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NECESSARY CONDITIONS FOR BOUNDARY CONTROL PROBLEMS GOVERNED BY PARABOLIC VARIATIONAL INEQUALITIES

BY

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0. **Introduction.** In this paper we propose studying necessary conditions of optimality for boundary control problems with convex cost criterion and whose state is given by the solution y of the unilateral problem (the „obstacle problem")

$$y(x, t) \geqslant \psi(x) \quad \text{on} \quad Q = \Omega \times]0, T[,$$

$$y_t + A_0 y = f \quad \text{on} \quad \{(x, t) \in Q \,;\, y(x, t) > \psi(x)\},$$

$$y_t = \max\{f - A_0\psi, 0\} \quad \text{on} \quad \{(x, t) \in Q \,;\, y(x, t) = \psi(x)\}, \tag{0.1}$$

$$\alpha_1 y + \alpha_2 \frac{\partial y}{\partial \nu} = u \quad \text{on} \quad \Sigma = \Gamma \times]0, T[,$$

$$y(x, 0) = y_0(x) \quad \text{on} \quad \Omega,$$

where A_0 is a second order elliptic operator on Ω, $\alpha_i \geqslant 0$; $i = 1, 2$, $\alpha_1 + \alpha_2 \neq 0$ and ψ is a given function on Ω. Here Ω is a bounded and open domain of the Euclidean space R^N with the boundary Γ. The control u which is exercised through the boundary is taken from the space $L^2(\Sigma)$. The necessary conditions of optimality are found in terms of quasi variational evolution inequalities which generalize the classical Lagrange equations. The paper is much in the spirit of the author works [1], [2] where the present approach has been used in the study of distributed control problems governed by elliptic and parabolic variational inequalities. For recent other results on necessary conditions for control problems governed by nonlinear parabolic equations, we refer the reader to [10] and [12]. The plan of the paper is the following:

In Section 1 we give the basic notations. Section 2 contains a functional description of a general class of nonlinear boundary control systems. It is shown that such sys ems can be reduced to a class of nonlinear evolution inequations whose existence is discussed. In Section 3 we give one of the main results of this paper, Theorem 1, which is concerned with a control problem governed by Eq. (0.1), where $\alpha_2 = 0$, i.e. the Dirichlet problem with interior unilateral constraint. The proof is given in Section 4. In Section 5 is considered the case $\alpha_2 \neq 0$. In both situations the neces-sary conditions for optimality are obtained for sufficiently regular optimal controls u^*

$$u^* \in W_q^{2-1/q,1-1/2q}(\Sigma) \text{ if } \alpha_2=0 \text{ and } u^* \in H^{1,2,1/4}(\Sigma) \text{ if } \alpha_2\neq 0.$$

1. Basic notations. Ω is a bounded and open subset of R^N with a suf-ficiently smooth boundary (it will be enough to assume that Γ is regular of class C^2). Let m, r, s be real numbers and let $1 \leqslant p \leqslant \infty$. We set $Q =$ $= \Omega \times]0, T[$, $\Sigma = \Gamma \times]0, T[$ and denote by $W_p^m(\Omega)$, $W_p^m(\Gamma)$, $W_p^{r,s}(Q)$ and $W_p^{r,s}(\Sigma)$ the usual Sobolev spaces on Ω, Γ, Q and Σ, respectively (see e.g. [7] Chapter II). We set $W_2^m(\Omega) = H^m(\Omega)$, $W_2^m(\Gamma) = H^m(\Gamma)$, $W_2^{r,s}$ $(Q) = H^{r,s}(Q)$ and $W_2^{r,s}(\Omega) = H^{r,s}(\Sigma)$. By $H_0^m(\Omega)$ we shall denote the space of all elements $y \in H^m(\Omega)$ of trace zero on Γ. By $C^{(k)}(\bar{Q})$ and $C^{(k)}(\Sigma)$, k positive integer, we shall denote the space of all continuously differen-tiable functions up to order k on the closures $\bar{Q}$ and $\bar{\Sigma}$ of Q and Σ, respec-tively. By $L^p(Q)$ and $L^p(\Sigma)$, $1 \leqslant p \leqslant \infty$, we shall denote the usual spaces of integrable functions on Q and Σ respectively. By $\mathfrak{D}'(Q)$ we shall denote the space of distributions on Q and by $M(Q)$ and $M(\Sigma)$ we shall denote the space of bounded Radon measures on $\bar{Q}$ and Σ, respectively. Given a Banach space E, we shall denote by $L^p(0, T; E)$, $1 \leqslant p \leqslant \infty$, the space of all p-integrable functions from $[0, T]$ to E. Let $\varphi : E \to \bar{R} =$ $]-\infty, +\infty]$ be a lower semicontinuous convex function on E. We shall denote by $\partial\varphi(y) \subset E^*$ (the dual space of E) the set of all subgradients of φ at y (see e.g. [3] p. 89). The mapping $\partial\varphi : E \to E^*$ is the subdifferential of φ. If φ is Gâteaux differentiable at y then $\partial\varphi(y)$ consists of a single element, namely the gradient $\nabla\varphi(y)$ of φ at y. All the spaces considered in this paper are real.

If E is a Hilbert space, then for each $\varepsilon > 0$ the function $\varphi_\varepsilon : E \to$ $]-\infty, +\infty[$ defined by ($|.|$ denotes the norm of E)

$$\varphi_\varepsilon(y) = \inf\{\varepsilon^{-1}|y - z|^2 + \varphi(z);\ z \in H\} \tag{1.1}$$

is Fréchet differentiable on E and (see [3], p. 107, [4] p. II. 21)

$$\varphi(y) \geqslant \varphi_\varepsilon(y) = \frac{\varepsilon}{2}|\partial\varphi_\varepsilon(y)|^2 + \varphi((1 + \varepsilon\,\partial\varphi)^{-1}y) \tag{1.2}$$

$$\varepsilon^{-1}(y - (1 + \varepsilon\,\partial\varphi)^{-1}y) = (\partial\varphi)_\varepsilon(y) = \partial\varphi_\varepsilon(y) \text{ for all } y \in E. \tag{1.3}$$

2. Nonlinear boundary control systems. Let H be a Hilbert space with norm $|\cdot|$ and inner product $(.\,,\,.)$ and let V be a Hilbert space densely and continuously embedded in H. Denote by $\|.\|$ the norm in V and by V' the dual of V. We have

$$V \subset H \subset V'$$

algebraically and topologically.

Let A be a linear continuous operator from V to V' which satisfies

$$(Au, u) \geqslant \omega\|u\|^2 + \alpha|u|^2 \text{ for all } u \in V,$$

where $\omega > 0$.

Finally, let U be another Hilbert space with inner product $<\,.\,,\,.\,>$ and norm $|||\cdot|||$ and let B be a linear continuous operator from U to H.

The boundary control system we shall consider is

$$\begin{aligned} &z' + Az + Fy \ni f \text{ on } [0, T], \\ &y = z + Aw_u, \\ &z(0) = z_0, \end{aligned} \tag{2.1}$$

where w_u is the solution to

$$\begin{aligned} &w_u' + Aw_u = Bu, \text{ a.e. } t \in]0, T[, \\ &w_u(0) = A^{-1}y_0. \end{aligned} \tag{2.2}$$

Here $f \in L^2(0, T; H)$ and $z_0\ y_0 \in H$ are given, the input u belongs to L^2 $(0, T; U)$ and $F = \partial\varphi$, where φ is a lower semicontinuous convex func-tion from H to $\bar{R}$. By $'$ we have denoted the differentiation symbol $\frac{d}{dt}$. We notice that our assumptions on A imply that for each $u \in L^2(0, T; U)$ the Cauchy problem (2.2) has a unique solution $w_u \in C(0, T; H) \cap L^2$ $(0, T; V)$ such that w_u', $Aw_u \in L^2(0, T; H)$ (see e.g. [3], p. 22, [9] p. 69). In this way the system (2.1), (2.2) reduces to

$$\begin{aligned} &z' + Az + F(z + Aw_u) \ni f \text{ on } [0, T], \\ &z(0) = z_0. \end{aligned} \tag{2.3}$$

In general, the time dependent evolution equation (2.3) does not admit strong solutions. However if $\zeta = Aw_u$ is sufficiently regular, we have:

Proposition 1. *Assume that A is symmetric. Let $\zeta = Aw_u \in L^2(0, T;$ $H)$ and $z_0 \in V$ be such that $\zeta' \in L^2(0, T; H)$ and $\varphi(z_0 + \zeta(0)) < +\infty$. Suppose in addition that there exists a function $C_0 \in L^1(0, T)$ such that*

$$(\partial\varphi_\varepsilon(z + \zeta(t)), Az) \geqslant C_0(t) - \frac{1}{2}|\partial\varphi_\varepsilon(z + \zeta(t))|^2 \tag{2.4}$$

for all $\varepsilon > 0$ and $z \in V$ with $Az \in H$.

Then Eq. (2.3) *has a unique strong solution z which satisfies*

$$z' \in L^2(0, T; H),\ Az \in L^2(0, T; H). \tag{2.5}$$

Proof. The proof is standard (see e.g. [4], [6]) but we shall outline it for the convenience of the reader.

We start with the approximating equations

$$\begin{aligned} &z' + Az + \partial\varphi_\varepsilon(z + \zeta) = f \text{ a.e. } t \in]0, T[, \\ &z(0) = z_0. \end{aligned} \tag{2.6}$$

Since $\partial\varphi_\varepsilon$ is Lipschitzian, Eq. (2.6) has a unique solution $z_\varepsilon \in C([0, T];$ $H)$ such that $z_\varepsilon' \in L^2(0, T; H)$ and $Az_\varepsilon \in L^2(0, T; H)$. By multiplying

Eq. (2.6) (where $z = z_\varepsilon$) with $\partial\varphi_\varepsilon(z_\varepsilon + \zeta)$ and using condition (2.4) along with the formula

$$(\partial \varphi_\varepsilon(w),\ w') = ((\varphi_\varepsilon w))' \text{ a.e. on }]0,\ T[,$$

we get

$$\varphi_\varepsilon(z_\varepsilon(t) + \zeta(t)) + \frac{1}{2}\int_0^t |\partial\varphi_\varepsilon(z_\varepsilon + \zeta)|^2\,ds \leq \varphi_\varepsilon(z_0 + \zeta(0)) + \int_0^t C_0(s)\,ds +$$

$$+ \int_0^t |f + \zeta'|\ |\partial\varphi_\varepsilon(z_\varepsilon + \zeta)|\,dt, \quad t \in [0,\ T].$$

Since φ_ε are uniformly bounded from below by an affine function, the latter yields

$$\int_0^T |\partial\varphi_\varepsilon(z_\varepsilon + \zeta)|^2\,dt \leq C, \quad \varepsilon > 0. \tag{2.7}$$

In particular it follows that $\{z_\varepsilon' + Az_\varepsilon\}$ is bounded in $L^2(0,\ T;\ H)$. Since $z_0 \in V$, we may infer that $\{z_\varepsilon'\}$ and $\{Az_\varepsilon\}$ are bounded in $L^2(0,\ T;\ H)$ (see e.g. [9] p. 22). Therefore

$$|z_\varepsilon(t)|^2 + \int_0^T (|z_\varepsilon'(t)|^2 + |Az_\varepsilon(t)|^2 + \|z_\varepsilon(t)\|^2\,dt \leq C, \tag{2.8}$$

where C is independent of ε.

Using (2.7) and the monotonicity of $\partial\varphi$, we get that for all $\varepsilon, \lambda > 0$,

$$|z_\varepsilon(t) - z_\lambda(t)|^2 + \int_0^t \|z_\varepsilon - z_\lambda\|^2\,ds \leq C(\varepsilon + \lambda),\ t \in [0,\ T].$$

Hence there exists $z : [0,\ T] \to V$ such that

$$\begin{aligned} &z_\varepsilon \to z \quad \text{strongly in } C([0,\ T];\ H) \text{ and } L^2(0,\ T;\ V),\\ &z_\varepsilon' \to z' \quad \text{weakly in } L^2(0,\ T;\ H),\\ &Az_\varepsilon \to Az \text{ weakly in } L^2(0,\ T;\ H),\\ &\partial\varphi_\varepsilon(z_\varepsilon + \zeta) \to w \text{ weakly in } L^2(0,\ T;\ H). \end{aligned} \tag{2.9}$$

By a standard argument, the latter implies that

$$w(t) \in \partial\varphi(z(t) + \zeta(t)) \text{ a.e. } t \in]0,\ T[$$

and therefore z is a strong solution to (2.3), i.e.

$$z'(t) + Az(t) + \partial\varphi(z(t) + \zeta(t)) \ni f(t) \text{ a.e. } t \in]0,\ T[.$$

The uniqueness of z is an immediate consequence of the monotonicity of $\partial\varphi$.

We shall conclude this section with two examples

Example 1 (Mixed Dirichlet problem). Consider on Ω, the second order differential operator

$$A_0 y = -\sum_{i,j=1}^{N} (a_{ij}(x)\,y_{x_i})_{x_j} + a_0(x)\,y, \tag{2.10}$$

where $a_{ij} \in C^1(\overline{\Omega})$, $a_0 \in L^\infty(\Omega)$ $a_{ij} = a_{ji}$ and

$$\sum_{i,j=1}^{N} a_{ij}(x)\,\xi_i\xi_j \geq \omega\,|\xi|^2 \text{ a.e. } x \in \Omega, \quad \xi \in R^N, \tag{2.11}$$

(ω is a positive constant). We shall also assume that $a_0 \geq 0$ a.e. on Ω.

Let $F = \partial\varphi$ where $\varphi : L^2(\Omega) \to R$ is a lower semicontinuous convex function. Consider the boundary control system

$$\begin{aligned} &y_t + A_0 y + Fy \ni f && \text{on } Q = \Omega \times]0,\ T[,\\ &y = u && \text{on } \Sigma = \Gamma \times]0,\ T[,\\ &y(x,\ 0) = y_0(x) && x \in \Omega. \end{aligned} \tag{2.12}$$

System (2.12) can be brought into the form (2.1) where $H = L^2(\Omega)$ $V = H_0^1(\Omega)$. $U = L^2(\Gamma)$ and $A : V \to V'$ defined by

$$(Ay,\ \chi) = a(y,\ \chi) = \sum_{i,j=1}^{N} \int_\Omega (a_{ij}(x)\,y_{x_i}\chi_{x_j} + a_0\,y\chi)\,dx \tag{2.13}$$

$$\text{for all } y,\ \chi \in H_0^1(\Omega).$$

The operator $B : L^2(\Gamma) \to L^2(\Omega)$ is defined by $Bu = \delta_u$, where $\delta_u \in L^2(\Omega)$ is the weak solution to nonhomogeneous Dirichlet problem

$$\begin{aligned} &A_0\,\delta_u = 0 && \text{on } \Omega,\\ &\delta_u = u && \text{on } \Gamma, \end{aligned} \tag{2.14}$$

i.e.,

$$\int_\Omega \delta_u A_0 \chi\,dx + \int_\Gamma u\frac{\partial\chi}{\partial\nu}\,d\sigma = 0 \text{ for all } \chi \in H_0^1(\Omega) \cap H^2(\Omega).$$

Here

$$\frac{\partial\chi}{\partial\nu} = \sum_{i,j} a_{ij}\frac{\partial\chi}{\partial x_j}\cos(n,\ x_i) \text{ on } \Gamma.$$

Example 2. This is a mixed nonlinear Neumann problem with boundary control. Namely

$$\begin{aligned} &y_t + A_0 y + Fy \ni f && \text{on } Q,\\ &\alpha y + \frac{\partial y}{\partial\nu} = u && \text{on } \Sigma,\\ &y(x,\ 0) = y_0(x) && \text{on } \Omega, \end{aligned} \tag{2.15}$$

where A_0, F are as in Example 1 and $\alpha \geq 0$. To put (2.15) into the form (2.1), we take $H = L^2(\Omega)$, $V = H^1(\Omega)$ and A defined by

$$(Ay, \varsigma) = a(y, \varsigma) + \alpha \int_\Gamma y\varsigma \, d\sigma, \text{ for } y, \varsigma \in H^1(\Omega), \tag{2.16}$$

$U = L^2(\Gamma)$ and $B: L^2(\Gamma) \to L^2(\Omega)$ is given by $Bu = \xi_u$, where $\xi_u \in H^1(\Omega)$ is the solution to

$$\begin{aligned} A_0 \xi_u &= 0 && \text{on } \Omega, \\ \alpha\xi_u + \frac{\partial \xi_u}{\partial \nu} &= u && \text{on } \Gamma. \end{aligned} \tag{2.17}$$

3. Mixed Dirichlet problem with unilateral constraines. Consider the boundary control system

$$\begin{aligned} y_t + A_0 y + \partial \varphi(y) &\ni f && \text{on } Q, \\ y &= u && \text{on } \Sigma, \\ y(\cdot, 0) &= y_0 && \text{on } \Omega. \end{aligned} \tag{3.1}$$

Here A_0 is an elliptic, symmetric differential operator of the form (2.10), $y_0 \in W_q^{2-2/q}$, $f \in L^2(Q)$. The function $\varphi: L^2(\Omega) \to \overline{R}$ is given by

$$\varphi(y) = \begin{cases} 0 \text{ if } y(x) \geq \psi(x) & \text{a.e. } x \in \Omega \\ +\infty \text{ otherwise,} \end{cases} \tag{3.2}$$

where $\psi \in H^2(\Omega)$ is given such that $\psi(x) \leq 0$ a.e. $x \in \Gamma$. Formally, Eq. (3.1) can be written as (0.1). As seen in Example 1, Eq. (3.1) can be put into the form (2.1) where $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $U = L^2(\Gamma)$, $z_0 = 0$ and A is defined by (2.13). It is easy to see that in this case $\zeta_u = Aw_u$ is the solution to

$$\begin{aligned} (\zeta_u)((\zeta_u)_t) + A_0 \zeta_u &= 0 && \text{on } Q, \\ \zeta_u &= u && \text{on } \Sigma, \\ \zeta_u(\cdot, 0) &= y_0 && \text{on } \Omega. \end{aligned} \tag{3.3}$$

In the sequel we shall denote by X_q the space $W^{2-1/q, 1-2/q}(\Sigma)$ with the natural norm $\|\cdot\|_q$. By X_q^* we shall denote the dual space of X_q. Let us assume that $u \in X_q$ where $q \geq 2$ and $u(\cdot, 0) = y_0$. Then $\zeta_u \in W_q^{2,1}(Q)$ (see [7] p. 389) and

$$\|\zeta_u\|_{W_q^{2,1}(Q)} \leq C\|u\|_q + C_1 \quad \forall u \in X_q. \tag{3.4}$$

Further, assume that

$$y_0(x) \geq \psi(x) \quad \text{a.e. } x \in \Omega, \tag{3.5}$$

and

$$u(\sigma, t) \geq \psi(\sigma) \text{ a.e. } \sigma \in \Gamma, \quad t \in [0, T]. \tag{3.6}$$

It should be noticed that

$$\partial \varphi_\varepsilon(y) = -\varepsilon^{-1}(y-\psi)^-, \quad \forall y \in L^2(\Omega).$$

Since by (3.6) $(z + \zeta_u - \psi)^- = 0$ a.e. on Γ for every $z \in H_0^1(\Omega)$, by Green's formula it follows that

$$\begin{aligned} (Az, \partial\varphi_\varepsilon(z+\zeta_u)) = a(z, \partial\varphi_\varepsilon(z+\zeta_u)) = a(z+\zeta_u-\psi, \partial\varphi_\varepsilon(z+{}\\ {}+\zeta_u)) - a(\zeta_u - \psi, \partial\varphi_\varepsilon(z+\zeta_u)). \end{aligned} \tag{3.7}$$

Since the function $y \to -y^-$ is monotone (non decreasing), we see that $a(z+\zeta_u-\psi, \partial\varphi_\varepsilon(z+\zeta_u)) \geq 0$ a.e. $t \in]0, T[$. On the other hand, using the Green formula in Eq. (3.3) we see that

$$a(\zeta_u - \psi, \partial\varphi_\varepsilon(z+\zeta_u)) = -\int_\Omega ((\zeta_u)_t + A_0\psi)\,\partial\varphi_\varepsilon(z+\zeta_u)dx.$$

Since $(\zeta_u)_t \in L^2(Q)$ and $\psi \in H^2(\Omega)$ we may therefore conclude by (3.7) that condition (2.4) is satisfied with

$$C_0(t) = 2\int_\Omega ((\zeta_u)_t + A_0\psi)^2 \, dx.$$

Hence according to Proposition 1, for $y_0 \in W_q^{2-2/q}(\Omega)$ satisfying (3.5) and $u \in X_q$, satisfying condition (3.6), Eq. (3.1) has a unique strong solution $y = z + \zeta_u$. Moreover $\zeta_u \in W_q^{2,1}(Q)$ and $z \in H^{2,1}(Q)$ is the limit in the sense of (2.9) of a sequence $\{z_\varepsilon\} \subset H^{2,1}(Q)$ defined by

$$\begin{aligned} (z_\varepsilon)_t + Az_\varepsilon - \varepsilon^{-1}(z_\varepsilon + \zeta_u - \psi)^- &= f && \text{a.e. on } Q, \\ z_\varepsilon(x, 0) &= 0 && \text{a.e. on } \Omega. \end{aligned} \tag{3.7}$$

(For a general $u \in L^2(\Sigma)$ it follows from the results of M i g n o t and P u e l [11] that the control system (3.1) has a minimum weak solution $y \in L^2(Q)$).

Hence in order to have existence and uniqueness in the control system (3.1), we must restrict the input functions u to the class U_0 defined by

$$U_0 = \{u \in X_q;\ u(\sigma, t) \geq \psi(\sigma),\ u(\sigma, 0) = y_0(\sigma) \text{ a.e. } (\sigma, t) \in \Sigma\}. \tag{3.8}$$

The optimal control problem we consider here is: *Minimize*

$$\frac{1}{2}\int_Q |y(x, t) - y_d(x)|^2 \, dx\,dt + \int_\Sigma g(\sigma, u(\sigma, t)) \, d\sigma\,dt \tag{3.9}$$

over all $u \in U_0$ *and* $y \in L^2(Q)$ *subject to state system* (3.1).

Here y_d is a fixed element of $L^2(\Omega)$ and $g: \Gamma \times R \to \overline{R}$ is a convex normal integrand on $\Gamma \times R$ (see [13]). In other words,

(i) For each $\sigma \in \Gamma$, $g(\sigma, .)$ is a lower semicontinuous convex function on R and nonidentically $+\infty$.

(ii) g is measurable with respect to the σ-field of subsets of $\Gamma \times R$ generated by products of Lebesgue sets in Γ and Borel sets in R.

(iii) There exists $u^0 \in U_0$ such that $g(\sigma, u^0) \in L^1(\Sigma)$. There exist the functions $\alpha_1 \in L^2(\Gamma)$ and $\alpha_2 \in L^1(\Gamma)$ such that

$$g(\sigma, u) \geq \alpha_1(\sigma) u + \alpha_2(\sigma) \text{ a.e. } \sigma \in \Gamma,\ u \in R. \tag{3.10}$$

Theorem 1. *Let $(y^*, u^*) \in H^{2,1}(Q) \times X_q$ be an optimal pair in problem (3.9) where $q > (N+2)/2$. Then there exists a function $p \in C(0, T; L^1(\Omega) + H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ such that*

$$p_t - A p \in M(Q),\ \frac{\partial p}{\partial \nu} \in X_q^*, \tag{3.11}$$

$$(p_t - A p)_a = y^* - y_d \text{ a.e. on } \{(x, t) \in Q;\ y^*(x, t) > \psi(x)\}. \tag{3.12}$$

$$p(0, T) = 0 \text{ on } \Omega, \tag{3.13}$$

$$p(y_t^* + A_0 y^* - f) = 0 \text{ a.e. on } Q, \tag{3.14}$$

$$\frac{\partial p}{\partial \nu}(u^* - v) \geq \Phi(u^*) - \Phi(v) \text{ for all } v \in U_0. \tag{3.15}$$

Here $(p_t - A p)_a$, denotes the absolutely continuous part of the measure $p_t - A p$, $\Phi : X_q \to \bar{R}$ is defined by

$$\Phi(u) = \int_\Sigma g(\sigma, u(\sigma, t))\, d\sigma\, dt,\quad u \in X_q \tag{3.16}$$

and $\partial p/\partial \nu\, (u^* - v)$ denotes the value of $\partial p/\partial \nu \in X_q^*$ at $(u^* - v) \in X_q$. The normal derivative $\partial p/\partial \nu$ of p is defined by

$$\int_\Sigma y \frac{\partial p}{\partial \nu}\, d\sigma\, dt = (p_t - A p)(y) + \int_Q p(y_t + A_0 y)\, dx\, dt$$

for all $y \in C^1(\bar{Q})$ such that $y(.,0) = 0$ on Ω. Here $(p_t - A p)(y)$ is the value of the measure $p_t - A p$ at y.

Let us denote by $\bar{\Phi} : X_q \to \bar{R}$ the function

$$\bar{\Phi}(u) = \begin{cases} \Phi(u) & \text{if } u \in U_0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then Eq. (3.15) can be equivalently written as

$$\frac{\partial p}{\partial \nu} \in \partial \bar{\Phi}(u^*),$$

where $\partial \Phi : X_q \to X_q^*$ is the subdifferential of Φ. We recall that if $(\text{int } U_0) \cap D(\Phi)$ or $(\text{int } (D(\Phi)) \cap U_0 \neq \emptyset$ then

$$\partial \bar{\Phi}(u) = \partial \Phi(u) + N(u) \text{ for all } u \in U_0, \tag{3.17}$$

where $N(u)$ is the cone of normals to U_0 at u.

Suppose now that $g(\sigma, u) \in L^1(\Gamma)$ for any $u \in R$. Then $\partial \Phi(u^*) \in L^1(\Sigma)$ (see e.g. [5], Corollary 1) and therefore $\partial \Phi(u^*) = \{w \in L^1(\Sigma);$

$w(\sigma, t) \in \partial g(\sigma, u^*(\sigma, t))$, a.e. $(\sigma, t) \in \Sigma\}$ (∂g is the subdifferential of $g(t, \cdot)$. Then by (3.15) and (3.17) we see that there exists $w^* \in L^1(\Sigma)$ such that

$$w^*(\sigma, t) \in \partial g(t, u^*(\sigma, t)),\ \text{a.e. } (\sigma, t) \in \Sigma, \tag{3.18}$$

$$\left(\frac{\partial p}{\partial \nu} - w^*\right)(u^* - v) \geq 0 \text{ for all } v \in U_0.$$

The latter implies that the distribution $w^* - \partial p/\partial \nu$ is positive on Q. Hence $\partial p/\partial \nu$ is a Radon measure on Q and Eq. (3.15) reduces to

$$\frac{\partial p}{\partial \nu} - w^* \leq 0 \qquad \text{on } \Sigma,$$

$$\frac{\partial p}{\partial \nu} - w^* = 0 \qquad \text{on } \{(\sigma, t) \in \Sigma;\ u^*(\sigma, t) > \psi(\sigma)\},$$

where $w^* \in L^1(\Sigma)$ satisfies (3.18).

It should be noticed that since $y^* \in H^{2,1}(Q)$, Eq. (3.14) can be written as

$$p = 0 \text{ a.e. on } \{(x, t) \in Q;\ A_0 \psi - f \neq 0\}$$

Finally, if $f \in L^q(Q)$ then $y^* \in W_q^{2,1}(Q) \subset C(\bar{Q})$ (see [7], p. 389) Suppose in addition that $\psi \in C(\bar{\Omega})$. Then we shall see later that Eq. (3.12) can be strengthened to

$$(p_t - Ap - y^* + y_d)(y^* - \psi) = 0 \text{ in } \mathcal{D}'(Q) \tag{3.19}$$

and therefore we have

$$p_t - Ap = y^* - y_d \text{ on } \{y^* > \psi\}. \tag{3.12$'$}$$

4. **Proof. of Theorem 1.** We start with the approximating control problem: Minimize

$$\frac{1}{2}\int_Q |y - y_d|^2\, dx dt + \int_\Sigma g_\varepsilon(\sigma, u)\, d\sigma\, dt + \frac{1}{2}\| u - u^* \|_q^2 \tag{4.1}$$

over all $u \in U_0$ and $y \in L^2(Q)$ subject to

$$\begin{aligned} &z_t + Az + \beta^\varepsilon(z + \zeta_u - \psi) = f && \text{a.e. on } Q, \\ &z(0) = 0 && \text{a.e. on } \Omega, \\ &y = z + \zeta_u, \end{aligned} \tag{4.2}$$

where g_ε is given by (1.1), i.e.

$$g_\varepsilon(\sigma, u) = \inf\left\{\frac{|u - v|^2}{2\varepsilon} + g(\sigma, v);\ v \in R\right\}. \tag{4.3}$$

Here ζ_u is the solution to (3.3) and β^ε is the C^∞ real valued function defined by

(4.4) $$\beta^\varepsilon(y) = -\varepsilon^{-1}\int_{-\infty}^{\infty}(y-\varepsilon\theta)^-\rho(\theta)\,d\theta \quad \textit{for all } y\in R.$$

Here ρ is a C_0^∞ function on R such that

support $\rho\subset[-1,1]$, $\int_{-\infty}^{\infty}\rho(\theta)\,d\theta=1$, $\rho\geqslant 0$ and $\rho(0)=\rho(-\theta)$ for $\theta\in R$.

Since β^ε is Lipschitzian, for each $u\in L^2(\Sigma)$ and $\varepsilon>0$, Eq. (4.2) has a unique solution $z\in H^{2,1}(Q)$. On the other hand, for each $u\in X_q$ the boundary value problem (3.3) has a unique solution $\zeta_u\in W_q^{2,1}(Q)$ and the mapping $u\to\zeta_u$ is weakly continuous from X_q to $W_q^{2,1}(Q)$. Using this fact one can prove via a standard argument that for each $\varepsilon>0$, problem (4.1) has at least one solution $(y^\varepsilon, u^\varepsilon)\in L^2(Q)\times X_q$. We shall denote by $z^\varepsilon\in H^{2,1}(Q)$ the solution to Eq. (4.2) corresponding to $u=u^\varepsilon$, i.e., $z^\varepsilon=y^\varepsilon-\zeta_{u^\varepsilon}$. We shall prove that $(y^\varepsilon, u^\varepsilon)\to(y^*, u^*)$ strongly in $L^2(Q)\times L^2(\Sigma)$. To this end we need the following technical lemma

Lemma 1. *Let* $w_\varepsilon\in H^{2,1}(Q)$ *be the solution to*

(4.5) $$(w_\varepsilon)_t + Aw_\varepsilon + \beta^\varepsilon(w_\varepsilon+\zeta_\varepsilon-\psi) = f \text{ on } Q,\qquad w_\varepsilon(0)=0,$$

where, for $\varepsilon\to 0$, $\{\zeta_\varepsilon\}$ *is weakly convergent to* ζ *in* $W_q^{2,1}(Q)$, $q\geqslant 2$. *Suppose that* $\zeta_\varepsilon(0)=y_0$. $y_0\geqslant\psi$ a.e. *on* Ω *and*

(4.6) $$\zeta_\varepsilon(\sigma, t)\geqslant\psi(\sigma) \text{ a.e. } (\sigma, t)\in\Sigma.$$

Then for $\varepsilon\to 0$ *one has*

(4.7) $w_\varepsilon\to z$ *strongly* in $C([0,T]; L^2(\Omega))\cap L^2(0,T; H_0^1(\Omega))$.

and weakly in $H^{2,1}(Q)$,

where $z\in H^{2,1}(Q)$ *is the solution to*

(4.8) $$z_t + Az + \partial\varphi(z+\zeta)\ni f \quad \text{a.e. on } Q,\qquad z(0)=0.$$

Proof. Observe firstly that

(4.9) $$|\beta^\varepsilon(y)+\varepsilon^{-1}y^-|\leqslant C \text{ for all } \varepsilon>0,\ y\in R.$$

Then writing E . (5) as

(4.10) $$(w_\varepsilon)_t + Aw_\varepsilon - \varepsilon^{-1}(w_\varepsilon+\zeta_\varepsilon-\psi)^- = g_\varepsilon,\qquad w_\varepsilon(0)=0,$$

where $g_\varepsilon = f-\varepsilon^{-1}(w_\varepsilon+\zeta_\varepsilon-\psi)^- - \beta^\varepsilon(w_\varepsilon+\zeta_\varepsilon-\psi)$, it follows by (4.6) that condition (2.4) is satisfied with some C_0 independent on ε. (This follows by formula (3.7) using the fact that $\{(\zeta_\varepsilon)_t\}$ is bounded in $L^2(Q)$.) Then by the proof of Proposition 1 it follows that $\{w_\varepsilon\}$ is bounded in $C([0,T]; L^2(\Omega))\cap L^2(0,T; V)$ and $\{Aw_\varepsilon\}$, $\{(w_\varepsilon)_t\}$ are bounded in $L^2(Q)$. Thus extracting a subsequence, if necessary, we may assume that

(4.11) $$w_\varepsilon\to z \text{ weakly in } L^2(0,T; H_0^1(\Omega)\cap H^2(\Omega)),\qquad (w_\varepsilon)_t\to z_t \text{ weakly in } L^2(Q).$$

Furthermore, by the Arzelà-Ascoli theorem $\{w_\varepsilon\}$ is compact in $C([0,T];L^2(\Omega))$ and in $L^2(0,T; H_0^1(\Omega))$ (see [8]). Hence we have

(4.12) $$w_\varepsilon\to z \text{ strongly in } C([0,T]; L^2(\Omega))\cap L^2(0,T; H_0^1(\Omega)).$$

Since $\{\beta^\varepsilon(w_\varepsilon+\zeta_\varepsilon-\psi)\}$ is bounded in $L^2(Q)$ we may also assume that

(4.13) $$\beta^\varepsilon(w_\varepsilon+\zeta_\varepsilon-\psi)\to g \text{ weakly in } L^2(Q).$$

By (4.4) we see that

(4.14) $$\beta^\varepsilon(w_\varepsilon+\zeta_\varepsilon-\psi)(w_\varepsilon+\zeta_\varepsilon-v)\geqslant j^\varepsilon(w_\varepsilon+\zeta_\varepsilon-\psi) - j^\varepsilon(v-\psi) \text{ for all } v\in R$$

where

$$j^\varepsilon(v) = (2\varepsilon)^{-1}\int_{-\infty}^{\infty}((v-\varepsilon\theta)^-)^2\rho(\theta)\,d\theta.$$

Since $H^{2,1}(Q)$ is compact in $L^2(Q)$ we may suppose that $\zeta_\varepsilon\to\zeta$ strongly in $L^2(Q)$. We set

$$\varphi^\varepsilon(y) = \int_Q j^\varepsilon(y-\psi)\,dx\,dt,\quad y\in L^2(Q).$$

It is easy to see that $\varphi^\varepsilon(y)\to\varphi(y)$ for every $y\in L^2(Q)$ where φ is defined by (3.2). Along with (4.12), (4.13) and (4.14) the latter implies

$$\int_Q g(z+\zeta-v)\,dx\,dt\geqslant\varphi(z+\zeta)-\varphi(v)\quad\forall v\in L^2(Q).$$

Hence $g\in\partial\varphi(z+\zeta)$ as claimed.

Remark 1. If $f\in L^q(Q)$ then $w_\varepsilon\in W_q^{2,1}(Q)$ and arguing as in [12], it follows that instead of (4.7) we have

$$w_\varepsilon\to z \text{ weakly in } W_q^{2,1}(Q).$$

In particular it follows that if $q>(N+2)/2$, then

$$w_\varepsilon+\zeta_\varepsilon\to y = z+\zeta \text{ in } C(\bar{Q}).$$

Lemma 2. *For* $\varepsilon\to 0$ *one has*

(4.15) $$u^\varepsilon\to u^* \quad \text{strongly in } X_q.$$

(4.16) $y^\varepsilon \to y^*$ strongly in $L^2(Q)$ and weakly in $H^{2,1}(Q)$.

Proof. We notice the inequality

$$\frac{1}{2}\int_Q |y^\varepsilon - y_d|^2\,dx\,dt + \int_\Sigma g_\varepsilon(\sigma, u^\varepsilon)\,d\sigma\,dt + \frac{1}{2}\|u^\varepsilon - u^*\|_q^2 \leq \tag{4.17}$$

$$\leq \frac{1}{2}\int_Q |\tilde{y}^\varepsilon - y_d|^2\,dx\,dt + \int_\Sigma g(\sigma, u^*)\,d\sigma\,dt,$$

where $\tilde{y}^\varepsilon = \tilde{z}^\varepsilon + \zeta_{u^*}$ and $\tilde{z}^\varepsilon$ is the solution to

$$(\tilde{z}^\varepsilon)_t + A\tilde{z}^\varepsilon + \beta^\varepsilon(\tilde{z}^\varepsilon + \zeta_{u^*} - \psi) = f \text{ on } Q,$$

$$\tilde{z}^\varepsilon(0) = 0.$$

By Lemma 1, we know that for $\varepsilon \to 0$,

(4.18) $\tilde{z}^\varepsilon \to z^* = y^* - \zeta_{u^*}$ strongly in $L^2(Q)$ and weakly in $H^{2,1}(Q)$.

On the other hand by condition (iii) (part (3.9)) it follows that

$$\int_\Sigma g_\varepsilon(\sigma, u^\varepsilon)\,d\sigma\,dt \geq -C\|u^\varepsilon\|^2_{L^2(\Sigma)} \text{ for all } \varepsilon > 0,$$

where C is some positive constant independent of ε. Along with (4.17) the latter implies that $\{u^\varepsilon\}$ remain in a bounded subset of X_q. Since, the mapping $u \to \zeta_u$ is weakly continuous from X_q to $W_q^{2,1}(Q)$ we may extract a subsequence convergent to zero, again denoted $\{\varepsilon\}$ such that

$$\begin{cases} u^\varepsilon \to \tilde{u}^* & \text{weakly in } X_q, \\ \zeta_{u^\varepsilon} \to \zeta_{\tilde{u}^*} & \text{weakly in } W_q^{2,1}(Q). \end{cases} \tag{4.19}$$

Since all conditions of Lemma 1 are satisfied, we may infer that

$$\tilde{y}^\varepsilon \to \tilde{y}^* \quad \text{strongly in } L^2(Q),$$

where $\tilde{y}^*$ is the solution to Eq. (3.1) where $\tilde{u} = u^*$. Next by assumptions (i), (ii) we know that the function $u \to \int_\Sigma g(\sigma, u)\,d\sigma\,dt$ is convex and lower semicontinuous on $L^2(\Sigma)$ and consequently on X_q. Then by (1.2), (1.3) and (4.19) it follows that

$$\liminf_{\varepsilon\to 0}\left\{\int_\Sigma g_\varepsilon(\sigma, u^\varepsilon)\,d\sigma\,dt + \frac{1}{2}\|u^\varepsilon - u^*\|_q^2\right\} \geq$$

$$\geq \int_\Sigma g(\sigma, \tilde{u})\,d\sigma\,dt + \frac{1}{2}\|\tilde{u}^* - u^*\|_q^2 \tag{4.20}$$

(for details see [3], p. 230). Since (u^*, y^*) is a minimum point for functional (3.8) we deduce by (4.17) that $\tilde{u}^* = u^*$ and $\tilde{y}^* = y^*$ thereby completing the proof of the lemma. As noticed earlier, the function $v \to g_\varepsilon(t, v)$ is differentiable on R. Our assumptions on g along with relations (1.3) imply that $\nabla g_\varepsilon(t, u) \in L^2(\Sigma)$ for all $u \in L^2(\Sigma)$.

Since the space X_q is uniformly convex and β^ε is differentiable we see that the functional (4.1) (where y is defined by (4.2)) is Fréchet differentiable on X_q. Inasmuch as u^ε is a minimum point of this functional on U_0, we deduce by a standard procedure, the existence of $p_\varepsilon \in H^{2,1}(Q) \cap L^2(0,T; H_0^1(\Omega))$ satisfying together with y^ε and u^ε the system

$$\begin{aligned} &(p_\varepsilon)_t - Ap_\varepsilon - (\beta^\varepsilon)'(y^\varepsilon - \psi)p_\varepsilon = y^\varepsilon - y_d \text{ on } Q, \\ &p_\varepsilon(x, T) = 0 \qquad x \in \Omega, \end{aligned} \tag{4.21}$$

$$\frac{\partial p_\varepsilon}{\partial \nu} = \nabla g_\varepsilon(\sigma, u^\varepsilon) + F_q(u^\varepsilon - u) + \gamma_\varepsilon \quad \text{a.e. on } \Sigma, \tag{4.22}$$

where $F_q : X_q \to X_q^*$ is the duality mapping of the space X_q and $\gamma_\varepsilon \in X$ belongs to the cone of normals to U_0 at u^ε. In other words,

$$\gamma_\varepsilon(u^\varepsilon - v) \geq 0 \text{ for all } v \in U_0. \tag{4.23}$$

We notice that since $p_\varepsilon \in H^{2,1}(Q)$, $\partial p/\partial\nu$ is well defined and belongs to $H^{1/2,3/4}(\Sigma)$ (see [9], p. 9).

Now we multiply Eq. (4.21) by p_ε and integrate over Q. Taking into account the fact that $(\beta^\varepsilon)' \geq 0$, we get the estimate

$$\|p_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^t \|p_\varepsilon(s)\|^2_{H_0^1(\Omega)}\,ds \leq C \text{ for } t \in [0, T]. \tag{4.24}$$

Next we multiply Eq. (4.21) by sgn p_ε (more precisely by a C^∞-approximation of sgn p_ε) and integrate over Q. We get

$$\int_Q |(\beta^\varepsilon)'(y^\varepsilon - \psi)p_\varepsilon|\,dx\,dt \leq C \text{ for all } \varepsilon > 0. \tag{4.25}$$

In particular, this implies that $\{(p_\varepsilon)_t - Ap_\varepsilon\}$ is bounded in $L^1(Q)$. Since $\{Ap_\varepsilon\}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$ we deduce that $\{(p_\varepsilon)_t\}$ is bounded in $L^1(0, T; L^1(\Omega) + H^{-1}(\Omega))$ and therefore it is bounded in some $L^1(0, T; H^{-s}(\Omega))$ with a sufficiently large s. Then arguing as in [2] (Lemma 7) we deduce that $\{p_\varepsilon\}$ is a precompact subset of $L^2(Q)$. Thus extracting a subsequence if necessary, we may assume that there exists $p \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \cap C(0, T; L^1(\Omega) + H^{-1}(\Omega))$ such that

$$\begin{aligned} p_\varepsilon \to p \text{ } &\textit{strongly in } L^2(Q) \textit{ and weakly in } L^2(0, T; H_0^1(\Omega)) \cap \\ &\cap L^\infty(0, T; L^2(\Omega)). \end{aligned} \tag{4.26}$$

Next, by (4.25) it follows the existence of a bounded measure $\mu_p \in M(Q)$ such that

(4.27) $$(\beta^\varepsilon)'(y^\varepsilon - \psi)\, p_\varepsilon \to \mu_p \text{ weak-star in } M(Q)$$

(μ_p can be also regarded as an element of the dual space $L^\infty(Q))^*$ of $L^\infty(Q)$.) Then passing to limit in Eq. (4.21) we see that p satisfies in the sense of distributions the equation

(4.28) $$\begin{aligned} &p_t - A\, p - \mu_p = y^* - y_d && \text{on } Q,\\ &p(\cdot, T) = 0 && \text{on } \Omega. \end{aligned}$$

In other words,

(4.29) $$\int_Q p\, \chi_t\, dx\, dt + \int_0^T a(\chi, p)\, dt + \mu_p(\chi) = -\int_Q (y^* - y_d)\, \chi\, dx\, dt$$

for all $\chi \in C^1(\bar{Q}) \cap L^2(0, T; H_0^1(\Omega))$ such that $\chi(0, x) = 0$ for $x \in \Omega$.

On the other hand, by definition of β^ε we see that

(4.30) $$\begin{aligned} &|p_\varepsilon \beta^\varepsilon(y^\varepsilon - \psi) - (\beta^\varepsilon)'(y^\varepsilon - \psi)\, p_\varepsilon(y^\varepsilon - \psi)| =\\ &= |p_\varepsilon| \int_{\varepsilon^{-1}(y^\varepsilon - \psi)}^{1} \rho(\theta)\, d\theta \leqslant \varepsilon\, |(\beta^\varepsilon)'(y^\varepsilon - \psi) p_\varepsilon|. \end{aligned}$$

This yields

(4.31) $$\begin{aligned} &|p_\varepsilon \beta^\varepsilon(y^\varepsilon - \psi)| \leqslant 2\varepsilon\, |(\beta^\varepsilon)'(y^\varepsilon - \psi)\, p_\varepsilon|\, (\lambda_\varepsilon +\\ &+ \varepsilon^{-1}|y^\varepsilon - \psi|\, \xi_\varepsilon) \text{ a.e. on } Q, \end{aligned}$$

where

$$\lambda_\varepsilon(x, t) = \begin{cases} 0 & \text{if } |y^\varepsilon(x, t) - \psi(x)| \geqslant \varepsilon,\\ 1 & \text{if } |y^\varepsilon(x, t) - \psi(x)| > \varepsilon, \end{cases}$$

and

$$\xi_\varepsilon(x, t) = \begin{cases} 0 & \text{if } y^\varepsilon(x, t) - \psi(x) \geqslant -\varepsilon,\\ 1 & \text{if } y^\varepsilon(x, t) - \psi(x) \leqslant -\varepsilon. \end{cases}$$

Since $\{(\beta^\varepsilon)'(y^\varepsilon - \psi)\, p_\varepsilon\}$ is bounded in $L^1(Q)$ and $\varepsilon^{-1}\xi_\varepsilon(y^\varepsilon - \psi) = \xi_\varepsilon \beta^\varepsilon(y^\varepsilon - \psi)$ is bounded in $L^2(Q)$ we see by (4.31) that on some subsequence $\varepsilon \to 0$, we have

(4.32) $$p_\varepsilon \beta^\varepsilon(y^\varepsilon - \psi) \to 0 \text{ a.e. on } Q.$$

On the other hand, it follows by Lemma 2 that

$$\beta^\varepsilon(y^\varepsilon - \psi) \to f - y_t^* - A_0\, y^* \text{ weakly in } L^2(Q).$$

Along with (4.26) and (4.32) the latter yields

(4.33) $$p(y_t^* + A_0\, y^* - f) = 0 \quad \text{a.e. on } Q.$$

Now by (4.31) it follows that

(4.34) $$(\beta^\varepsilon)'(y^\varepsilon - \psi)\, p_\varepsilon\, (y^\varepsilon - \psi) \to 0 \text{ in } L^1(Q).$$

On the other hand by the Egorov theorem and Lemma 2, for each $\eta > 0$

there exists a measurable subset E_η of Q such that $m(Q \setminus E_\eta) \leqslant \eta$, $y^* - \psi \in L^\infty(E_\eta)$ and

(4.35) $$y^\varepsilon - \psi \to y^* - \psi \text{ uniformly on } E_\eta,$$

(m denotes the Lebesgue measure). Let $\mu_p = (\mu_p)_a + (\mu_p)_s$ be the Lebesgue decomposition of μ_p (which will be regarded as an element of $(L^\infty(Q))^*$) into the singular part $(\mu_p)_s$ and the absolutely continuous part $(\mu_p)_a \in L^1(Q)$. By (4.27) ~ (4.34) and (4.35) it follows that

$$\mu_p(y^* - \psi) = 0 \text{ on } E_\eta,$$

and therefore

(4.36) $$\int_{E_\eta} (y^* - \psi)(\mu_p)_a\, \chi\, dx\, dt + (\mu_p)_s\, ((y^* - \psi)\, \chi) = 0.$$

for all $\chi \in L^\infty(Q)$ which vanish outside E_η. On the other hand, the „singularity" of $(\mu_p)_s$ implies the existence of a nondecreasing sequence of measurable subsets Q_k of Q such that $m(Q \setminus Q_k) \leqslant 1/k$ and $(\mu_p)_s = 0$ on $L^\infty(Q_k)$. By (4.36) we see that

$$\int_{E_\eta \cap Q_k} (y^* - \psi)(\mu_p)_a\, \chi\, dx\, dt = 0,$$

for all $\chi \in L^\infty(Q)$ which vanish outside of $Q_k \cap E_\eta$. Hence

$$(y^* - \psi)(\mu_p)_a = 0 \text{ a.e. on } Q,$$

and by (4.28) we may conclude that

(4.37) $$(p_t - A_0 p)_a = y^* - y_d \text{ a.e. on } \{y^* > \psi\}.$$

By Remark 1 and the proof of Proposition 2 it follows that if $f \in L^q(Q)$ where $q > (N + 2)/2$ then $y^\varepsilon \in C(\bar{Q})$ and $y^\varepsilon \to y^*$ uniformly on $\bar{Q}$. Thus if $\psi \in C(\bar{\Omega})$ we have

$$y^\varepsilon - \psi \to y^* - \psi \text{ in } C(\bar{Q}).$$

Along with (4.27) and (4.34) the latter implies that $\mu_p(y^* - \psi) \in M(Q)$ and

$$\mu_p(y^* - \psi) = 0 \text{ on } Q,$$

thereby proving (3.19).

So far we have shown that the function p defined by (4.26) satisfies relations (3.12), (3.13) and (3.14) in Theorem 1. Turning to the proof of (3.15) we notice that by Green's formula we have

(4.38) $$\int_Q (((p_\varepsilon)_t - A_0 p_\varepsilon)\, \chi + (\chi_t + A_0\, \chi) p_\varepsilon\, dx\, dt = \int_\Sigma \frac{\partial p_\varepsilon}{\partial \nu}\, \chi_\varepsilon\, d\sigma\, dt$$

for all $\chi \in W_q^{2,1}(Q)$ such that $\chi(x, 0) = 0$, a.e. $x \in \Omega$.

Let $\chi \in W_q^{2,1}(Q)$ be the solution to

$$\varkappa_t + A_0 \varkappa = 0 \quad \text{on } Q,$$
$$\varkappa = \varsigma \quad \text{on } \Sigma,$$
$$\varkappa(.,0) = 0 \quad \text{on } \Omega,$$

where ς is an arbitrary element of X_q. Since $q > (N+2)/2$ it follows that $\varkappa \in C(\bar{Q})$ and $\|\varkappa\|_{C(\bar{Q})} \leq C\|\varsigma\|_q$. Then we deduce by (4.21), (4.25) and (4.38),

$$\left|\int_\Sigma \frac{\partial p_\varepsilon}{\partial \nu} \varsigma \, d\sigma \, dt\right| \leq \int_Q (|(\beta^\varepsilon)'(y^\varepsilon - \psi) p_\varepsilon| + |f|) |\varkappa| \, dx dt \leq C\|\varsigma\|_q.$$

Hence $\{\partial p_\varepsilon/\partial \nu\}$ is bounded in X_q^*. Thus extracting further subsequence we have

(4.39) $$\frac{\partial p_\varepsilon}{\partial \nu} \to \pi_p \text{ weakly in } X_q^*.$$

On the other hand, by (4.27) and (4.38) we see that

(4.39)' $$\pi_p(\varkappa) = \mu_p(\varkappa) + \int_Q p(\varkappa_t + A_0\varkappa) \, dx \, dt + \int_Q (y^* - y_d) \varkappa \, dx dt,$$

for all $\varkappa \in C^1(\bar{Q})$ such that $\varkappa(x, 0) = 0$.
(We have denoted again by $\varkappa$ the „trace" of $\varkappa$ on Σ.) Formula (4.39)' shows that $\pi_p = \partial p/\partial \nu$.

Next, observe by (4.15) that $F_\varepsilon(u^\varepsilon - u^*) \to 0$ strongly in X_q^* and by (4.20)

$$\liminf_{\varepsilon \to 0} \int_\Sigma g_\varepsilon(\sigma, u^\varepsilon) \, d\sigma \, dt \geq \int_\Sigma g(\sigma, u^*) \, d\sigma \, dt.$$

Then by (4.22), (4.23) and (4.39) it follows that

$$\frac{\partial p}{\partial \nu}(u^* - v) \geq \int_\Sigma (g(\sigma, u^*) - g(\sigma, v)) \, d\sigma \, dt \text{ for all } v \in U_0.$$

This completes the proof.

5. Mixed Neumann problem with unilateral constraint. We shall consider here the boundary control system

(5.1) $$\begin{aligned} &y_t + A_0 y + \partial\varphi(y) \ni f && \text{on } Q, \\ &\alpha y + \frac{\partial y}{\partial \nu} = u && \text{on } \Sigma, \\ &y(.,0) = y_0 && \text{on } \Omega. \end{aligned}$$

Here $\alpha \geq 0$, φ is given by (3.2) where $\psi \in H^2(\Omega)$ and $y_0 \in H^1(\Omega)$, $f \in L^2(Q)$.

As seen in Example 2 we may put Eq. (5.1) into the form (2.1) where $H = L^2(\Omega)$, $V = H^1(\Omega)$, $U = L^2(\Gamma)$ and A is defined by (2.16).

We shall denote by Y the space

(5.2) $$Y = H^{1/2,1/4}(\Sigma);$$

and by Y^* the dual space of Y.

For every $u \in Y$ the solution ζ_u to

(5.3) $$\begin{aligned} &(\zeta_u)_t + A\zeta_u = 0 && \text{on } Q, \\ &\alpha\zeta_u + \frac{\partial \zeta_u}{\partial \nu} = u && \text{on } \Sigma, \\ &\zeta_u(., 0) = y_0 && \text{on } \Omega, \end{aligned}$$

lies in $H^{2,1}(Q)$ (see e.g. [9]). On the other hand, it follows after some calculations involving Green's formula and Eqs. (2.16), (2.13),

$$(Az, \partial\varphi_\varepsilon(z + \zeta_u)) \geq \int_\Omega ((\zeta_u)_t + A_0\psi) \, \partial\varphi_\varepsilon(z + \zeta_u) \, dx -$$
$$- \int_\Gamma \left(u - \alpha(z + \zeta_u) - \frac{\partial\psi}{\partial\nu}\right) \partial\varphi_\varepsilon(z + \zeta_u) \, dx, \text{ a.e. } t \in]0, T[$$

for all $z \in D(A) = \{z \in H^2(\Omega)\,;\ \alpha z + \partial z/\partial\nu = 0 \text{ on } \Gamma\}$.

The latter implies that condition (2.4) is in particular satisfied for all $u \in U_a$ where

(5.4) $$U_a = \left\{u \in Y\,;\ u(\cdot, 0) = \alpha y_0 + \frac{\partial y_0}{\partial \nu},\ u \geq \alpha\psi + \frac{\partial\psi}{\partial\nu} \text{ a.e on } \Sigma\right\}.$$

Hence for every $u \in U_a$ and $y_0 \in H^1(\Omega)$ satisfying

(5.5) $$y_0(x) \geq \psi(x) \text{ a.e. } x \in \Omega,$$

Eq. (5.1) has a unique strong solution $y = \zeta_u + z \in H^{2,1}(Q)$.

The optimal control problem we consider here is: Minimize

(5.6) $$\frac{1}{2}\int_Q |y - y_d|^2 \, dx \, dt + \int_\Sigma g(\sigma, u) \, d\sigma \, dt$$

over all $u \in U_a$ and $y \in H^{2,1}(Q)$ subject to state system (5.1).

Here $y_d \in L^2(Q)$ is fixed and function g satisfies assumptions (i), (ii), (iii) in Section 2.

Theorem 2. *Let (y^*, u^*) be an optimal pair of control problem (5.6). Then there exists a function $p \in C(0, T; L^1(\Omega) + (H^1(\Omega))') \cap L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, such that*

(5.7) $$p_t - Ap \in M(Q),$$

(5.8) $$(p_t - Ap)_a = y^* - y_d \quad \text{a.e. on } \{y^* > \psi\},$$

(5.9) $$p(\cdot, T) = 0 \quad on\ \Omega,$$

(5.10) $$p(y_t^* + A_0 y^* - f) = 0 \quad a.e.\ on\ Q,$$

(5.11) $$\int_\Sigma p(u^* - v)\, d\sigma\, dt \geq \Phi(u^*) - \Phi(v)\ for\ all\ v \in U_a,$$

where $\Phi(u) = \int_\Sigma g(\sigma, u)\, d\sigma\, dt$ for $u \in Y$.

We recall that the operator $A : H^1(\Omega) \to (H^1(\Omega))'$ is defined by formula

$$(Ap, \chi) = a(\chi, p) + \alpha \int_\Gamma p\chi\, d\sigma \text{ for all } p, \chi \in H^1(\Omega),$$

and the Neumann boundary condition $\alpha p + \partial p / \partial \nu = 0$ on Γ, is implicitely incorporated.

Proof. Since the proof is entirely similar to that of Theorem 1, it will be only outlined. We start with the approximating problem

Minimize

(5.12) $$\frac{1}{2} \int_Q |y - y_d|^2\, dx\, dt + \int_\Sigma g_\varepsilon(\sigma, u)\, d\sigma\, dt + \frac{1}{2} \| u^* - u \|_Y^2$$

over all $u \in U_a$ and $y \in L^2(Q)$ subject to state system (4.2) where ζ_u is the solution to Eq. (5.3). Let $(y^\varepsilon, u^\varepsilon)$ be a solution of the approximating problem. By the same reasoning as in the proof of Lemma 2, we infer that

(5.13) $$u^\varepsilon \to u^* \quad \text{strongly in } Y,$$

(5.14) $$y^\varepsilon \to y^* \quad \text{strongly in } L^2(Q) \text{ and weakly in } H^{2,1}(Q) \text{ for } \varepsilon \to 0.$$

Next, one finds $p_\varepsilon \in H^{2,1}(Q)$ solution to

$$(p_\varepsilon)_t - A_0 p_\varepsilon - (\beta^\varepsilon)'(y^\varepsilon - \psi)\, p_\varepsilon = y^\varepsilon - y_d \quad \text{a.e. on } Q,$$

$$p_\varepsilon(x, T) = 0 \quad \text{a.e. } x \in \Omega,$$

$$\frac{\partial p_\varepsilon}{\partial \nu} + \alpha p_\varepsilon = 0 \quad \text{a.e. on } \Sigma,$$

and satisfying

(5.15) $$p_\varepsilon(\sigma, t) = \nabla g_\varepsilon(\sigma, u^\varepsilon(\sigma, t)) + F^1(u^\varepsilon - u^*) + \delta_\varepsilon \quad \text{a.e. } (\sigma, t) \in \Sigma.$$

Here $F^1 : Y \to Y^*$ is the duality mapping of the space Y and δ_ε belongs to the cone of normals to U_a at u^ε.

Proceeding as in the proof of Theorem 1 one finds the following estimates

(5.16) $$\| p_\varepsilon(t) \|_{L^2(\Omega)} + \int_0^T \| p_\varepsilon(t) \|^2_{H^1(\Omega)} \leq C \text{ for } t \in [0, T],$$

(5.17) $$\int_Q | (\beta^\varepsilon)'(y^\varepsilon - \psi)\, p_\varepsilon |\, dx\, dt \leq C.$$

Thus we may infer that there exists $p \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ which is continuous from $[0, T]$ to $L^1(\Omega) + (H^1(\Omega))'$, and such that on some subsequence $\varepsilon \to 0$ we have

(5.18) $$p_\varepsilon \to p \text{ strongly in } L^2(Q) \text{ and weakly in } L^2(0, T; H^1(\Omega)).$$

By the trace theorem it follows that

$$p_\varepsilon \to p \text{ weakly in } H^{1/2, 0}(\Sigma) \subset L^2(\Sigma).$$

and by (5.13), (5.15) we deduce that

$$\int_\Sigma p(u^* - v)\, d\sigma\, dt \geq \Phi(u^*) - \Phi(v) \quad \forall v \in U_a$$

where Φ is defined by formula (3.16). By (5.17) we see that

(5.19) $$(\beta^\varepsilon)'(y^\varepsilon - \psi)\, p_\varepsilon \to \mu_p \text{ weak star in } M(Q)$$

where μ_p is a bounded measure on Q.

Hence p satisfies, in the sense of distributions, the equation

$$p_t - A_0 p - \mu_p = y^* - y_d.$$

It is also obvious that $p(\cdot, T) = 0$ (which makes sense as an element of $L^1(\Omega) + (H^1(\Omega))'$. Eqs. (5.8) and (5.9) follow by the same device as in the proof of Theorem 1.

REFERENCE

1. Barbu, V. — *Necessary conditions for nonconvex distributed control problems governed by elliptic variational inequalities*, J. Math. Anal. Appl. (to appear).
2. Barbu, V. — *Necessary conditions for distributed control problems governed by parabolic variational inequalities*, SIAM J. Control (to appear).
3. Barbu, V. and Precupanu, Th. — *Convexity and Optimization in Banach Spaces*, Sijthoff & Noordhoof & Publishing House of Romanian Academy, 1978.
4. Brézis, H. — *Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*, Math. Studies, 5, North-Holland, 1973.
5. Brézis H. — *Integrales convexes dans les espaces de Sobolev*, Israel J. Math. 13 (1972), 9—23.
6. Brézis, H. — *Problèmes unilatéraux*, J. Math Pures Appl. 51 (1972), 1—164.
7. Ladyzenskaya, O. A., Solonnikov, V. A. Uraltzeva, N. N. — *Linear and Quasilinear Equations of Parabolic Type*, (russian) Moskow 1967.
8. Lions, J. L. — *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod-Gauthier-Villars, Paris 1969.
9. Lions J. L. and Magenes, E. — *Non-Homogeneous Boundary Value Problems and Applications*, volume II, Springer-Verlag Berlin, Heidelberg, New York, 1972

10. Mignot, F. — *Contrôle dans les inéquations variationnelles elliptiques*, J. Functional Analysis 22 (1976), 130—185.
11. Mignot F. and Puel J. P. — *Inéquations d'évolution paraboliques avec convexes dépendent du temps. Applications aux inéquations quasi-variationnelles d'évolution*, Arch. Rat. Mech. Anal. 61 (1977), 59—91.
12. Moreno, C. Saguez, Ch. — *Dependence par rapport aux données de la frontière libre associée à certaines inequations variationnelles d'evolution*, Rapport Laboria, Mai 1978.
13. Rockafellar, R.T. — *Integral functional, normal integrands and measurable selections*, in *Nonlinear Operators and the Calculus of Variations Bruxelles 1975*, Lecture Notes in Mathematics, J. P. Gossez ed. Springer-Verlag 1976.

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OSCILLATION THEOREMS FOR SECOND ORDER ORDINARY DIFFERENTIAL INEQUALITIES AND EQUATIONS WITH ALTERNATING COEFFICIENTS

BY

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1. The oscillation problem for differential equations with alternating coefficients is of special importance in applications. Therefore, there has been an increassing interest in the study of the oscillatory behavior of solutions of such differential equations. Among numerous papers dealing with this problem, we choose to refer to [1]—[12]. The equations considered in these papers are the following ones :

$$x''(t) + p(t)\, x(t) = 0, \quad t \geqq t_0. \tag{1}$$

$$x''(t) + p(t) f[x(t)] = 0, \quad t \geqq t_0. \tag{2}$$

$$[r(t)x'(t)]' + p(t) f[x(t)] = 0, \quad t \geqq t_0. \tag{3}$$

where the real-valued functions p, f and r are subject to appropriate assumptions.

The results obtained there establish sufficient conditions in order that all solutions be oscillatory, and they can be classified in two categories. The first category (cf. [1], [4], [7]—[10] and [12]) concerns sufficient conditions of the form

$$\int^{\infty} \varphi(t)\, p(t)\, dt = \infty.$$

while the second one (cf. [2], [3], [5], [6] and [11]) concerns conditions involving the averaging of the function

$$\int_{T_0}^{t} \varphi(s)\, p(s)\, ds, \quad T_0 \geqq \max\{t_0, 0\},$$

where, in both cases, φ is an appropriate function.

For the equation (1), Wintner (cf. [11]) introduced the averaging condition

$$\lim_{t\to\infty} \frac{1}{t} \int_{T_0}^{t} ds \int_{T_0}^{s} p(\tau)\, d\tau = \infty.$$