graph $H_6 = (X, \mathcal{F})$, where X is the vertices set of G, and $\mathcal{F}$ is the family of all subsets F of X such that $[F]_G$ is P-perfect, gives a characterization of the maximum induced P-perfect subgraph of G.

Acknowledgements. I would like to thank Professors I. Drăgan and E. Olaru for guidance into the graph theory as well as for certain useful discussions. I also wish to thank Mr. I. Gorgos for bringing Zykov's paper to my attention.

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Received 27.IX.1978

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVI, s. I a, 1980, fasc. 1

ON THE GRADABLE DIGRAPHS

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Let $G = < \mathbf{N}, \mathbf{A} >$ be a *directed finite graph (digraph)*, where $N = \{n_1, n_2, \ldots, n_p\}$ is the set of *nodes* and $\mathbf{A}$ the set of *arcs*. A *path of leght r* from node n to node m, is a sequence of r arcs: $< n = x_{i_0}, n_{i_1} >, < n_{i_1}, n_{i_2} >, \ldots, < n_{i_{r-1}}, n_{i_r} = m >$. We denote a path by $\gamma[n, m]$ and its length by $d\gamma(n, m)$.

A *circuit* is a path for which $n = m$. A *loop* is an arc $< n, n >$. For an arbitrary node $n_0 \in \mathbf{N}$ we consider the set $\mathbf{D}(n_0) = \{m \in \mathbf{N} \mid$ there exists $\gamma[n_0, m]\} \cup \{n_0\}$.

A digraph without circuits is *gradable* [1] if there exist the non-negative integers $\mathbf{M}[n_1], \mathbf{M}[n_2], \ldots, \mathbf{M}[n_p]$ associated to the nodes $n_1, n_2, \ldots, n_p$, so that for every arc $< n, m >$ of A, we have: $\mathbf{M}[m] - \mathbf{M}[n] = 1$.

We mention that a diagraph with circuits cannot be gradable (see theorem 10.1, [1]).

A theory of gradable diagraphs and their practical importance are detailed in []1.

Theorem 1. *Let $G = < \mathbf{N}, \mathbf{A} >$ be a digraph without circuits so that there exists an unique node $n_0 \in \mathbf{N}$ with $\mathbf{D}(n_0) = \mathbf{N}$. G is gradable if and only if for every two paths $\gamma_1[n, m]$ and $\gamma_2[n, m]$ we have $d\gamma_1(n, m) = d\gamma_2(n, m)$, for all $n, m \in \mathcal{N}$.*

Proof. Obviously, if G is gradable, for every path $\gamma[n, m]$ we have $d\gamma(n, m) = \mathbf{M}[m] - \mathbf{M}[n]$. Hence, $d\gamma_1(n, m) = d\gamma_2(n, m) = \mathbf{M}[m] - M[n]$.

Now, we attach to n_0 an arbitrary non-negative integer $\mathbf{M}[n_0]$, and to every $m \in \mathbf{N}$, $m \neq n_0$, the number $\mathbf{M}[m] = \mathbf{M}[n_0] + d(n_0, m)$, where $d(n_0, m)$ is the lenght of a path from n_0 to m, (all the paths have the same lenght), and let $< n_\alpha, n_\beta >$ be an arbitrary arc of $\mathbf{A}$. Since G is without circuits and n_α is reachable from n_0 it is impossible to have $n_\alpha \neq n_0$ and $n_\beta = n_0$. If $n_\alpha = n_0$ and $n_\beta \neq n_0$, we have: $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = \mathbf{M}[n_0] + d(n_0, n_\beta) - \mathbf{M}[n_0] = d(n_0, n_\beta) = 1$. If $n_\alpha \neq n_0$ and $n_\beta \neq n_0$, we have: $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = \mathbf{M}[n_0] + d(n_0, n_\beta) - \mathbf{M}[n_0] - d(n_0, n_\alpha) = d(n_0, n_\beta) - d(n_0, n_\alpha)$. But, $d(n_0, n_\beta) = d(n_0, n_\alpha) + 1$. Hence, $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = 1$, and the graph G is gradable.

The node n_0 is unique because G is without circuits.

A *chain* of length r between n and m denoted by $L[n, m]$, is a sequence of r edges: $(n = n_{i_0}, n_{i_1}), (n_{i_1}, n_{i_2}), \ldots, (n_{i_{r-1}}, n_{i_r} = m)$ so that $< n_{i_{j-1}},$

$n_{i_j} > \in A$ or $< n_{i_j}, n_{i_{j-1}} > \in \mathbf{A}$, for all $j = 1, 2, \ldots, r$. For a chain $L[n, m]$ we denote by $N(L, n, m)$ the number of forward arcs and by $N(L, m, n)$ the number of backward arcs.

Theorem 2. *Let $G = <\mathbf{N}, \mathbf{A}>$ be a connected digraph without loops. If G contains a node $n_0 \in \mathbf{N}$ so that for every $m \in \mathbf{N}$, the number $N(L, n_0, m) - N(L, m, n_0)$ does not depend on the choice of the chain $L[n_0, m]$ (i.e. is constant), then the digraph G is gradable.*

Proof. We attach to n_0 an arbitrary non-negative integer $\mathbf{M}[n_0]$, and to every $m \in \mathbf{N}$ $m \neq n_0$, the number $\mathbf{M}[m] = \mathbf{M}[n_0] + \Omega(n_0, m)$ where $\Omega(n_0, m) = N(L, n_0, m) - N(L, m, n_0)$, and let $< n_\alpha, n_\beta >$ be an arbitrary arc of $\mathbf{A}$.

If $n_\alpha = n_0$ and $n_\beta \neq n_0$, we have: $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = \mathbf{M}[n_0] + \Omega(n_0, n_\beta) - \mathbf{M}[n_0] = \Omega(n_0, n_\beta) = 1$.

If $n_\alpha \neq n_0$ and $n_\beta = n_0$, we have: $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = \mathbf{M}[n_0] - (\mathbf{M}[n_0] + \Omega(n_0, n_\alpha)) = \mathbf{M}[n_0] - \mathbf{M}[n_0] - \Omega(n_0, n_\alpha) = -\Omega(n_0, n_\alpha) = 1$. Suppose now that $n_\alpha \neq n_0$, $n_\beta \neq n_0$, and let $L[n_0, n_\alpha]$ be an arbitrary chain from n_0 to n_α. (Such a chain exists because G is connected. If the chain $L[n_0, n_\alpha]$ does not contain the node n_β, we consider the chain $L^*[n_0, n_\beta] = L \cup < n_\alpha, n_\beta >$, for which we have:

$$(1) \qquad N(L^*, n_0, n_\beta) = N(L, n_0, n_\alpha) + 1,$$

$$(2) \qquad N(L^*, n_\beta, n_0) = N(L, n_\alpha, n_0).$$

From (1) and (2) we obtain: $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = \mathbf{M}[n_0] + \Omega(n_0, n_\beta) - \mathbf{M}[n_0] - \Omega(n_0, n_\alpha) = \Omega(n_0, n_\beta) - \Omega(n_0, n_\alpha) = N(L^*, n_0, n_\beta) - N(L^*, n_\beta, n_0) - (N(L, n_0, n_\alpha) - N(L, n_\alpha, n_0)) = N(L, n_0, n_\alpha) + 1 - N(L, n_\alpha, n_0) - N(L, n_0, n_\alpha) + N(L, n_\alpha, n_0) = 1$.

If L contains the node n_β as a last but one node in L we consider the chain $L^{**}[n_0, n_\beta] = L - < n_\alpha, n_\beta >$, for which we have:

$$(3) \qquad N(L, n_0, n_\alpha) = N(L^{**}, n_0, n_\beta),$$

$$(4) \qquad N(L, n_\alpha, n_0) = N(L^{**}, n_\beta, n_0) + 1.$$

From (3) and (4) we obtain: $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = \mathbf{M}[n_0] + \Omega(n_0, n_\beta) - \mathbf{M}[n_0] - \Omega(n_0, n_\alpha) = \Omega(n_0, n_\beta) - \Omega(n_0, n_\alpha) = N(L^{**}, n_0, n_\beta) - N(L^{**}, n_\beta, n_0) - (N(L, n_0, n_\beta) - N(L, n_\alpha, n_0)) = N(L^{**}, n_0, n_\beta) - N(L^{**}, n_\beta, n_0) - N(L, n_0, n_\beta) + N(L, n_\beta, n_0) + 1 = 1$.

If L contains the node n_β (not necessarily the last but one in L) we consider the subchains $\tilde{L}[n_0, n_\beta]$ and $\bar{L}[n_\beta, n_\alpha]$ of L, so that $L = \tilde{L} \cup \bar{L}$.

We define the chain $L^{***}[n_0, n_\alpha] = \tilde{L} \cup < n_\alpha, n_\beta >$, for which we have:

$$(5) \qquad N(L^{***}, n_0, n_\alpha) = N(\tilde{L}, n_0, n_\beta),$$

$$(6) \qquad N(L^{***}, n_\alpha, n_0) = N(\tilde{L}, n_\beta, n_0) + 1.$$

From (5) and (6) we obtain: $\mathbf{M}[n_\beta] - \mathbf{M}[n_\alpha] = \mathbf{M}[n_0] + \Omega(n_0, n_\beta) - \mathbf{M}[n_0] - \Omega(n_0, n_\alpha) = \Omega(n_0, n_\beta) - \Omega(n_0, n_\alpha) = N(\bar{L}, n_0, n_\beta) - N(\bar{L}, n_\beta, n_0) - (N(L^{***}, n_0, n_\alpha) - N(L^{***}, n_\alpha, n_0)) = N(\tilde{L}, n_0, n_\beta) - N(\tilde{L}, n_\beta, n_0) - N(\bar{L}, n_0, n_\beta) + N(\bar{L}, n_\beta, n_0) + 1 = 1$.

Because $\mathbf{M}[n_\beta] > M[n_\alpha]$, G does not contain circuits (see [1]), hence, G is a gradable digraph.

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Received 30.XI.1978

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