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Faculté de Mathématiques
Université de Iassy
R.S. Roumanie

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DUALITY IN BEST APPROXIMATION PROBLEM

BY

TEODOR PRECUPANU

The best approximation problem in normed linear spaces may be regarded as a special optimization problem. Thus, many known properties of the best approximation problem appear as particular cases of some general results of optimization theory. For instance, it is the case for the duality properties of the convex best approximation for which the convex analysis offers strong methods of investigation. Using various results from the optimization theory, many authors have obtained interesting results in the abstract approximation theory or in the approximation theory for concret spaces of functions. A unitary treatment of the best approximation problem as an optimization problem can be found in the book of P.-J. Laurent [7].

In the present paper we intend to point out a relationship between the existence of the elements of best approximation by a closed set and the property that the sum of two closed sets is also closed. Consequently, we obtain different characterizations of the proximinal sets of a normed linear space.

§ 1. Some duality results. Let C be a non empty closed set of the normed linear space X. Let us consider the *problem of the best approximation*

$$(\mathcal{P}_u) \qquad \min_{x \in C} \| x - u \|$$

of an element $u \in X$ by the elements of the set C.

The solutions of this problem (if they exist) are called the *elements of the best approximation* of u by the elements of C. The problem $\mathcal{P}_u$ can be considered as a *problem without constraints*

$$\min_{x \in X} \{ \| x - u \| + I_C(x) \} \tag{1.1}$$

or as a *minimax problem*

$$\min_{x \in C} \sup_{x^* \in S^*} (x^*, u - x), \tag{1.2}$$

where I_A denotes the *indicator function* of a set A and

$$S^* = \{ x^* \in X^* ; \| x^* \| \leqslant 1 \}$$

is the closed unit ball of the dual X^* of the normed linear space X.

Thus we can naturaly define the *dual of the problem* $\mathcal{P}_u$ considering the *duality by conjugacy* for (1.1)

(1.3) $$\max_{x^* \in X^*} \{(x^*, u) - I_C^*(x^*) - I_{S^*}(x^*)\}$$

or *minimax duality* for (1.2)

(1.4) $$\max_{x^* \in S^*} \inf_{x \in C} (x^*, u - x).$$

Since we have

$$I_C^*(x^*) = \sup_{x \in C} (x^*, x), \quad x^* \in X^*,$$

it follows that the two dual problems (1.3) and (1.4) coincide. Consequently we can obtain duality results for the best approximation problem, using results of the duality theory by conjugacy or using results of the duality theory by minimax. Thus, if C is a nonempty closed convex set, applying *Fenchel Theorem* ([1], p. 189) we find

(1.5) $$\inf_{x \in C} \| x - u \| = \max_{x^* \in S^*} \{(x^*, u) - I_C^*(x^*)\}, \quad u \in X,$$

because $f_u(x) = \| x - u \|$, $x \in X$, is a real-valued convex function, continuous on X for every $u \in X$, $g(x) = - I_C(x)$, $x \in X$, is a concave function upper-semicontinuous on X and, as conjugate functions, we have $f^*(-x^*) = -(x^*, u) + I_{S^*}(x^*)$, $x^* \in X^*$, $g^*(-x^*) = - I_C^*(x^*)$. From (1.5) it follows that *the dual* (1.3) *or* (1.4) *of the best approximation problem* $\mathcal{P}_u$ *has always solution and the two problems have equal values for each nonempty closed convex set. Furthermore, an element* $\bar{x} \in C$ *is a best approximation to* $u \in X$ *from* C, *if and only if there exists* $\bar{x}^* \in S^*$ *such that*

(1.6) $$\| \bar{x} - u \| = (\bar{x}^*, u) - I_C^*(\bar{x}^*).$$

If C is not convex, $\mathcal{P}_u$ can have solutions while (1.6) does not hold for any element $\bar{x}^* \in S^*$. Generally we have

(1.7) $$(x^*, u) - I_C^*(x^*) \leqslant \| x - u \| \quad \textit{for every} \quad [x^*, x] \in S^* \times C,$$

since always the value of the dual problem by conjugacy does not exceed the values of the primal problem. Hence, in the convex case, the relation (1.6) is equivalent to

$$(\bar{x}^*, x) \leqslant (\bar{x}^*, u) - \| \bar{x} - u \| \quad \textit{for all} \quad x \in C,$$

or

(1.8) $$(\bar{x}^*, x - \bar{x}) \leqslant (\bar{x}^*, u - \bar{x}) - \| \bar{x} - u \| \quad \textit{for all} \quad x \in C.$$

If $u \in C$, then $\bar{x} = u$ and we can take $\bar{x}^* = 0$. If $u \notin C$, then $\bar{x}^* \neq 0$ and necessarily $\| \bar{x}^* \| = 1$. It is easy to see that the property (1.8) is equivalent to the following two properties:

(1.9) $$(\bar{x}^*, u - \bar{x}) = \| \bar{x} - u \|,$$

(1.10) $$(\bar{x}^*, x - \bar{x}) \leqslant 0, \quad \textit{for any} \quad x \in C.$$

From (1.10) we obtain that $\bar{x}^* \in (\bar{x} - C)^+$, where

$$A^+ = \{x^* \in X^*;\ (x^*, x) \geqslant 0 \quad \textit{for all} \quad x \in A\}.$$

If A is a convex cone with vertex at origin, then $A^+ = - A^0$ and therefore (1.10) says that *if* C *is a closed convex cone with vertex at* $\bar{x}$, *we have* $\bar{x}^* \in (C - \bar{x})^0 \cap S^*$.

It is known that if $0 \in C$, then the conjugate of the indicator function coincides with the Minkowski functional associated to the polar, i.e. $I_C^* = p_{C^0}$. In this case the dual problem is

(1.11) $$\max_{x^* \in S^* \cap C^0} \{(x^*, u) - p_{C^0}(x^*)\}.$$

A detailed study of this dual problem can be found in [9].

If C is a cone with vertex at origin, then $p_{C^0} = I_{C^0}$ and the dual problem becomes

(1.12) $$\max_{x^* \in S^* \cap C^0} (x^*, u),$$

which is a linear optimization problem with convex constraints. If C is a linear subspace the dual problem can again be simplified. We remark that all these known results of the duality theory can be deduced by minimax duality using *minimax theorems* for the convex-concave function

$$K(x, x^*) = (x^*, u - x), [x, x^*] \in C \times S^*$$

since S^* is w^*-compact.

The pairs of solutions for $\mathcal{P}_u$ and (1.4) are the saddle points of the function K.

Consequently $\bar{x} \in C$ *is a solution for* $\mathcal{P}_u$ *if and only if there exists* $\bar{x}^* \in S^*$ *such that*

(1.13) $$(x^*, u - \bar{x}) \leqslant (\bar{x}^*, u - \bar{x}) \leqslant (\bar{x}^*, u - x), \text{ for all } [x, x^*] \in C \times S^*.$$

In fact it is easy to see that (1.13) coincides with the two properties (1.9) and (1.10) with $\bar{x}^* \in S^*$.

§ 2. **Characterizations of proximinal sets.** A nonempty set C of X is called *proximinal* if every element of X has a best approximation in C. It is clear that a proximinal set is necessarily closed. If C is nonempty, closed and convex we obtain dual characterization for the proximinality of C if the properties given in the previous section hold for every $u \in X$.

Now we point out a simple characteristic property for proximinality given by the condition that a certain set of $X \times R$ is closed.

Theorem 1. *A nonempty set* C *of* X *is proximinal if and only if the sum of the epigraph of the norm and* $C \times \{0\}$ (*the natural embedding of the set* C *in* $X \times R$) *is closed in* $X \times R$.

Proof. We recall that

$$\operatorname{epi} \|\cdot\| = \{[v, \alpha] \in X \times R;\ \| v \| \leqslant \alpha\}.$$

Let us denote

(2.1) $H = \operatorname{epi} \|\cdot\| + C \times \{0\} = \{[v + x, \alpha] \in X \times R ; \|v\| \leqslant \alpha, x \in C\}$.

Suppose that H is closed and consider an arbitrary element $u \in X$.

Denote

$$d_C(u) = \inf_{x \in C} \|u - x\|. \tag{2.2}$$

There exists a sequence $(x_n) \subset C$ such that $\|u - x_n\| \leqslant d_C(u) + 1/n$, that is $[u, d_C(u) + 1/n] \in H$. Since H is closed we have $[u, d_C(u)] \in H$. From (2.1) it follows that there exists $\tilde{x} \in C$ such that $\|u - \tilde{x}\| \leqslant d_C(u)$, which says that $\tilde{x}$ is an element of the best approximation of u by the elements of C.

Conversely, if $\mathcal{P}_u$ has solution for every $u \in X$ and $[u_n, \alpha_n] \in H$, $n \in N$, such that $\lim_{n \to \infty} [u_n, \alpha_n] = [u_0, \alpha_0]$, then there exists $\tilde{x}_0 \in C$ such that $\|u_0 - \tilde{x}_0\| = d_C(u_0)$. For every $\varepsilon > 0$ there exists $n_0 \in N$ such that $\|u_{n_0} - u_0\| \leqslant \varepsilon/2$, $|\alpha_{n_0} - \alpha_0| \leqslant \varepsilon/2$. Since $[u_{n_0}, \alpha_{n_0}] \in H$ there exists $\tilde{x}_{n_0} \in C$ such that $\|u_{n_0} - \tilde{x}_{n_0}\| \leqslant \alpha_{n_0}$. We have $\|u_0 - \tilde{x}_0\| \leqslant \|u_0 - \tilde{x}_{n_0}\| \leqslant \|u_0 - u_{n_0}\| + \alpha_{n_0} \leqslant \alpha_0 + \varepsilon$ for every $\varepsilon > 0$, that is $\|u_0 - \tilde{x}_0\| \leqslant \alpha_0$ with $\tilde{x}_0 \in C$, from which we obtain $[u_0, \alpha_0] \in H$. Hence H is closed in $X \times R$. This completes the proof.

We remark that the epigraph of the norm coincides with the cone generated by $S(0; 1) \times \{1\}$ and hence

$$H = \{[\alpha v + x, \alpha] \in X \times R ;\ \alpha \geqslant 0,\ v \in S(0; 1),\ x \in C\} = \tag{2.3}$$
$$= \operatorname{con}(S(0; 1) \times \{1\}) + C \times \{0\}.$$

or

$$H = \bigcup_{\alpha \geqslant 0} \bigcup_{x \in C} S(x; \alpha) \times \{\alpha\}. \tag{2.4}$$

In the convex case Theorem 1 with H given by (2.4) was established in [10] as a special case of the global stability of a minimax problem. Also this result is related to the results of P. L e v i n e - J.-Ch. P o m e r o l [8].

Corollary. *For a Banach space X the following properties are equivalent.*

(i) *X is reflexive;*

(ii) *$\operatorname{epi} \|\cdot\| + (\ker x^*) \times \{0\}$ is closed in $X \times R$ for every $x^* \in S^*$;*

(iii) *$\operatorname{epi} \|\cdot\| + C \times \{0\}$ is closed in $X \times R$ for every closed convex set C in X.*

Proof. It is well known that a Banach space is reflexive if and only if every convex closed nonempty set is proximinal and hence (i) $\Leftrightarrow$ (iii), or, if and only if, the closed hyperplanes are proximinal that is (i) $\Leftrightarrow$ (ii) ([12], p. 99).

The properties (ii), (iii) are related to the characteristic property of reflexive spaces that the sum of two bounded closed convex sets is closed ([6], p. 322). But in our case the two sets can be unbounded since $\operatorname{epi} \|\cdot\|$ is a cone.

Remark. If C is convex, the norm topology can be replaced by one which is compatible with natural duality (X, X^*) (for example the weak

topology). In the nonconvex case supposing that the property from the hypotheses of Theorem 1 holds for a compatible topology we obtain only a sufficient condition for proximinality.

By virtue of Theorem 1 the property that a set is proximinal can be reduced to the property that the sum of two closed sets is a closed set. A general criterion along these lines was established by J. D i e u d o n n é [5]: *If the asymptotic cones of two convex closed sets of a separated locally convex space have in common only the origin and in addition one of the sets is locally compact then the difference of the two sets is a closed set.*

In our case to apply this criterion we must ask that C is convex. Then it follows that H is convex and hence H is closed if and only if H is weak closed. We recall that the *asymptotic cone* of a convex closed set A [2] is defined by

$$A_\infty = \bigcap_{\lambda > 0} \lambda (A - a) \tag{2.5}$$

where a is an element of the set A.

We can show that A_∞ is independent on the element a arbitrarily chosen in A [2]. It is easy to see that the asymptotic cones of the sets $\operatorname{epi} \|\cdot\|$ and $(-C) \times \{0\}$ have in common only the origin of $X \times R$. But $\operatorname{epi} \|\cdot\|$ is weakly locally compact if and only if $S(0; 1)$ is weak compact, that is X is reflexive. Thus we find again the known result that *every convex closed set of a reflexive space is proximinal*. We also see that from the *Dieudonné's criterion* it follows that *weakly locally compact convex closed sets are proximinal*. But this result can be directly obtained if we observe that the best approximation problem of an element $u \in X$ by elements of a set C is equivalent to the best approximation problem by the elements of $C \cap S(u; d_C(u) + \varepsilon)$, with $\varepsilon > 0$, which is weakly compact if C is weakly locally compact because it contains no half-line ([6], p. 340); it is obvious that, for weakly compact sets, $\mathcal{P}_u$ has solutions.

Consequently, the application of Dieudonné's criterion leads to the refinding of some known sufficient conditions for the proximinality of convex closed sets. Recently, J. P. D e d i e u [3], [4] has shown that in Dieudonné's criterion the conditions of convexity and of locally compacity may be relaxed.

For an arbitrary closed set A, the *asymptotic cone* is given by

$$A_\infty = \bigcap_{\varepsilon > 0} \overline{]0, \varepsilon]\, A}. \tag{2.6}$$

If there exist $a \in A$ and $\varepsilon \in]0, 1]$ such that $]0, \varepsilon]\,(A - a) \subset A - a$ (in particular if A is convex) the definition considered here coincides with the definition given by (2.5). A set A is called *asymptotically compact* if there exist $\varepsilon > 0$ and a neighborhood V of the origin such that $(]0, \varepsilon]\, A) \cap V$ is relatively compact. For instance the locally compact convex closed sets and the locally compact closed cones are asymptoticaly compact [4].

Theorem 2 [4]. *If the asymptotic cones of two closed sets have in common only the origin and in addition one of them is asymptotically compact, then their difference is a closed set.*

From this theorem and Theorem 1 we easily obtain

Theorem 3. *Every asymptotically compact closed set is proximinal.*

Corollary. *The locally compact closed cones are proximinal.*

Remark. Since the weakly closed sets are closed, the results remain valid if we consider topologies on X compatible with the natural duality (X, X^*). Also, as it follows from [4], *every weakly closed set with the interior of the polar nonempty with respect to Mackey topology is proximinal.*

Now, let us remark that the set H may be considered as the projection of the epigraph of the function $F_C : X \times X \to \overline{R}$ defined by

$$F_C(x, u) = \| u - x \| + I_C(x), \quad [x, u] \in X \times X. \tag{2.7}$$

Indeed

$$\text{epi } F_C = \{[x, u, \alpha] \in X \times X \times R ; \| u - x \| \leqslant \alpha, \ x \in C\}.$$

If we denote by $\text{Proj}_{2,3}$ the projection of the space $X \times (X \times R)$ on $X \times R$ we see that

$$H = \text{Proj}_{2,3} (\text{epi } F_C). \tag{2.8}$$

Since C is closed it follows that I_C is lower-semicontinuous and hence epi F_C is closed. Therefore, by virtue of Theorem 1, *a nonempty closed set C is proximinal if and only if the projection on $X \times R$ of the epigraph of the function F_C defined by (2.7), is a closed set.*

P. Levine-J. Ch. Pomerol [8] have obtained several sufficient conditions that the projection of weakly closed convex set of a dual space is weakly closed. Thus we can easily obtain different criteria for the proximinality of a convex set of X^*.

§3. Dual characterizations of the proximinal convex sets.

Theorem 4. *A nonempty convex set C of X is proximinal if and only if for every $[x_0, \alpha_0] \in X \times R_+$ with $C \cap S(x_0 ; \alpha_0) = \varnothing$ there exists $\tilde{x}_0^* \in X^*$ such that*

$$\inf_{[u,x] \in X \times C} \{(\tilde{x}_0^*, x_0 - u) + \| u - x \|\} > \alpha_0 . \tag{3.1}$$

Proof. It is clear that the relation (3.1) can be equivalently written as

$$(\tilde{x}_0^*, x_0) - \sup_{u \in X} \{(\tilde{x}_0^*, u) - d_C(u)\} > \alpha_0 ,$$

from which, according to the definition of the conjugate of function d_C, we obtain

$$(\tilde{x}_0^*, x_0) - d_C^*(\tilde{x}_0^*) > \alpha_0 . \tag{3.2}$$

If C is proximinal and $C \cap S(x_0 ; \alpha_0) = \varnothing$, that is $\| x_0 - x \| > \alpha_0$ for every $x \in C$, we also have $d_C(x_0) > \alpha_0$. On the other hand, since the set C is convex and closed, the bifunction F_C defined by (2.5) is properly convex and lower-semicontinuous on $X \times X$. Hence d_C is convex and lower-semicontinuous on X [11]. It follows that $d_C = d_C^{**}$. According to the definition of biconjugate, if we take $\varepsilon > 0$ such that $0 < \varepsilon < d_C(x_0) - \alpha_0$ there exists $\tilde{x}_0^* \in X^*$ such that

$$(\tilde{x}_0^*, x_0) - d_C^*(\tilde{x}_0^*) > d_C^{**}(x_0) - \varepsilon = d_C(x_0) - \varepsilon > \alpha_0 .$$

Hence (3.2) is proved, i.e. the inequality (3.1) holds.

Conversely, if C is not proximinal, there exists $x_0 \in X$ such that $d_C(x_0) < \| x_0 - x \|$ for every $x \in C$. Hence $C \cap S(x_0 ; d_C(x_0)) = \varnothing$ and consequently for $\alpha_0 = d_C(x_0)$ there exists $\tilde{x}_0^* \in X^*$ such that $(\tilde{x}_0^*, x_0) - d_C^*(\tilde{x}_0^*) > d_C(x_0)$ which contradicts the definition of the conjugate.

Theorem 5. *A convex set which contains the origin is proximinal if and only if $C \cap S(u, \alpha) \neq \varnothing$ whenever*

$$\sup_{x^* \in C^0} \{(x^*, u) - \alpha \| x^* \|\} \leqslant 1. \tag{3.3}$$

Proof. By virtue of the polar definition of a set we have

$$H^0 = \{[x^*, \beta] \in X^* \times R ; (x^*, u) + \alpha \beta \leqslant 1 \text{ for every } [u, \alpha] \in H\} =$$
$$= \{[x^*, \beta] \in X^* \times R ; \alpha[(x^*, v) + \beta] + (x^*, x) \leqslant 1 \text{ for all } \alpha \geqslant 0,$$
$$x \in C, \ v \in S(0 ; 1)\} = \{[x^*, \beta] \in X^* \times R ; \| x^* \| \leqslant -\beta, \ x^* \in C^0\}.$$

Hence $[u, \alpha] \in H^{00}$ if and only if

$$(x^*, u) + \alpha \beta \leqslant 1 \text{ for every } [x, \beta] \in X^* \times R^- \text{ with } x^* \in C^0 \cap S^*(0 ; -\beta). \tag{3.4}$$

It is clear that $\alpha \geqslant 0$ and then the property (3.4) is equivalent to

$$(x^*, u) - \alpha \| x^* \| \leqslant 1 \text{ for all } x^* \in C^0$$

which is just (3.3).

According to Theorem 1 the set C is proximinal if and only if H is closed. But H contains the origin and it is convex and then from the bipolars theorem we have that H is closed if and only if $H = H^{00}$, which is equivalent to $H \supset H^{00}$.

Since $C \cap S(u ; \alpha) \neq \varnothing$ if and only if $[u, \alpha] \in H$ the theorem is completely showed.

We easily see that Theorem 4 may be established also taking into account that the convex set H is closed if and only if every point of the complement of H can be strongly separated by a closed hyperplane by H.

We also observe that (3.1) is equivalent to

$$\| \tilde{x}_0^* \| \leqslant 1 \text{ and } (\tilde{x}_0^*, x_0) - I_C^*(\tilde{x}_0^*) > \alpha_0 \tag{3.5}$$

obtained replacing $u - x$ by u. On the other hand, (3.3) may be written as

$$\sup_{x^* \in C^0} \inf_{v \in S(0; 1)} \{(x^*, u + \alpha v)\} \leqslant 1. \tag{3.6}$$

In the case of the cones, since $I_C^* = I_{C^0}$, the theorems 4 and 5 can be simplified.

Theorem 6. *A convex cone C with vertex at origin is proximinal if and only if it satisfies one of the following equivalent properties:*

(i) *for every $[x_0, \alpha_0] \in X \times R_+ \setminus H$ there exists $\tilde{x}_0^* \in C^0 \cap S^*$ such that $(\tilde{x}_0^*, x_0) > \alpha_0$;*

(ii) *if*

$$\sup_{x^* \in C^0 \cap S^*} \{(x^*, x_0) - \| x^* \|\} \leqslant 0,$$

then $(x_0, 1) \in H$.

Proof. By virtue of Theorem 4, taking into account (3.5), the proximinality of the convex cone with the vertex at origin is equivalent to (i). From Theorem 5, since H is also a cone and

$$H^{00} = \{[x, \alpha] \in X \times R;\ (x^*, x) + \alpha\beta \leqslant 0 \text{ for all } [x^*, \beta] \in H^0\},$$

it follows the equivalence to (ii). On the other hand $C \cap S(u; \alpha) \neq \emptyset$ with $\alpha > 0$ if and only if $C \cap S(1/\alpha u; 1) \neq \emptyset$.

In the case C a linear subspace of X, the polar set C^0 becomes $C^{\perp} = \{x^* \in X^*;\ (x^*, x) = 0 \text{ for all } x \in C\}$.

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Faculty of Mathematics
University of Iasi
6600 Iasi, R.S. Romania

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FUZZY PROXIMITIES AND FUZZY COMPLETELY REGULAR SPACES I

BY

A. K. KATSARAS

1. **Introduction**. In [8] the author gave the notion of a fuzzy proximity on a set X. In this paper we will change one of the axioms of the definition of a fuzzy proximity. One of the advantages of this change will be that the corresponding fuzzy topology will not necessarily consist of ordinary subsets of X as it happens for a proximity in the sense of [8]. Also, every fuzzy proximity in the sense of [8] will also be a fuzzy proximity in the new sense.

Section 2 or this paper is preliminary. In section 3 we give the definition of a fuzzy proximity δ on a set X and we prove that to δ corresponds a fuzzy topology. In section 4 we define the fuzzy completely regular spaces. We show that, if G is a family of functions from X into the fuzzy unit interval, then the coarsest fuzzy topology τ on X with respect to which each member of G is continuous is completely regular. Also, every fuzzy completely regular topology on X is obtained in this way. A T_1 fuzzy topological space X is completely regular iff X is homeomorphic to a subspace of some fuzzy cube.

Finally in section 5 we examine the question of when a given fuzzy topology τ is compatible with some fuzzy proximity. We show that there exists a fuzzy proximity compatible with τ iff τ is completely regular.

2. **Preliminaries**. In this section we will recall some of the definitions related to fuzzy sets and fuzzy topological spaces.

Let X be a set and I the unit interval. A fuzzy set in X is an element of the set I^X of all functions μ from X into I. If f is a function from X to Y and $\mu \in I^Y$, then $f^{-1}(\mu)$ is the element of I^X which is defined by $f^{-1}(\mu)(x) = \mu(f(x))$. Also, for $\sigma \in I^X$, $f(\sigma)$ is the member of I^Y defined by $f(\sigma)(y) = 0$, if $y \notin f(X)$ and $f(\sigma)(y) = \sup\limits_{x \in f^{-1}(y)} \sigma(x)$ if $y \in f(X)$. A fuzzy topology on X is a subset τ of I^X such that

(i) $0, 1 \in \tau$,
(ii) If $\mu, \rho \in \tau$, then $\mu \wedge \rho \in \tau$,
(iii) If $\mu_i \in \tau$ for each $i \in A$, then $\sup\limits_{i \in A} \mu_i \in \tau$.

If τ is a fuzzy topology on X, then the pair (X, τ) is called a fuzzy topological space and the members of τ are the open fuzzy sets of this topo-