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A NOTE ON POSITIVE OPERATORS ON BANACH SPACES

BY

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1. Introduction : A complex (real) vector space X is called a complex (real) semi-inner product space if, corresponding to arbitrary pair of elements x, $y \in X$, there is defined a complex (real) number (x, y) which satisfies the following properties, for $x, y, z \in X$, λ complex (real).

(i) $(x + y, z) = (x, z) + (y, z)$, $(\lambda x, y) = \lambda (x, y)$,

(ii) $(x, \lambda y) = \bar{\lambda} (x, y)$,

(iii) $(x, x) > 0$ for $x \neq 0$,

(iv) $|(x, y)|^2 \leqslant (x, y) (y, y)$,

Giles has shown [4, Theorem 1] that every normed vector space can be represented as a s.i.p. space.

Definition 1.1. *Let X be a Banach space and (,) be any consistent s.i.p. on X. Then numerical range of an operator T on X is defined by* $W(T) = \{(T x, x) : \| x \| = 1\}$.

Definition 1.2. *We recall that an operator T is Hermitian iff $W(T)$ is real.*

Berberian [1] has proved that, if T be a Hermitian operator on a Hilbert space, then T is positive iff $\sigma(T)$ is non-negative. In this note we generalize this result to Banach spaces.

2. Positive operators. We now define positive operators on Banach spaces.

Definition 2.1. *Let X be a Banach space. An operator T is said to be positive iff $(Tx, x) \geqslant 0$ for all $x \in X$, where (,) denotes consistent s.i.p. on X.*

Lumer [5, Theorem 14] has shown that all determinations of the numerical range of an operator have the same convex hull: therefore we can say that if T is positive in one determination then T is positive for all determinations.

Definition 2.2. (Lumer [6]) : *A operator T on a Banach space X is said to be normal iff $T = R + iJ$, where R and J are Hermitians, and $RJ = JR$.*

Theorem 2.1. *Let T be a normal operator on a Banach space X. Then T is positive (or Hermitian) iff σ (T) is non-negative (or real).*

Proof : We shall prove the 'positive' case. 'Hermitian' case will follow similarly. First let T be positive. Then by theorem 4 of Lumer [5], $\partial\sigma(T) \subseteq W(T)$, $\sigma(T)$ is non-negative.

Conversely, before we prove the 'if part' we first state a result of Douglas [2, Cor—2.36, p. 45] as follows : If B is a commutative Banach algebra and x is in B then $\sigma(x) =$ range Γx and $|\sigma(x)| = \|\Gamma x\|_\infty$ where Γ denotes the Gelfand transform.

Proof of the 'if part' : Let $\sigma(T)$ be non-negative. Since T is normal, $T = R + iJ$ where R and J are Hermitians, and $RJ = JR$.

Suppose, M be a maximal commutative sub-algebra generated by R and J and hence $T \in M$. Since J is Hermitian, by proposition 2 of Sinclair [7] $\|J\| = |\sigma(J)|$.

Now the spectrum $\sigma(T)$ of T is the same as an operator or an element of M. Therefore by above result of Douglas [2], $J = 0$. Hence $T = R$ is Hermitian. By propositon 2 of Sinclair [7]

$$\|T\| = |\sigma(T)| = |W(T)|. \tag{1}$$

Now let $k = g.l.b. \{(Tx, x) : \|x\| = 1\}$. Then by lemma 20 of Lumer [5] and the relation (i), $k \in \sigma(T)$. This completes our result.

Corollary 2.1. *If T is Hermitian then T is positive iff $\sigma(T)$ is non-negative.*

Corollary 2.2 *If $T - \lambda I$ is positive for some positive scalar λ then T is invertible and $\|T^{-1}\| \leqslant 1/\lambda$.*

Remark 2.1. In the above theorem 2.1 if we replace normal operator by a more general operator namely, scalar type spectral operator, [3], then the theorem is no longer valid because a non-self adjoint idempotent operator on a Hilbert space would be a scalar type special operator with non-negative spectrum, but it would not be positive. As for example, let $T = \begin{bmatrix} 1 & -9 \\ 0 & 0 \end{bmatrix}$ be an operator on a two-dimensional Hilbert space X. Then T is a non-self adjoint idempotent operator with non-negative spectrum but not positive. That T is a scalar type spectral operator follows from the following result of Dunford [3, p. 2173.].

Theorem : *If X is a weakly sequentially complete Banach space and $T \in B(X)$ has real spectrum, then T is a scalar type spectral operator iff there exists $M > 0$ such that for every polynomial p one has $\|p(T)\| \leqslant M$.* sup $p(\lambda)$, $\lambda \in \sigma(T)$.

Remark 2.2. In the above theorem 2.1. if we replace 'positive. by, positive equivalent' (i.e. positive in some equivalent norm) and Hermitian by Hermitian equivalent (i.e. Hermitian in some equivalent norm), then the condition that T be a normal operator can be replaced by the condition that T is a scalar type spectral operator. This follows from the following result of Berkson [6] : Any scalar type spectral operator T can be decomposed as $T = R + iJ$, where R and J are Hermitian equivalents and $RJ = JR$.

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