section for a particular class of hypergraphs.

3. The Main Result. Definition. *A hypergraph $H = (E, \mathcal{F})$ such that $F_1 \subseteq F_2 \in \mathcal{F}$ implies $F_1 \in \mathcal{F}$, is called a hereditary hypergraph.*

To find a good characterization for maximum edges of a hereditary hypergraph is in fact the „packing problem". In the following, H will denote a hereditary hypergraph.

Let F be any edge of H, and let us denote

$$\mathcal{G}'_F = \{G : G \subseteq E,\ F \Delta G \in \mathcal{F} \text{ and } |G| = 2|F \cap G| + 1\}.$$

Lemma 2. *F is a maximum edge of H if and only if $\mathcal{G}'_F = \varnothing$.*

Proof. If F is a maximum edge, then $\mathcal{G}_F = \varnothing$ (cf. lemma 1) and since $\mathcal{G}'_F \subseteq \mathcal{G}_F$, we have $\mathcal{G}'_F = \varnothing$.

Conversely, let F be any edge such that $\mathcal{G}'_F = \varnothing$ and suppose that F is not a maximum edge. According to lemma 1, we have $\mathcal{G}_F \neq \varnothing$ and therefore, there is a subset $G \subseteq E$ such that $F \Delta G \in \mathcal{F}$ and $|G| > 2\,|F \cap G|$. Consider $G' = F \cap G \cup G_1$, where $G_1 \subseteq G - F$ and $|G_1| = |F \cap G| + 1$ (this is possible since $|G| > 2\,|F \cap G|$). We have $F \Delta G' \subseteq F \Delta G \in \mathcal{F}$ and therefore, since H is hereditary, $F \Delta G' \in \mathcal{F}$. On the other hand $|G'| = 2\,|G' \cap F| + 1$, hence we have $G' \in \mathcal{G}'_F$, contradicting $\mathcal{G}'_F = \varnothing$.

We can effectively construct a subfamily W_F of $\mathcal{G}'_F$ which allows to characterize the maximum edges of H.

The next approach is a generalization of works [1] and [2].

Definition. *An alternating chain of H relative to the edge F is a sequence $w = (e_2, e_2, \ldots, e_n)$ of distinct vertices of H, which belong alternatively to $E - F$ and F, constructed with the following algorithm.*

Algorithm 1.

Step 1. Chose an arbitrary vertex $e_1 \in E - F$ and let $w_1 = (e_1)$ and $k = 0$; go to step 2.

Step 2. If $w_{2k+1} \Delta E \in \mathcal{F}$ let $w = w_{2k+1}$ and go to step 4; otherwise, choose arbitrarily $e_{2k+2} \in F - w_{2k+1}$, put $w_{2k+2} = (e_1, e_2, \ldots, e_{2k+1}, e_{2k+2})$, $p = k + 1$, and go to step 3.

Step 3. If $(w_{2p} \Delta (E - F)) \cup A \in \mathcal{F}$, *for any* $A \subseteq F - w_{2p}$, let $w = w_{2p}$ and go to step 4; otherwise, try to choose a vertex $e_{2p+1} \in E - F - w_{2p}$ such that $(w_{2p} - F) \cup e_{2p+1} \in \mathcal{F}$. If such a vertex does not exist, let $w = w_{2p}$ and go to step 4; otherwise let $w_{2p+1} = (e_1 e_2, \ldots, e_{2p}, e_{2p+1})$, $k = p + 1$ and go to step 2.

Step 4. Terminate, w is an alternating chain of H relative to the edge F.

An alternating chain is *odd* if it has an odd number of vertices. Let us denote by W_F the set of all odd alternating chains of H relative to the edge. The family W_F can be obtained with the algorithm 1 combined with a backtrakct procedure; let us denote this, **Back 1.**

Remarks.

i) In the above definition we have assumed $F \neq E$.

ii) Notice that one of the sets A considered in Step 3 is the empty set.

iii) The family W_F is a subfamily of $\mathcal{G}'_F$, since an odd alternating chain is obtained in step 4 of algorithm 1 with the „go to" from step 2.

iv) In step 2 of àlgortihm 1, we choose elements from $F - w_{2k+1}$, ussuming this set nonempty. This fact is a consequence of the remark i) above and the following argument. If $F - w_{2k+1} = \varnothing$, we have $w_{2k+1} \Delta F = (w_{2k+1} - F) \cup (F - w_{2k+1}) = w_{2k+1} - F$. But, from step 3 of the algorithm we obtain $w_{2k+1} - F = w_{2k+1} \cap (E - F) \in \mathcal{F}$ hence $w_{2k+1} \Delta F \in \mathcal{F}$ which contradicts the initial test of step 2.

v) In some particular cases (as we shall show in section 4), the step 3 of the algorithm can be very much simplified and in these cases the choosing of the elements in step 2 and 3, can be done using some rules, which give an important improvement to the algorithm.

Now, we are able to state our main theorem.

Theorem 1. *Let $H = (E, \mathcal{F})$ be a hereditary hypergraph and F any of his edges. F is a maximum edge if and only if $W_F = \varnothing$.*

Proof. If F is a maximum edge, then inasmuch as $W_F \subseteq \mathcal{G}'_F$ we have (by lemma 2) $W_F = \varnothing$.

Conversely, let F be an edge of H for which $W_F = \varnothing$ and F is not a maximum edge. Let G be an edge of maximum cardinality of H such that $F \Delta G = \min \{|F \Delta G'| ; G' \text{ is a maximum edge of } H\}$. Obviously, $|F \Delta G| > 0$ (otherwise, F is a maximum edge).

Let $H_1 = (E_1, \mathcal{F}_1)$ be a hypergraph with $E_1 = F \Delta G$ and $\mathcal{F}_1 = \{F_1 ; F_1 \subseteq E_1 \text{ and } F_1 \cup (F \cap G) \in \mathcal{F}\}$. We have $F - G \in \mathcal{F}_1$, $G - F \in \mathcal{F}_1$ and furthermore $|G - F| > |F - G| \geqslant 0$.

The hypergraph H_1 is hereditary. Indeed, if $F_0 \subset F_1 \in \mathcal{F}_1$ then $F_0 \subset E_1$ and since $F_0 \cup (F \cup G) \cup (F \cap G) \in \mathcal{F}$, the heredity of H implies $F_0 \cup (F \cap G) \in \mathcal{F}$ i.e., $F_0 \in \mathcal{F}_1$.

Now, we can apply the algorithm 1 to construct an alternating chain w of the hereditary hypergraph H_1 relative to the edge $F - G$ (this is possible since $G - F \neq \varnothing$). The chain w is even since otherwise, w is an odd alternating chain of H relative to the edge. F. This fact is a consequence of the statement of the algorithm 1, and of the remarks (a), (b), (c), and (d) bellow. Notations are from algorithm 1.

(a) $w_{2k+1} \Delta (F - G) \in \mathcal{F}_1$ implies $w_{2k+1} \Delta F \in \mathcal{F}$. Indeed, from $w_{2k+1} \Delta (F - G) \in \mathcal{F}_1$ we have $(w_{2k+1} \Delta (F - G)) \cup (F \cap G) \in \mathcal{F}$ which implies $(w_{2k+1} - (F - G)) \cup (((F - G) - w_{2k+1}) \cup (F \cap G)) \in \mathcal{F}$. Since $w_{2k+1} \subset F \Delta \Delta G$ we obtain $(w_{2k+1} - F) \cup (F - w_{2k+1}) \in \mathcal{F}$, that is $w_{2k+1} \Delta F \in \mathcal{F}$.

(b) $w_{2k+1} \Delta(F - G) \notin \mathcal{F}_1$ implies $w_{2k+1} \Delta F \notin \mathcal{F}$. Indeed, $w_{2k+1} \Delta (F - G) \notin \mathcal{F}_1$ implies $(w_{2k+1} \Delta(F - G)) \cup (F \cap G) \notin \mathcal{F}$ and as above, we have $w_{2k+1} \Delta F \notin \mathcal{F}$.

(c) If there is $A_0 \subset (F - G) - w_{2p}$ such that $(w_{2p} \Delta (G - F)) \cup A_0 \notin \mathcal{F}_1$; then there is $A \subset F - w_{2p}$ such that $(w_{2p} \Delta(E - F)) \cup A \notin \mathcal{F}$:

Indeed, let $A = A_0 \cup F \cap G$. Then $(w_{2p} \Delta (E - F)) \cup A = ((w_{2p} - (G - F)) \cup (G - F - w_{2p})) \cup (E - (F \cup G)) \cup A = ((w_{2p}\ (G - F)) \cup A_0 \cup F \cap G) \cup (E - (F \cup G)) \notin \mathcal{F}$ since from the hypothesis $(w_{2p}\ (G - F)) \cup A_0 \cup F \cap G \notin \mathcal{F}$.

(d) The algorithm 1 cannot finish by the second „go to" of the step 3. Indeed, for any $e_{2p+1} \in G - F - w_{2p+1}$ we have $w_{2p} - (F - G) \cup \{e_{2p+1}\} \subset G - F$ an since $G - F \in \mathcal{F}_1$, by the heredity of H_1 we have $w_{2p} - (F - G) \cup \{e_{2p+1}\} \in \mathcal{F}_1$; furthermore $G - F - w_{2p+1} \neq \varnothing$ since $|G - F| > |F - G|$.

We have obtained that w is an even alternating chain of H_1 relative to the edge $F - G$ and hence taking $A = \varnothing$ in step 3 of the algorithm, we have $(G - F) \Delta w \in \mathcal{F}_1$ that is, $G' = (G - F) \Delta w \cup (F \cap G) \in \mathcal{F}$, from which we obtain $G' = G \Delta w \in \mathcal{F}$. Since w is even we have $|G'| = |G|$; on the other hand, $|F \Delta G'| < |F \Delta G|$ which contradicts the choosing of G. End of proof.

Now, the Theorem 1 justifies the following algortihm for finding of a maximum edge of a hereditary hypergraph H.

Algorithm 2.

Step. 1. Let $F_0 = F$ (F is an arbitrary edge of H) and $k = 0$.

Step 2. Apply **Back 1** to F_k; if $W_{F_k} = \emptyset$, go to step 3, otherwise, if w is the first odd alternating chain obtained, let $F_{k+1} = F_k \Delta w$, put $k = k + 1$ and repeat step 2.

Step 3. Terminate. F_k is a maximum edge.

Unfortunately, the task can still be one exponential order, depending on what further directions are given.

4. Applications. 1. Definition. *A hypergraph $H = (E, \mathcal{F})$ is called binary hereditary if:*

(a) *it is hereditary, and*

(b) $\forall F_1, F_2 \in \mathcal{F}$, $F_1 \Delta F_2 \in \mathcal{F}$ *implies* $F_1 \cup F_2 \in \mathcal{F}$.

The word „binary" is justified by the following

Theorem 2. *$F \in \mathcal{F}$ if and only if for any distinct vertices $e_i, e_j \in F$ we have $\{e_i, e_j\} \in \mathcal{F}$.*

Proof. The necessity is trivial.

To prove the sufficiency, let $F \in E$ with the property that for any $e_i, e_j \in F$, $\{e_i, e_j\} \in \mathcal{F}$. Suppose that $F \notin \mathcal{F}$ and let $F_0 \subset F$ be a minimal subset of F with the same property, i.e. $F_0 \notin \mathcal{F}$ and for any F_1 properly contained in F_0, $F_1 \in \mathcal{F}$.

Clearly $|F_0| \geqslant 3$. Let $F_1 = F_0 - \{e_1\}$, $F_2 = \{e_1, e_2\}$ $(e_1, e_2 \in F_0)$. We have $F_1, F_2 \in F$ and $F_1 \Delta F_2 = F_0 - \{e_2\} \subset F_0$. Hence $F_1 \Delta F_2 \in \mathcal{F}$ and by the second condition of the definition of a binary hereditary hypergraph we have $F_1 \cup F_2 = F_0 \in \mathcal{F}$ which provides a contradiction. Therefore $F \in \mathcal{F}$ and the theorem is proved.

If H is a binary hereditary hypergraph, then the Step 3 of Algorithm 1 can be replaced with the following one:

Step 3'. If $w_{2p} \Delta (E - F) \in \mathcal{F}$, let $w = w_{2p}$ and go to step 4; otherwise try to choose a vertex $e_{2p+1} \in E - F - w_{2p}$ such that $(w_{2p} - F) \cup \{e_{2p+1}\} \in \mathcal{F}$. If such a vertex does not exist, let $w = w_{2p}$ and go to step 4; ortherwise let $w_{2p+1} = (e_1, e_2, \ldots, e_{2p}, e_{2p+1})$, $k = p + 1$ and go to step 2.

In other words, the only set A needed in the first test of step 3 is $A = \emptyset$.

Let us denote *Algorithm* 1' this new algorithm obtained, and with W'_F the family of all odd alternating chains of H relative to the edge F, produced by Algorithm 1' combined with a backtrack procedure. Now, we can state a theorem corresponding to the theorem 1.

Theorem 1. *Let $H = (E, \mathcal{F})$ be a binary hereditary hypergraph and F any of his edges. F is a maximum edge if and only if $W'_F = \emptyset$.*

Proof. The proof of this theorem is essentially the proof of theorem 1 with the only modification of the remark c) and their proof, which must be replaced with:

c') $w_{2p} \Delta(G - F) \notin \mathcal{F}_1$ implies $w_{2p} \Delta(E - F) \notin \mathcal{F}$. Indeed, if $w_{2p} \Delta (E - F) \in \mathcal{F}$, then by the heredity of H we have $w_{2p} \Delta (G - F) \in \mathcal{F}$.

Now, let $F_0 = w_{2p} \Delta (G - F) \cup (F \cap G)$. If $F_0 \notin \mathcal{F}$, then there exist $e_1, e_2 \in F_0$ such that $\{e_1, e_2\} \notin \mathcal{F}$ (by theorem 2). Clearly one of the e_i $i = 1, 2$ must belong to $F \cap G$ (otherwise $w_{2p} \Delta (G - F) \notin \mathcal{F}$). Let us consider $e_1 \in F \cap G$. Then $e_2 \notin F \cap G$ since $F \cap G \in \mathcal{F}$; if $e_2 \in w_{2p} - (G - F)$ then $\{e_1, e_2\} \subset F$ and hence $\{e_1, e_2\} \in \mathcal{F}$. Therefore $e_2 \in (G - F) - w_{2p}$; we have obtained $\{e_1, e_2\} \subset G$ and $\{e_1, e_2\} \notin \mathcal{F}$, a contradiction. Hence $F_0 \in \mathcal{F}$, that is $w_{2p} \Delta (G - F) \in \mathcal{F}_1$, which contradicts the working hypothesis. End of proof.

Furthermore, the theorem 2 gives a rule of choice in steps 2 and 3 of our algorithms; if $w_{2k+1} \Delta F \notin \mathcal{F}$, there are vertices $e \in F - w_{2k+1}$ and $e' \in w_{2k+1} - F$ such that $\{e, e'\} \notin \mathcal{F}$, then choose $e_{2k+2} = e$. Similarly in step 3. This rule diminishes substantially the searching for odd alternating chains in Algorithm 2.

Remark. If we compare Algorithm 1 with Algorithm 1' it is obvious that Algorithm 1' is faster. However if we consider Algorithm 2 and Algorithm 2' (Algorithm 2' is obtained from Algorithm 2 using a combination of Algorithm 1' with a backtrack procedure, instead of Back 1) then the Algorithm 2 is faster.

Examples: Let $G = (X, U)$ be a given graph. Let us consider $H_1 = (X, \mathcal{F})$ where $\mathcal{F}$ is the family of all stables of G. F_1 is a binary hereditary hypergraph and Theorem 1' gives a characterization of a maximum stable of G; this is even Theorem 2, page 264, from [1]. Let us consider $H_2 = (U, \mathcal{F})$ where $\mathcal{F}$ is the family of all matchings of G. H_2 is a binary hereditary hypergraph and Theorem 1' is the wellknown alternating chains theorem due to B e r g e for characterization of maximum matching. We remark that the hypergraph H_2 has remarkable property:

if $F \in \mathcal{F}$ and $u \in U - F$, *then* $|\{u'; u' \in F$ and $(u', u) \notin \mathcal{F}\}| \leqslant 2$. This is the essential fact which has permitted to E d m o n d s to construct his polynomyal algorithm for finding the maximum matching in a graph. We note that H_1 has above property iff G is $K_{1,3}$-free, and for such graphs the finding of a maximum independent set is therefore, solved.

2. Let $G = (X, U)$ be a simple graph. Consider $H_3 = (X, \mathcal{F})$ where $\mathcal{F}$ is the family of all subsets $F \subset X$ such that $[F]_G$ is planar. H_3 is then a hereditary hypergraph and Theorem 1 characterizes the maximum planar induced subgraph of G.

3. Let $H = (X, \mathcal{E})$ be a given hypergraph without multiple edges. Consider $H_4 = (\mathcal{E}, \mathcal{F})$ where $\mathcal{F}$ is the family of all b-matchings of F. The Theorem 1 applied to the hereditary hypergraph H_4 gives a characterization of a maximum b-matching of H. The Theorem 1' can not be apllied because H is not a binary hereditary hypergraph. This is the mistake. of [3].

4. Let P be a property of graphs and let us denote P^E, the *associate hereditary property of P, defined as follows: G has property P^E iff every induced subgraph G' of G has property P.* If for any graph G: G has property P implies G has property P^E, then P is said a *hereditary property* of graphs. For instance, the properties of graphs to be completely disjoint, to be clique, to be planar, etc., are hereditary.

If P is not a hereditary property then a graph G having the property P^E is called a *P-perfect* graph. Theorem 1 applied to the hereditary hypergraph $H_s = (X, \mathcal{F})$, where X is the vertices set of G, and $\mathcal{F}$ is the family of all subsets F of X such that $[F]_G$ is P-perfect, gives a characterization of the maximum induced P-perfect subgraph of G.

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ON THE GRADABLE DIGRAPHS

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Let $G = < \mathbf{N}, \mathbf{A} >$ be a *directed finite graph* (*digraph*), where $N = \{n_1, n_2, \ldots, n_p\}$ is the set of *nodes* and $\mathbf{A}$ the set of *arcs*. A *path of leght* r from node n to node m, is a sequence of r arcs : $< n = x_{i_0}, n_{i_1} >, < n_{i_1}, n_{i_2} >, \ldots, < n_{i_{r-1}}, n_{i_r} = m >$. We denote a path by $\gamma[n, m]$ and its length by $d\gamma(n, m)$.

A *circuit* is a path for which $n = m$. A *loop* is an arc $< n, n >$. For an arbitrary node $n_0 \in \mathbf{N}$ we consider the set $\mathbf{D}(n_0) = \{m \in \mathbf{N} \mid$ there exists $\gamma[n_0, m]\} \cup \{n_0\}$.

A digraph without circuits is *gradable* [1] if there exist the non-negative integers $\mathbf{M}[n_1], \mathbf{M}[n_2], \ldots, \mathbf{M}[n_p]$ associated to the nodes $n_1, n_2, \ldots, n_p$, so that for every arc $< n, m >$ of A, we have : $\mathbf{M}[m] - \mathbf{M}[n] = 1$.

We mention that a diagraph with circuits cannot be gradable (see theorem 10.1, [1]).

A theory of gradable diagraphs and their practical importance are detailed in []1.

Theorem 1. *Let $G = < \mathbf{N}, \mathbf{A} >$ be a digraph without circuits so that there exists an unique node $n_0 \in \mathbf{N}$ with $\mathbf{D}(n_0) = \mathbf{N}$. G is gradable if and only if for every two paths $\gamma_1[n, m]$ and $\gamma_2[n, m]$ we have $d\gamma_1(n, m) = d\gamma_2(n, m)$, for all $n, m \in \mathcal{N}$.*

Proof. Obviously, if G is gradable, for every path $\gamma[n, m]$ we have $d\gamma(n, m) = \mathbf{M}[m] - \mathbf{M}[n]$. Hence, $d\gamma_1(n, m) = d\gamma_2(n, m) = \mathbf{M}[m] - M[n]$.

Now, we attach to n_0 an arbitrary non-negative integer $\mathbf{M}[n_0]$, and to every $m \in \mathbf{N}$, $m \neq n_0$, the number $\mathbf{M}[m] = \mathbf{M}[n_0] + d(n_0, m)$, where $d(n_0, m)$ is the lenght of a path from n_0 to m, (all the paths have the same lenght), and let $< n_\alpha, n_\beta >$ be an arbitrary arc of $\mathbf{A}$. Since G is without circuits and n_α is reachable from n_0 it is impossible to have $n_\alpha \neq n_0$ and $n_\beta = n_0$. If $n_\alpha = n_0$ and $n_\beta \neq n_0$, we have : $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = \mathbf{M}[n_0] + d(n_0, n_\beta) - \mathbf{M}[n_0] = d(n_0, n_\beta) = 1$. If $n_\alpha \neq n_0$ and $n_\beta \neq n_0$, we have : $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = \mathbf{M}[n_0] + d(n_0, n_\beta) - \mathbf{M}[n_0] - d(n_0, n_\alpha) = d(n_0, n_\beta) - d(n_0, n_\alpha)$. But, $d(n_0, n_\beta) = d(n_0, n_\alpha) + 1$. Hence, $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = 1$, and the graph G is gradable.

The node n_0 is unique because G is without circuits.

A *chain* of length r between n and m denoted by $L[n, m]$, is a sequence of r edges : $(n = n_{i_0}, n_{i_1}), (n_{i_1}, n_{i_2}), \ldots, (n_{i_{r-1}}, n_{i_r} = m)$ so that $n_{i_{j-1}}$,