Proof. By virtue of Theorem 4, taking into account (3.5), the proximinality of the convex cone with the vertex at origin is equivalent to (i). From Theorem 5, since H is also a cone and

$$H^{00} = \{[x, \alpha] \in X \times R\,;\ (x^*, x) + \alpha\beta \leqslant 0 \text{ for all } [x^*, \beta] \in H^0\},$$

it follows the equivalence to (ii). On the other hand $C \cap S(u\,;\,\alpha) \neq \varnothing$ with $\alpha > 0$ if and only if $C \cap S(1/\alpha\, u\,;\,1) \neq \varnothing$.

In the case C a linear subspace of X, the polar set C° becomes $C^\perp = \{x^* \in X^*\,;\ (x^*, x) = 0 \text{ for all } x \in C\}$.

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Faculty of Mathematics
University of Iasi
6600 Iasi, R.S. Romania

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FUZZY PROXIMITIES AND FUZZY COMPLETELY REGULAR SPACES I

BY

A. K. KATSARAS

1. **Introduction**. In [8] the author gave the notion of a fuzzy proximity on a set X. In this paper we will change one of the axioms of the definition of a fuzzy proximity. One of the advantages of this change will be that the corresponding fuzzy topology will not necessarily consist of ordinary subsets of X as it happens for a proximity in the sense of [8]. Also, every fuzzy proximity in the sense of [8] will also be a fuzzy proximity in the new sense.

Section 2 of this paper is preliminary. In section 3 we give the definition of a fuzzy proximity δ on a set X and we prove that to δ corresponds a fuzzy topology. In section 4 we define the fuzzy completely regular spaces. We show that, if G is a family of functions from X into the fuzzy unit interval, then the coarsest fuzzy topology τ on X with respect to which each member of G is continuous is completely regular. Also, every fuzzy completely regular topology on X is obtained in this way. A T_1 fuzzy topological space X is completely regular iff X is homeomorphic to a subspace of some fuzzy cube.

Finally in section 5 we examine the question of when a given fuzzy topology τ is compatible with some fuzzy proximity. We show that there exists a fuzzy proximity compatible with τ iff τ is completely regular.

2. **Preliminaries**. In this section we will recall some of the definitions related to fuzzy sets and fuzzy topological spaces.

Let X be a set and I the unit interval. A fuzzy set in X is an element of the set I^X of all functions μ from X into I. If f is a function from X to Y and $\mu \in I^Y$, then $f^{-1}(\mu)$ is the element of I^X which is defined by $f^{-1}(\mu)(x) = \mu(f(x))$. Also, for $\sigma \in I^X$, $f(\sigma)$ is the member of I^Y defined by $f(\sigma)(y) = 0$, if $y \notin f(X)$ and $f(\sigma)(y) = \sup\limits_{x \in f^{-1}(y)} \sigma(x)$ if $y \in f(X)$. A fuzzy topology on X is a subset τ of I^X such that

(i) $0, 1 \in \tau$,

(ii) If $\mu, \rho \in \tau$, then $\mu \wedge \rho \in \tau$.

(iii) If $\mu_i \in \tau$ for each $i \in A$, then $\sup\limits_{i \in A} \mu_i \in \tau$.

If τ is a fuzzy topology on X, then the pair (X, τ) is called a fuzzy topological space and the members of τ are the open fuzzy sets of this topological space. A fuzzy set μ is called closed if $1-\mu$ is open. A subfamily α of τ is called:

a) a base for τ if every member of τ is a union of members of α;

b) a subbase for τ if the family of all finite intersections of members of α is a base for τ.

A closure operator on I^X is a map $\mu \to \bar{\mu}$, from I^X to I^X, such that for all μ, ρ in I^X we have

(1) $\mu \leqq \bar{\mu}$, (3) $\overline{\mu \vee \rho} = \bar{\mu} \vee \bar{\rho}$

(2) $\bar{\bar{\mu}} = \bar{\mu}$, (4) $\bar{0} = 0$.

Given a closure operator on I^X, the collection $\{\mu \in I^X : \overline{1-\mu} = 1-\mu\}$ is a fuzzy topology on X.

Next we recall the definition of the fuzzy unit interval which was given in [6]. Let J_φ denote the set of all increasing functions $\alpha : [0,1] \to [0,1]$. On J_φ we consider the equivalence relation defined by $\alpha_1 \sim \alpha_2$ iff $\alpha_1(t+) = \alpha_2(t+)$ and $\alpha_1(t-) = \alpha_2(t-)$ for all $t \in [0,1]$. The fuzzy unit interval is the set $I_\varphi = [0,1]_\varphi$ of all equivalence classes in which J_φ is partioned by $\sim$. For $\alpha \in J_\varphi$, we will denote also by α the equivalence class to which α belongs. For each $t \in (0,1)$ we define L_t and R_t on I_φ by

$$L_t(\alpha) = 1-\alpha(t-),\quad R_t(\alpha) = \alpha(t+).$$

The fuzzy topology on I_φ generated by the family $\{L_t, R_t : t \in (0,1)\}$ is called the usual fuzzy topology on I_φ. Throughout the paper, we will consider on I_φ the usual fuzzy topology unless it is stated explicitly otherwise.

By the usual order on I_φ we will mean the partial order $\geqq$ defined by

$$\alpha_1 \geqslant \alpha_2 \text{ iff } \alpha_1(t+) \geqq \alpha_2(t+) \text{ and } \alpha_1(t-) \geqq \alpha_2(t-) \text{ for all } t.$$

3. **Fuzzy Proximity Spaces. Definition 3.1.** *A binary relation δ on I^X is called a fuzzy proximity on X if δ satisfies the following axioms:*

(FP 1) *$\mu \,\delta\, \rho$ implies $\rho \,\delta\, \mu$.*

(FP 2) *$(\mu \vee \rho)\,\delta\,\sigma$ iff $\mu\,\delta\,\sigma$ or $\rho\,\delta\,\sigma$,*

(FP 3) *$\mu\,\delta\,\rho$ implies $\mu \neq 0$ and $\rho \neq 0$,*

(FP 4) *$\mu\,\delta\,\rho$ implies $\mu \leqq 1-\rho$.*

(FP 5) *$\mu\,\delta\,\rho$ implies that there exists $\sigma \in I^X$ such that $\mu\,\delta\,\sigma$ and $(1-\sigma)\,\rceil\delta\,\rho$. The pair (X, δ) is called a fuzzy proximity space.*

Example 3.2. If we defined δ on I^X by $\mu\,\rceil\delta\,\rho$ iff $\mu \leqq 1-\rho$, then μ is a fuzzy proximity on X. We will refer to this fuzzy proximity as the *discrete* fuzzy proximity on X.

We omit the proof of the following easily established

Lemma 3.3. *Let δ be a fuzzy proximity on X. Then:*

1) *If $\mu\,\delta\,\rho$, $\mu_1 \geqq \mu$, $\rho_1 \geqq \rho$, then $\mu\,\delta\,\rho_1$.*

2) *If there exists an $x \in X$ with $\mu(x) > 1/2$, then $\mu\,\delta\,\mu$.*

3 *If $\mu \neq 0$, then $\mu\,\delta\,1$.*

Definition 3.4. *Let δ be a fuzzy proximity on X. For $\mu \in I^X$ we define $\bar{\mu} = \bar{\mu}_\delta$ by $\bar{\mu} = 1 - \sup\{\rho \in I^X : \rho\,\rceil\delta\,\mu\}$.*

The proof of the next Theorem is analogous to the proof of Theorem 3.3 in [8] and we omit it.

Theorem 3.5. *Let (X, δ) be a fuzzy proximity space. Then the map $\mu \to \bar{\mu} = \bar{\mu}_\delta$ is a closure operator on I^X. Therefore the collection $\tau(\delta) = \{\mu : \overline{1-\mu} = 1-\mu\}$ is a fuzzy topology on X.*

We will refer to the fuzzy topology $\tau(\delta)$ as the fuzzy topology which corresponds to the fuzzy proximity δ. If δ is the discrete fuzzy proximity on X, then $\bar{\mu}_\delta = \mu$ and thus the corresponding fuzzy topology is the discrete fuzzy topology on X.

If X is a fuzzy topological space and μ a fuzzy set in X, we will denote by μ^0 the interior of μ and by $\bar{\mu}$ the closure of μ. If the topology τ of X is the fuzzy topology which corresponds to some fuzzy proximity δ on X, then $\bar{\mu} = \bar{\mu}_\delta$.

Proposition 3.6. *Let δ be a fuzzy proximity on X and $\mu, \rho \in I^X$ with $\mu\,\rceil\delta\,\rho$. Then:*

a) $\bar{\mu} \leqq 1-\rho$, b) $\mu \leqq (1-\rho)^0$.

Proof. (a) It is obvious.

b) *$\mu\,\rceil\delta\,\rho$ implies that $\mu\,\rceil\delta\,\bar{\rho}$ and thus $\mu \leqq 1-\bar{\rho} \leqq 1-\rho$. Since $1-\bar{\rho}$ is open contained in $1-\rho$, the result follows.*

Example 3.7. For μ, $\rho \in I^X$, we define $\mu\,\delta\,\rho$ iff $\mu \neq 0$ and $\rho \neq 0$. It is easy to see that δ is a fuzzy proximity on X. We will call this proximity the indiscrete fuzzy proximity on X. For this δ we have

$$\bar{\mu}_\delta = \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } \mu \neq 0. \end{cases}$$

Thus $\tau(\delta) = \{0, 1\}$ is the indiscrete fuzzy topology on X.

4. **Completely Regular Fuzzy Topological Spaces. Definition 4.1.** *A fuzzy topological space X is called completely regular if for each open fuzzy set μ in X and every $x \in X$ the following condition is satisfied: If $\mu(x) > \alpha$, then there exists a continuous function f from X to the fuzzy unit interval I_φ such that $f(y)(0+) \leqq \mu(y)$ for each $y \in X$ and $f(x)(1-) > \alpha$.*

Proposition 4.2. *If $(X_\alpha)_{\alpha \in A}$ is a family of completely regular fuzzy topological spaces, then $X = \prod_{\alpha \in A} X_\alpha$ with the product fuzzy topology is completely regular.*

Proof. Let μ be an open fuzzy set in X and let $x = (x_\alpha) \in X$ with $\mu(x) > \beta$. There are $\alpha_1, \ldots, \alpha_n \in A$ and open fuzzy sets $\mu_1, \ldots, \mu_n$ in $X_{\alpha_1}, \ldots, X_{\alpha_n}$ respectively with $\rho \leqq \mu$ and $\rho(x) > \beta$ where ρ is defined on X by $\rho(y) = \min\{\mu_1(y_{\alpha_1}), \ldots, \mu_n(y_{\alpha_n})\}$, $y = (y_\alpha) \in X$. For each i, $i = 1, \ldots, n$, there exists $f_i : X_{\alpha_i} \to I_\varphi$ continuous, such that $f_i(x_{\alpha_i})(1-) > \beta$ and $f_i(z)(0+) \leqq \mu_i(z)$ for all $z \in X_{\alpha_i}$. Let $\pi_{\alpha_i} : X \to X_{\alpha_i}$ be the canonical projection and $f = \inf\{f_i \circ \pi_{\alpha_i} : i = 1, \ldots, n\}$. Then f is a continuous function from X to I_φ and satisfies the inequalities

$$f(y)(0+) = \inf\{f_i(y_{\alpha_i})(0+) : i = 1, \ldots, n\} \leqq$$
$$\inf\{\mu_i(y_{\alpha_i}) : i = 1, \ldots, n\} \leqq \mu(y),$$
$$f(x)(1-) = \inf\{f_i(x_{\alpha_i}) : i = 1, \ldots, n\} > \beta.$$

This completes the proof.

Lemma 4.3. *Let X be a fuzzy topological space and let B be a subbase for its topology. Then X is completely regular iff the following condition is satisfied: Given $\mu \in \mathrm{B}$, $x \in X$ and θ a real number with $\mu(x) > \theta > 0$, there exists a continuous function f from X into the fuzzy unit interval, such that $f(x)(1-) > 0$ and $f(y)(0+) \leqq \mu(y)$ for all y in X.*

Proof. Suppose that the condition is satisfied and let μ be open in X and $x \in X$ with $\mu(x) > \theta > 0$. There are $\mu_1, \dots, \mu_n \in \mathrm{B}$ such that $\mu_1 \wedge \dots \wedge \mu_n \leqq \mu$ and $\min \{\mu_i(x) : i = 1, \dots, n\} > \theta$. For each i, there exists $f_i : X \to I_\varphi$ continuous such that $f_i(x)(1-) > \theta$ and $f_i(y)(0+) \leqq \mu_i(y)$ for all y. Let $f : X \to I_\varphi$ be defined by

$$f(y)(s) = \min \{f_i(y)(s) : i = 1, \dots, n\}, \quad y \in X, \ s \in [0, 1].$$

Then f is continuous. In fact, for $t \in (0, 1)$, we have

$$f^{-1}(L_t)(y) = 1 - f(y)(t-) = 1 - \min \{f_i(y)(t-) ; i = 1, \dots, n\} =$$
$$= \max \{1 - f_i(y)(t-) : i = 1, \dots, n\} = [f_1^{-1}(L_t) \vee \dots \vee f_n^{-1}(L_t)](y),$$

which proves that $f^{-1}(L_t)$ is open in X. Also

$$f^{-1}(R_t)(y) = [f_1^{-1}(R_t) \wedge \dots \wedge f_n^{-1}(R_t)](y)$$

and so $f^{-1}(R_t)$ is open. Hence f is continuous.

Moreover, $f(x)(1-) = \min \{f_i(x)(1-) : i = 1, \dots, n\} > \theta$ and

$$f(y)(0+) = \min \{f_i(y)(0+) : i = 1, \dots, n\} \leqq \min \{\mu_i(y) : i = 1, \dots, n\} \leqq \mu(y).$$

This completes the proof.

Theorem 4.4. *Let X be a fuzzy topological space and $G = C(X, I_\varphi)$ the family of all continuous functions from X into the fuzzy unit interval I_φ. Then X is completely regular iff the fuzzy topology τ of X coincides with the coarsest fuzzy topology on X for which each member of G is continuous.*

Proof. Suppose that X is completely regular and let τ_1 be a fuzzy topology on X with respect to which each member of G is continuous. Let $\mu \in \tau$. To show that μ is in τ_1, it suffices to show that if $x \in X$ and if $\mu(x) > \theta > 0$, then there exists $\rho \in \tau_1$ with $\rho \leqq \mu$ and $\rho(x) > \theta$. So let $\mu(x) > \theta > 0$. Since X is completely regular, there exists $f \in G$ with $f(x)(1-) > \theta$ and $f(y)(0+) \leqq \mu(y)$ for each y in X. By hypothesis f is τ_1-continuous and hence $\rho = f^{-1}(R_{1/2}) \in \tau_1$.

Moreover $\rho(x) = f(x)(1/2+) \geqq f(x)(1-) > \theta$, and

$$\rho(y) = f(y)(1/2+) \leqq f(y)(0+) \leqq \mu(y),$$

for each y. This proves that $\mu \in \tau_1$ and hence τ is coarser than τ_1.

Conversely, assume that τ coincides with the coarsest fuzzy topology on X for which each member of G is continuous. Let
$\mathrm{B} = \{f^{-1}(L_t) : t \in (0, 1), f \in G\} \cup \{f^{-1}(R_t) : t \in (0, 1), f \in G\} \cup \{0, 1\}$.
Then B is a subbase for τ. Let $f \in G$. Define $g : X \to I_\varphi$ by $g(y)(s) = 1 - f(y)(1-s)$. Then g is continuous since, for $t \in (0, 1)$, $g^{-1}(L_t) = f^{-1}(R_{1-t})$ and $g^{-1}(R_t) = f^{-1}(L_{1-t})$.

It follows that

$$\mathrm{B} = \{f^{-1}(L_t) : t \in (0, 1), f \in G\} \cup \{0, 1\}.$$

Let now $\mu \in B$ and $x \in X$ with $\mu(x) > \theta > 0$. If $\mu \neq 1$, there exists $f \in G$ and $t_0 \in (0, 1)$ with $\mu = f^{-1}(R_{t_0})$. For each $y \in X$ we define

$$g(y) : [0, 1] \to [0, 1], \ g(y)(s) = f(y)(t_0 + s(1 - t_0)).$$

The function $g : X \to I_\varphi$ is continuous. In fact

$$g^{-1}(L_t)(y) = 1 - g(y)(t-) = 1 - f(y)(t_0 + t(1 - t_0)-) = f^{-1}(L_{t_0 + t(1-t_0)})$$

and $g^{-1}(R_t) = f^{-1}(R_{t_0 + t(1-t_0)})$. Also $g(y)(0+) = f(y)(t_0+) = \mu(y)$. Since

$$\theta < \mu(x) = g(x)(0+) = \sup_{t>0} g(x)(t),$$

there exists $t_1 > 0$ with $g(x)(t_1-) > \theta$. Define $h : X \to I_\varphi$ by $h(y)(s) = g(y)(st_1)$. Then h is continuous since $h^{-1}(L_t) = g^{-1}(L_{tt_1})$ and $h^{-1}(R_t) = g^{-1}(R_{tt_1})$ for all $t \in (0, 1)$. Also $h(y)(0+) = g(y)(0+) = \mu(y)$ and $h(x)(1-) = g(x)(t_1-) > \theta$. If $\mu = 1$, we take $h : X \to I_\varphi$, $h(x)(s) = 1$ for all $x \in X$, $s \in [0, 1]$. We proved that if $\mu \in \mathrm{B}$ and $x \in X$ with $\mu(x) > \theta > 0$, there exists a continuous function $h : X \to I_\varphi$ such that $h(x)(1-) > \theta$ and $h(y)(0+) = \mu(y)$ for all $y \in X$. In view of Lemma 4.3, X is completely regular and the result follows.

The preceding Theorem gives us a way to construct completely regular fuzzy topologies on a given set X as the following Proposition shows.

Proposition 4.5. *Let X be a set and H a family of functions from X into the fuzzy unit interval I_φ. Let τ denote the coarsest fuzzy topology on X, with respect to which, each member of H is continuous. Then (X, τ) is completely regular.*

Proof. Let $G = C(X, I_\varphi)$ and let τ_1 denote the coarsest fuzzy topology on X with respect to which each member of G is continuous. It is clear that $\tau = \tau_1$ and hence τ is completely regular by the preceding Theorem.

Proposition 4.6. *The fuzzy unit interval is a completely regular space.*

Proof. The family $\mathrm{B} = \{L_t : t \in (0, 1)\} \cup \{R_t : t \in (0, 1)\} \cup \{0, 1\}$ is a subbase for the topology of I_φ. We will prove the result using Lemma 4.3. Let $\mu \in \mathrm{B}$, $\alpha \in I_\varphi$ and $\theta > 0$, such that $\mu(\alpha) > \theta$.

Case I. $\mu = L_{t_0}$. Since $\mu(\alpha) = 1 - \alpha(t_0-) > \theta$, there exists $0 < t_1 < t_0$ such that $1 - \alpha(t_1+) > \theta$. For each $\beta \in I_\varphi$, define $f(\beta) : [0, 1] \to [0, 1]$ by $f(\beta)(s) = 1 - \beta(t_0 - \gamma s)$ where $\gamma = t_0 - t_1$. Then $f(\beta) \in I_\varphi$. The map $f : I_\varphi \to I_\varphi$ is continuous since

$$f^{-1}(L_s) = R_{t_0 - \gamma s} \text{ and } f^{-1}(R_s) = L_{t_0 - \gamma s},$$

for each $s \in (0, 1)$. Also $f(\beta)(0+) = 1 - \beta(t_0-) = \mu(\beta)$, for each $\beta \in I_\varphi$, and $f(\alpha)(1-) = 1 - \alpha(t_1+) > \theta$.

Case II. $\mu = R_{t_0}$. Since $\mu(\alpha) = \alpha(t_0+) > \theta$, there exists $t_0 < t_1 < 1$, such that $\alpha(t_1-) > \theta$. Define: $I_\varphi \to I_\varphi$ by $f(\beta)(s) = \beta(t_0 + \gamma s)$ where $\gamma = t_1 - t_0$. Then f is continuous, $f(\alpha)(1-) = \alpha(t_1-) > \theta$ and $f(\beta)(0+) = \beta(t_0+) = \mu(\beta)$ for all $\beta \in I_\varphi$.

Case III. $\mu = 1$. In this case we take $f : I_\varphi \to I_\varphi$, $f(\beta)(s) = 1$. Clearly f is continuous, $f(\alpha)(1-) = 1 > \theta$ and $f(\beta)(0+) = \mu(\beta)$ for each $\beta \in I_\varphi$.

Thus the result follows from Lemma 4.3.

Recall that a fuzzy topological space X is called a T_1-space [10], if for all $x \neq y$ in X there are open fuzzy sets μ, ρ in X with $\mu(x) > 0$, $\rho(y) > 0$ and $\mu(y) = \rho(x) = 0$.

Definition 4.7. *A fuzzy topological space X is called a Tychonoff space if X is completely regular and T_1.*

Proposition 4.8. *If X is a Tychonoff fuzzy topological space, then the family $C(X, I_\varphi)$ separates the points of X.*

Proof. Let x, y be two distinct points of X. There is an open fuzzy set μ in X with $\mu(x) > 0$ and $\mu(y) = 0$. Let θ be a real number with $0 < \theta < \mu(x)$. There exists a continuous function f from X into I_φ such that $f(x)(1-) > \theta$ and $f(z)(0+) \leq \mu(z)$ for each $z \in X$. Now

$$f(y)(1-) \leq f(y)(0+) \leq \mu(y) = 0 \text{ and so } f(y)(1-) = 0.$$

It follows that $f(x) \neq f(y)$.

We omit the proof of the following easily established

Proposition 4.9. 1) *Let X be a fuzzy topological space and let Y be a subspace of X. Then:*

a) *If X is completely regular, so is Y.*

b) *If X is T_1, then the same is true for the space Y.*

2) *The product of a family of T_1 fuzzy topological spaces is a T_1 space.*

3) *Let X, Y be two homeomorphic fuzzy topological spaces. Then:*

a) *If X is completely regular, so is Y.*

b) *If X is T_1, the same holds for Y.*

Definition 4.10. *The cartesian product of a number of copies of the fuzzy unit interval, with the product fuzzy topology, is called a fuzzy cube.*

Theorem 4.11. *Let X be a fuzzy topological space and let $G = C(X, I_\varphi)$ be the family of all continuous functions from X to the fuzzy unit interval I_φ. For each $f \in G$, let Y_f denote the space I_φ. Let $Y = \prod_{f \in G} Y_f$ with the product fuzzy topology. If X is a Tychonoff space, then X is homeomorphic to a subspace of Y. More precisely, the map $e : X \to Y$, $e(x) = \hat{x}$, $\hat{x}_f = f(x)$, is a homeomorphism from X onto $e(X)$ when X is a Tychonoff space.*

Proof. Suppose that X is a Tychonoff space and consider the evaluation map

$$e : X \to Y, \quad x \to (f(x))_{f \in G} = \hat{x}.$$

In view of Proposition 4.8, e is 1—1. Also e is continuous. In fact if $\pi_f : Y \to Y_f$ denotes the projection map, then each composition $\pi_f \circ e : X \to Y_f$, $x \to f(x)$, is continuous and thus e is continuous. Let $Z = e(X)$. We will show that the map $e : X \to Z$ is a homeomorphism. Since e is 1—1, onto and continuous, it only remains to show that e is an open map. It is easy to see that if $(\mu_i)_{i \in A}$ is a family of fuzzy sets in X, then $e(\sup_{i \in A} \mu_i) = \sup_{i \in A} e(\mu_i)$. Also $e(\mu_1 \wedge \ldots \wedge \mu_n) = e(\mu_1) \wedge \ldots \wedge e(\mu_n)$. It follows that, in order to show that e is an open map, it suffices to show that if B is a subbase for the topology of X, then $e(\mu)$ is open in Z for each $\mu \in$ B. By Theorem 4.4, the family

$$\mathbf{B} = \{f^{-1}(\mu) : f \in G, \ \mu \text{ open in } I_\varphi\}$$

is a base for the topology of X. Let now $f \in G$, μ open in $I_\varphi = Y_f$ and $\rho = f^{-1}(\mu)$. We have

$$(\rho)(\hat{x}) = \rho(x) = \mu(f(x)) = \pi_f^{-1}(\mu)(\hat{x})$$

for each $x \in X$. Thus $e(\rho) = \pi_f^{-1}(\mu)|_Z$. Since $\pi_f : Y \to Y_f$ is continuous, it follows that $\pi_f^{-1}(\mu)$ is open in Y and thus $e(\rho)$ is open in Z. This completes the proof.

Theorem 4.12. *Let X be a T_1 fuzzy topological space. Then X is completely regular iff X is homeomorphic to a subspace of a fuzzy cube.*

Proof. The necessity of the condition follows from the preceding Theorem. The sufficiency follows from the fact that every fuzzy cube is completely regular (by Propositions 4.6 and 4.2) and from the Proposition 4.9.

5. Fuzzy Proximities Compatible with a Fuzzy Topology. Definition 5.1. *Let τ be a fuzzy topology on a set X and let δ be a fuzzy proximity on X. If $\tau = \tau(\delta)$, then we say that δ is compatible with τ.*

One of the questions which we will examine in this section is the following: Given a fuzzy topology τ on a set X, when does there exist a fuzzy proximity on X compatible with τ? We will prove that this happens iff the fuzzy topology τ is completely regular.

Definition 5.2. *Two fuzzy sets μ, ρ, in a fuzzy topological space X are said to be functionally distinguishable if there exists a continuous function f from X to the fuzzy unit interval, such that $\mu(x) \leq f(x)(1-) \leq f(x)(0+) \leq 1 - \rho(x)$ for all x in X.*

Theorem 5.3. *Let X be a completely regular fuzzy topological space. Then the binary relation δ on I^X, which is defined by $\mu \, \delta \, \rho$ iff μ, ρ are not functionally distinguishable, is a fuzzy proximity on X compatible with the topology τ of X.*

Proof. We will show first that δ satisfies the axioms (FP 1) — (FP 5) of the definition 3.1.

1) Suppose that $\mu \, \bar{\delta} \, \rho$. Then there exists a function $f \in C(X, I_\varphi)$ such that $\mu(x) \leq f(x)(1-) \leq f(x)(+) \leq 1 - \rho(x)$ for each $x \in X$. Define $g : X \to I_\varphi$ by $g(x)(s) = 1 - f(x)(1-s)$. It is easy to see that g is continuous. Also

$$g(x)(1-) = 1 - f(x)(0+) \geq \rho(x) \text{ and } g(x)(0+) = 1 - f(x)(1-) \leq 1 - \mu(x).$$

It follows that $\rho \, \bar{\delta} \, \mu$.

2) Clearly $(\mu \vee \rho) \, \bar{\delta} \, \sigma$ implies $\mu \, \bar{\delta} \, \sigma$ and $\rho \, \bar{\delta} \, \sigma$. On the other hand, assume that $\mu \, \bar{\delta} \, \sigma$ and $\rho \, \bar{\delta} \, \sigma$. There are $f_1, f_2 \in C(X, I_\varphi)$ such that $\mu(x) \leq f_1(x)(1-) \leq f_1(x)(0+) \leq 1 - \sigma(x)$ and $\rho(x) \leq f_2(x)(1-) \leq f_2(x)(0+) \leq 1 - \sigma(x)$, for all x in X. Let $f : X \to I_\varphi$ be defined by

$$f(x)(s) = \max\{f_1(x)(s), f_2(x)(s)\}.$$

Then $f(x)(t+) = \max\{f_1(x)(t+), f_2(x)(t+)\}$ and $1 - f(x)(t-) = \min\{1 - f_1(x)(t-), 1 - f_2(x)(t-)\}$. It follows that f is continuous and

$$(\mu \vee \rho)(x) \leq f(x)(1-) \leq f(x)(0+) \leq 1 - \sigma(x)$$

and hence $(\mu \vee \rho) \, \bar{\delta} \, \sigma$.

3) The function $f : X \to I_\varphi$, $f(x)(s) = 0$, is continuous. Moreover, *if* $\mu = 0$, *then* $\mu(x) \leq f(x)(1-) \leq f(x)(0+) \leq 1 - \rho(x)$ for each $\rho \in I^X$. Thus $\mu \, \delta \, \rho$ whenever $\mu = 0$.

4) It, is clear from the definition of δ, that $\mu \, \delta \, \rho$ implies $\mu \leq 1 - \rho$.

5) Suppose that $\mu \rceil\delta \rho$ and let $f \in C(X, I_\varphi)$ be such that $\mu(x) \leqq f(x)(1-) \leqq f(x)(0+) \leqslant 1-\rho(x)$ for all $x \in X$. The functions $h_1, h_2 : X \to I_\varphi$, $h_1(x)(s) = f(x)((1+s)/2)$, $h_2(x)(s) = f(x)(s/2)$, are continuous. Also

$$h_1(x)(1-) = f(x)(1-),\ h_1(x)(0+) = f(x)\left(\frac{1}{2}+\right) \leqq f(x)\left(\frac{1}{2}-\right).$$

$$h_2(x)(1-) = f(x)\left(\frac{1}{2}-\right) \text{ and } h_2(x)(0+) = f(x)(0+).$$

Setting $\sigma = f^{-1}(L_{1/2})$, we have

$$\mu(x) \leqq h_1(x)(1-) \leqq f(x)\left(\frac{1}{2}+\right) = h_1(x)(0+) \leqq 1-\sigma(x),$$

$$1-\sigma(x) = f(x)\left(\frac{1}{2}-\right) = h_2(x)(1-) \leqq h_2(0+) \leqslant 1-\rho(x).$$

Hence $\mu \rceil\delta \sigma$, $(1-\sigma) \rceil\delta \varphi$.

This proves that δ is a fuzzy proximity on X. It remains to show that δ is compatible with τ. Let μ be a τ-closed fuzzy set. We will show that $\bar{\mu}_\delta = \mu$. Suppose that $\mu \neq \bar{\mu}_\delta$. Then there exists $x \in X$ and a real number θ such that $\mu(x) < \theta < \bar{\mu}_\delta(x)$. The fuzzy set $\rho = 1-\mu$ is τ-open and $\rho(x) > 1-\theta > 0$. Since X is completely regular, there exists $f \in C(X, I_\varphi)$ such that $f(x)(1-) > 1-\theta$ and $f(y)(0+) \leqq \rho(y) = 1-\mu(y)$ for each $y \in X$. Let μ_1 be the fuzzy set in X defined by $\mu_1(y) = f(y)(1-)$. Then $\mu_1(y) = f(y)(1-) \leqq f(y)(0+) \leqq 1-\mu(y)$ for all $y \in X$ and so $\mu_1 \rceil\delta \mu$. It follows that $\bar{\mu}_\delta \leqq 1-\mu_1$ and hence $\bar{\mu}_\delta(x) \leqq 1-\mu_1(x) = 1-f(x)(1-) < \theta$ which is a contradiction. This contradiction proves that $\mu = \bar{\mu}_\delta$ and so μ is $\tau(\delta)$-closed.

Conversely, let μ be $\tau(\delta)$-closed. We will prove that $1-\mu$ is τ-open. In fact let $x \in X$ with $(1-\mu)(x) \neq 0$ and let θ be a positive real number with $\theta < 1-\mu(x)$. Since $\mu(x) = \bar{\mu}_\delta(x)$, there exists $\rho \in I^X$ with $\mu \rceil\delta \rho$ and $1-\rho(x) < 1-\theta$.

Since $\mu \delta \rho$, there exists $f \in C(X, I_\varphi)$ such that $\mu(y) \leqq f(y)(1-) \leqq f(y)(0+) \leqq 1-\rho(y)$, for all $y \in X$. The fuzzy set $\mu_1 = f^{-1}(L_{1/2})$ is τ-open, and

$$\mu_1(y) = 1-f(y)\left(\frac{1}{2}-\right) \leqq 1-f(y)(1-) \leqq 1-\mu(y).$$

Thus $\mu_1 \leqq 1-\mu$. Moreover

$$\mu_1(x) = 1-f(x)\left(\frac{1}{2}-\right) \geqq 1-f(x)(0+) \geqq \rho(x) > \theta.$$

It follows that $1-\mu$ is τ-open and hence μ is τ-closed. This proves that $\tau(\delta) = \tau$ and hence δ is compatible with τ.

Let δ be fuzzy proximity on X. For μ, ρ fuzzy sets in X, we write $\mu \ll \rho$ iff $\mu \rceil\delta (1-\rho)$. Then we have the following

Lemma 5.4. *Let μ, ρ be fuzzy sets in X with $\mu \ll \rho$. Then :*

(1) *If $\mu_1 \leqq \mu$ and $\rho_1 \geqq \rho$, then $\mu_1 \ll \rho_1$.*

(2) *$\bar{\mu} \ll \rho^0$ where ρ^0 is the interior of ρ with respect to the fuzzy topology $\tau(\delta)$.*

(3) $\bar{\mu} \leqq \rho^0$.

(4) *There exists an open fuzzy set μ_1 such that $\mu \ll \mu_1 \ll \rho$.*

Proof. It is clear that (1) is satisfied and that (2) implies (3). For the (2), we observe that $\mu \rceil\delta(1-\rho)$ implies $\bar{\mu} \rceil\delta(\overline{1-\rho})$ and that $\overline{1-\rho} = 1-\rho^0$. Finally, since $\mu \rceil\delta (1-\rho)$, there exists a fuzzy set μ_2 such that $\mu \rceil\delta (1-\mu_2)$ and $\mu_2 \rceil\delta (1-\rho)$. Let $\mu_1 = \mu_2^0$. Since $\mu \ll \mu_2$, we have $\mu \ll \mu_1$ by (3). Also since $\mu_1 \leqq \mu_2 \ll \rho$, we have $\mu_1 \ll \rho$. This completes the proof.

The next Theorem tells us that if a fuzzy topology τ on a set X is not completely regular, then τ is not compatible with any fuzzy proximity on X.

Theorem 5.5. *If δ is a fuzzy proximity on a set X, then the fuzzy topology $\tau = \tau(\delta)$ is completely regular.*

Proof. Let μ be open, $x \in X$ and θ a positive real number with $\mu(x) > \theta$. Since $\overline{1-\mu}(x) = 1-\mu(x) < 1-\theta$, there exists a fuzzy set λ with $(1-\mu) \rceil\delta \lambda$ and $1-\lambda(x) < 1-\theta$ and so $\lambda(x) > \theta$. Since $\lambda \ll \mu$, there exists (by the preceding Lemma) an open fuzzy set λ_0 with $\lambda \ll \lambda_0 \ll \mu$. Now $\lambda_0(x) \geqq \lambda(x) > \theta$. For each integer $n \geqq 0$, let

$$A_n = \left\{\frac{m}{2^n} : m = 0, 1, \dots, 2^n\right\},\quad A = \bigcup_{n=1}^{\infty} A_n.$$

We will define for $r \in A$ an open fuzzy set ρ_r such that the following conditions hold :

(1) $\rho_0 = \lambda_0$ and $\rho_1 = \mu$.

(2) If $r < s$, then $\rho_r \ll \rho_s$

To this end, suppose that we have already defined ρ_r for each $r \in A_n$, so that conditions (1) and (2) are satisfied. (We know how to define ρ_r for $r \in A_0$). We will define ρ_r for each $r \in A_{n+1}$ which is not in A_n. Such an r will be of the form $r = (2k+1)/2^{n+1}$ where $k \geqslant 0$ is an integer. Since $\rho_{k/2^n} \ll \rho_{(k+1/2^n)}$, there exists an open fuzzy set λ_1 with $\rho_{k/2^n} \ll \lambda_1 \ll \rho_{(k+1/2^n)}$. Define $\rho_r = \lambda_1$. In this way we define the family $\{\rho_r : r \in A_{n+1}\}$ of open fuzzy sets which clearly satisfies (1), (2). Inductively we define a family of open fuzzy sets $\{\rho_r : r \in A\}$ satisfying (1) and (2). Next we define the family $\{\sigma_t : t \in [0, 1]\}$ by taking

$$\sigma_t = \sup\{\rho_r : r \in A,\ r \leqq t\}.$$

Each σ_t is open and $\sigma_t = \rho_t$ if $t \in A$. If $s < t$, then $\bar{\sigma}_s \leqq \sigma_t$. In fact, let $r_1, r_2 \in A$ with $s < r_1 < r_2 < t$. Then $\sigma_s \leqq \sigma_{r_1} \leqq \sigma_{r_2} \leqq \sigma_t$ and thus $\sigma_s \leqq \ll \sigma_{r_1} \leqq \sigma_{r_2} \leqq \sigma_t$. Define $f : X \to I_\varphi$ by $f(x)(t) = \sigma_{1-t}(x)$, $x \in X$, $t \in [0, 1]$. If $t \in (0, 1)$ then for each $y \in X$, we have

$$\rho_0(y) \leqq f(y)(t) = \sigma_{1-t}(y) \leqq \mu(y)$$

and that $f(y)(0+) \leqq \mu(y)$. Also

$$\theta < \rho_0(x) \leqq \sigma_{1-t}(x) = f(x)(t)$$

and hence $f(x)(1-) \geqq \rho_0(x) > 0$. Finally f is continuous. In fact, if $t \in (0, 1)$, then for each $y \in X$ we have

$$f^{-1}(L_t)(y) = 1 - f(y)(t-) = 1 - \inf_{s<t} f(y)(s) = 1 - \inf_{s<t} \sigma_{1-s}(y) =$$
$$1 - \inf_{s>1-t} \sigma_s(y) = 1 - \inf_{s>1-t} \bar{\sigma}_s(y).$$

Hence $f^{-1}(L_t)$ is open. Similarly,

$$f^{-1}(R_t)(y) = f(y)(t+) = \sup_{s>t} \sigma_{1-s}(y)$$

and so $f^{-1}(R_t)$ is open. This proves that f is continuous and the result follows.

Theorem 5.6. *Let δ be a fuzzy proximity on X and equip X with the fuzzy topology $\tau = \tau(\delta)$. Let μ, ρ be fuzzy sets in X with $\mu \,\bar{\delta}\, \rho$. Then μ, ρ are functionally distinguishable.*

Proof. Since $\mu \ll 1-\rho$, there exist (by Lemma 5.4) open fuzzy sets ρ_0, ρ_1 such that $\mu \ll \rho_0 \ll \rho_1 \ll 1-\rho$. Working now as in the proof of the preceding Theorem, we get a function, $f : X \to I_\varphi$ continuous and such that $\rho_0(y) \leqq f(y)(t) \leqq \rho_1(y)$ for all y in X and all $t \in [0, 1]$. Thus

$$\mu(y) \leqq \rho_0(y) \leqq f(y)(1-) \leqq f(y)(0+) \leqq \rho_1(y) \leqq 1-\rho(y)$$

for all $y \in X$. This completes the proof.

Definition 5.7. *A mapping f, from a fuzzy proximity space (X, δ_1) to a fuzzy proximity space (Y, δ_2), is called a proximity mapping or a proximally continuous mapping if $\mu\, \delta_1\, \rho$ implies $f(\mu)\, \delta_2 f(\rho)$. Equivalently, f is a proximity map if $\mu_1 \bar{\delta}_2\, \mu_2$ implies $f^{-1}(\mu_1) \bar{\delta}_1 f^{-1}(\mu_2)$.*

If δ_1, δ_2 are two fuzzy proximities on the same set X and if the identity map from (X, δ_1) to (X, δ_2) is a proximity map, then we say that δ_1 is *finer* than δ_2 and we write $\delta_1 \geqq \delta_2$. In this case we also say that δ_2 is *coarser* than δ_1. Thus, $\delta_1 \geqq \delta_2$ iff $\mu\, \delta_1\, \rho$ implies $\mu\, \delta_2\, \rho$.

Proposition 5.8. *If $f : (X, \delta_1) \to (Y, \delta_2)$ is a proximity map, then f is continuous with respect to the corresponding fuzzy topologies $\tau(\delta_1)$ and $\tau(\delta_2)$.*

Proof. It suffices to show that for every $\tau(\delta_2)$-closed fuzzy set μ in Y, $f^{-1}(\mu)$ is $\tau(\delta_1)$-closed in X. In fact, suppose that μ is closed in Y and let $x \in X$ and 0 be a real number with $0 > f^{-1}(\mu)(x) = \mu(f(x))$. Since μ is closed, there exists a fuzzy set ρ in Y with $\mu \,\bar{\delta}_2\, \rho$ and $1-\rho(f(x)) > 0$. Since f is a proximity map, $f^{-1}(\mu) \bar{\delta}_1 f^{-1}(\rho)$ and thus

$$\overline{f^{-1}(\mu)}(x) \leqq 1 - f^{-1}(\rho)(x) = 1 - \rho(f(x)) > 0.$$

This proves that $\overline{f^{-1}(\mu)}(x) = f^{-1}(\mu)(x)$ and the result follows.

Corollary 5.9. *If the fuzzy proximity δ_1 is finer than δ_2, then the fuzzy topology $\tau(\delta_1)$ is finer than the fuzzy topology $\tau(\delta_2)$.*

In view of Theorems 5.3. and 5.7. if (X, τ) is a fuzzy completely regular space, then the relation δ on I^X, which is defined by $\mu\, \delta\, \rho$ iff μ and ρ are not functionally distinguishable, is the finest fuzzy proximity on X compatible with τ.

Theorem 5.11. *Let τ_1, τ_2 be two completely regular fuzzy topologies on a set X. Let δ_1 be a fuzzy proximity compatible with δ_1 and let δ_2 be the fuzzy proximity defined by: $\mu \,\bar{\delta}_2\, \rho$ iff μ, ρ are functionally distinguishable in (X, τ_2). Then, τ_2 finer that τ_1 implies that δ_2 is finer than δ_1.*

Proof. Suppose that $\mu \,\bar{\delta}_1\, \rho$. By 5.6, there exists a continuous function $f : (X, \tau_1) \to I_\varphi$ such that $\mu(x) \leqq f(x)(1-) \leqq f(x)(0+) \leqq 1 - \rho(x)$ for all $x \in X$. Since τ_2 is finer than τ_1, f is continuous with respect to the topology τ_2. Thus $\mu \,\bar{\delta}_2\, \rho$ and this completes the proof.

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Department of Mathematics
University of Ioannina
Ioannina, Greece