est bien défini, il est continu et il applique l'ensemble $\mathbf{K} \cap \{x \mid p(x) \leqslant c\}$ dans un ensemble compact. Le théorème Tichonoff-Singbal implique qu'il existe
$x_c \leqslant \mathbf{K} \cap \{x \mid p(x) \leqslant c\}$ tel que : $x_c = g_c(x_c)$. Nous avons : $p(x_c) = c$ et si nous posons : $\lambda_c = p(f_c(x_c))/c$ nous obtenons, $f(x_c) = \lambda_c x_c$ ce qui démontre le théorème.

Remarque. Un cas particulier du théorème 5 est démontré par H. Schaefer dans l'ouvrage [12].

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OPTIMAL IMPULSIVE CONTROL OF LINEAR EVOLUTIONARY PROCESSES WITH QUADRATIC PERFORMANCE INDEX IN HILBERT SPACES

BY

F. IACOB

§1. Introduction. One of the topics of contemporary interest in the control theory is that related to the impulsive control problems. These problems arise quite naturally from a number of mathematical models describing various physical, economic or biological processes in which certain modifications may often be well approximated by instantaneous changes of the system.

During recent years, some successful studies, investigating various aspects of particular impulsive control cases, both from a theoretical and a computational point of view, have been developed. The works of J. G. Pierce and A. Schumitzky [13, 14] are relevant. A good deal of attention has been given to this topic by J. L. Lions and his collaborators [10], who introduced the technique of quasi-variational inequalities for solving such problems. Finally, J. Noailles [11, 12] has very recently reported a couple of results relating to impulsive control in the area of evolution equations.

The purpose of this paper is to single out a particular type of problem — the linear impulsive regulator problem. Using a "semigroup and evolution operator" approach, an analytical treatment of a quadratic cost-linear dynamics impulsive control process is made. Concerning the existence of an optimal control law, the appropriate operator Riccati equation (in a mild version) is considered and studied.

The paper is divided into eight sections. In § 2, notations, definitions and some results which will be used in the succeding development are presented. Section 3 contains a number of models involving linear impulsive control problems with quadratic cost functional. The setup of the problem is described in detail in § 4 and the adequate optimality theorem is given in § 5. Section 6 deals with the synthesis of optimal control for the considered problem, introducing a suitable „Riccati equation" formulation for the feedback law. The next section is concerned with the respective operator equation. Finally, § 8 presents the linear hereditary systems case to which the general results are immediately applicable.

§2. Preliminaries. In order to establish the setting for the sequel, certain pertinent notations, definitions and results are presented in this section.

To begin with, let H and U be real Hilbert spaces with norms $|\cdot|_H$ $|\cdot|_U$ and inner products $<.,.>$ and $(.,.)$, respectively.

Let $\mathfrak{L}(H, U)$ denote the real Banach space of all continuous linear operators $L: H \to U$, endowed with the natural norm $\|L\|$; in particular $\mathfrak{L}(H)$ will stand for $\mathfrak{L}(H, H)$. The adjoint of L in $\mathfrak{L}(H, U)$ will be denoted by $L^* \in \mathfrak{L}(U, H)$. $L \in \mathfrak{L}(H)$ is said to be self-adjoint if $L = L^*$. A self-adjoint operator L will be said to be positive if $<Lh, h> \geq 0$ for all $h \in H$. The class of all positive self-adjoint operators in $\mathfrak{L}(H)$ will be called $\mathcal{K}(H)$. By I is denoted the unit operator on any space.

Relating to the elements of $\mathcal{K}(H)$, the following result will be needed:

Proposition 2.1. *If L_1 and L_2 are two arbitrary operators in $\mathcal{K}(H)$, then $I + L_1 L_2$ is invertible in H and moreover $L_2(I + L_1 L_2)^{-1}$ belongs to $\mathcal{K}(H)$.*

The proof is mostly similar to that corresponding to Lemma 1.2 in [15] and therefore will be omitted here (see also [8]).

Next, let $\mathbf{R}$ be the field of all real numbers and let $\mathbf{R}^n$ be the Euclidian real Hilbert space of finite dimension n. The symbol $\mathfrak{L}_{kn}$ will be used (instead of $\mathfrak{L}(\mathbf{R}^n, \mathbf{R}^k)$) for the collection of all real $k \times n$ matrices with a suitable matrix norm.

For $a < b$ in $\mathbf{R}$, the symbol $C(a, b; H)$ denotes the usual Banach space of continuous functions from the closed interval $[a, b]$ to H, and $\mathfrak{L}^p(a, b; H)$ means the vector space of all m-measurable (m, the Lebesgue measure on $\mathbf{R}$) maps $[a, b] \to H$ which are p-integrable, $1 \leq p < \infty$, or essentially bounded, $p = \infty$; the natural Banach space associated with $\mathfrak{L}^p(a, b; H)$ is denoted by $L^p(a, b; H)$ and the corresponding L^p-norm by $\|\cdot\|_p$.

Let the positive real number r be given. Then by $M^2(-r, 0; H)$ will be denoted the quotient space of $\mathfrak{L}^2(-r, 0; H)$ by the linear subspace of all f such that $\|f\|_{M^2} = 0$, where

$$\|f\|_{M^2} = (|f(0)|_H^2 + \|f\|_2^2)^{1/2}. \tag{2.1}$$

Endowed with the norm (2.1) and the inner product

$$(f, g)_{M^2} = <f(0), g(0)> + \int_{-r}^{0} <f(t), g(t)> dt, \tag{2.2}$$

$M^2(-r, 0; H)$ is a Hilbert space isometrically isomorphic to $H \times L^2(-r, 0; H)$ endowed with norm

$$\|[h^0, h^1]\| = \left(|h^0|_H^2 + \int_{-r}^{0} |h^1(t)|_H^2 dt\right)^{1/2} \tag{2.3}$$

and inner product

$$([h^0, h^1], [d^0, d^1]) = <h^0, d^0> + \int_{-r}^{0} <h^1(t), d^1(t)> dt. \tag{2.4}$$

By this isomorphism, the correspondent of $h \in M^2(-r, 0; H)$ is $[h(0), h]$ in $H \times L^2(-r, 0; H)$. For simplicity, h will be often identified with

the pair $[h^0, h^1] = [h(0), h]$. (Concerning the motivation in introducing $M^2(-r, 0; H)$ and additional details regarding this space, see [7] and its references).

For an operator $L \in \mathfrak{L}(M^2(-r, 0; H))$, the mentioned isomorphism will be used (as in [7]) to decompose L into a matrix of operators:

$$\begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}, \tag{2.5}$$

where $L_{00} \in \mathfrak{L}(H)$, $L_{01} \in \mathfrak{L}(L^2(-r, 0; H), H)$, $L_{10} \in \mathfrak{L}(H, L^2(-r, 0; H))$ and $L_{11} \in \mathfrak{L}(L^2(-r, 0; H))$.

Now, let $0 < T < +\infty$ and $n \geq 1$ be a fixed real number and an integer, respectively. The symbol $\{t_i\}_{i=0}^{n+1}$ will be used to denote a strict partition of the closed interval $[0, T]$, that is

$$0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T. \tag{2.6}$$

The class of all H-valued functions f on $[0, T]$ which have continuous restriction to $[t_i, t_{i+1}[$ and left-hand limit in t_{i+1}, denoted by $f(t_{i+1}^-)$, for every $i \in \{0, 1, \ldots, n\}$, will be conventionally called $C_{res}^-(0, T; H)$.

Let $\{A(t);\ 0 \leq t \leq T\}$ be a family of closed and densely defined linear operators in H, such that the Cauchy problem

$$\frac{dx}{dt}(t) = A(t)\, x(t), \quad s < t < t_{i+1}; \tag{2.7}$$

$$x(s) = h, \tag{2.8}$$

with s arbitrarily fixed in $[t_i, t_{i+1}[$, is well posed for every $h \in H$ and each $i \in \{0, 1, 2, \ldots, n\}$.

Let then $\mathcal{S}$ be a system whose state x, belonging to $C_{res}^-(0, T; H)$, satisfies (2.7), for $s = t_i$, $\forall i \in \{0, 1, \ldots, n\}$, and

$$x(0) = x_0, \tag{2.9}$$

with x_0 given in H.

Definition 2.1. *An impulsive control of $\mathcal{S}$ will be defined as the set of nodes (times) t_i and a set of elements $u_i \in U$, such that*

$$x(t_i) = x(t_i^-) + B_i u_i, \tag{2.10}$$

where $B_i \in \mathfrak{L}(U, H)$, $i = 1, 2, \ldots, n$.

Consequently, the dynamics of such a controlled system is described by the multipoint boundary-value problem

$$\begin{aligned} &\frac{dx}{dt}(t) = A(t)\, x(t), \quad t_i < t < t_{i+1},\ i = 0, 1, \ldots, n, \\ &x(t_i) = x(t_i^-) + B_i u_i, \quad i = 1, 2, \ldots, n, \\ &x(0) = x_0. \end{aligned} \tag{2.11}$$

On this line, an impulsive control problem will consist in the minimization of a given cost functional $F(x, u)$, subject to (2.11) and (possibly) to the other constraints on state $x \in C_{res}^{-}(0, T : H)$ and control $u=(u_1, u_2, \ldots, u_n) \in U^n$.

Such a problem, in which the performance index F is quadratic and the relations (2.11), having also involved a „distributed" control $u_0 \in L^2(0, T : U)$, are the only constraints, will be investigated here (§ 4—§ 8). The study of this particular case is important (see Examples 3.3 and 3.4 in §3), even if the optimization problems often have a limited control actually (as in Examples 3.1 and 3.2).

For some concepts and results pertaining to abstract linear evolution equations, used in the following, see [4] or [5], ch. 1. In the linear hereditary systems case, sufficient conditions for the existence of evolution operators are given by M. C. Delfour and S. K. Mitter (see [7] and its references).

In general, a strong evolution opertor $S(t, s)$, generated by $A(t)$, exists when the Cauchy problem

$$
(2.12) \qquad \begin{aligned} &\frac{dx}{dt}(t) = A(t)\, x(t), \qquad s < t \leqslant T, \\ &x(s) = x_0, \qquad 0 \leqslant s < T \end{aligned}
$$

has a unique continuously differentiable solution, for every $x_0 \in D(A(s))$. Then, the solution $x(t)$ can be expressed as

$$
(2.13) \qquad x(t) = S(t, s)x_0, \qquad \text{for} \qquad (t, s) \in \mathfrak{T},
$$

where

$$
\mathfrak{T} = \{(t, s) \; ; \; 0 \leqslant s \leqslant t \leqslant T\}.
$$

Concerning the inhomogeneous form of (2.12), that is

$$
(2.14) \qquad \frac{dx}{dt}(t) = A(t)x(t) + f(t), \qquad 0 < t \leqslant T, \qquad x(0) = x_0,
$$

it is known that, for sufficiently smooth f (see [4]), (2.14) has the unique continuously differentiable solution

$$
(2.15) \qquad x(t) = S(t, 0)x_0 + \int_0^t S(t, s) f(s)\, ds, \qquad 0 \leqslant t \leqslant T.
$$

If $f: [0, T] \to H$ is merely Bochner integrable, then (2.15) does not represent a solution, but $x(t)$ is still a well-defined H-valued continuous function which is usually called a *mild solution* of (2.14).

Finally, here is a perturbation result which will be used in what follows and whose proof can be found in [5], ch. 1.

Proposition 2.2. *Let $A(t)$ be a closed linear operator on H generating the strong evolution operator $S(t, s)$ with* $\operatorname*{ess\,sup}_{(t,s)\in \mathfrak{T}} \|S(t, s)\| < M_1$. *Let $L : [0, T] \to \mathfrak{L}(H)$ be a strongly measurable map such that* $\operatorname*{ess\,sup}_{t \in [0, T]} \|L(t)\| < M_2$.

Then the integral equation

$$
(2.16) \qquad V(t, s) = S(t, s) + \int_0^t S(t, \tau)\, L(\tau)\, V(\tau, s)\, d\tau, \qquad (t, s) \in \mathfrak{T},
$$

has a unique solution $V(t,s) \in \mathfrak{L}(H)$, which is a mild evolution operator with $\operatorname*{ess\,sup}_{(t,s)\in \mathfrak{T}} \|V(t, s)\| < M_1 e^{M_1 M_2 T}$.

Remark 2.1. $V(t, s)$ is called the mild evolution operator generated by $A(t)+L(t)$. If, in addition, $L(s) : H \to D(A(t))$ for all $s \leqslant t \in [0, T]$ and $\operatorname*{ess\,sup}_{(t,s)\in \mathfrak{T}} |A(t)L(s)h|_H$ exists for all h in H, then $V(t, s)$ is a strong evolution operator.

Such a perturbation result holds also in the time-independent case, when $A(t) \equiv A$ is the generator of a C_0-semigroup and $L(t)$ is required to be strongly continuously differentiable in t.

§3. Examples of quadratic cost-linear impulsive control problems. The abstract theory which will be developed in the next sections is fitted for a wide class of linear evolutionary processes, both finite and infinite-dimensional, including lumped, delay and distributed systems. To motivate the infinite-dimensional setting of such a theory, several simple examples of models involving impulsive control problems are given in this section.

Example 3.1. (*on the drug distribution in the human body*). Among the mathematical models describing the exchanges of drug between various lumped subsections or compartments of the body with, in the simplest case, a homogeneous, time-invariant, linear system of differential equations, it may be considered the Kruger-Thiemer model (presented in [13]. It is assumed that the drug, which is administered orally, is first dissolved into the gastro-intestinal tract, is then absorbed into the so-called apparent volume of distribution (a lumped compartment which accounts for blood, muscle, tissue, etc.), and finally is eliminated from the system by the kidneys.

Let $x_1(t)$ and $x_2(t)$ denote the amount of drug at time t in the gastro-intestinal tract and apparent volume of distribution, respectively. Taking into account a certain delay $r > 0$ and denoting by $c_i (i=1, 2, \ldots, 5)$ the relevant rate constants (with $c_1 \neq c_2$), the dynamical description of this slight extension of the Kruger-Thiemer model is given by

$$
(3.1) \qquad \begin{aligned} &x_1'(t) = -c_1 x_1(t) - c_3 x_1(t-r), \qquad 0 < t < T, \\ &x_2'(t) = -c_2 x_2(t) + c_1 x_1(t) - c_4 x_2(t-r) + c_5 x_1(t-r), \qquad 0 < t < T, \\ &x_1(\theta) = x_2(\theta) = 0 \quad \text{for all} \quad \theta \in [-r, 0]. \end{aligned}
$$

The delay r is supposed to represent the time lag between the moment of prescription and the moment of ingestion of drug.

Assuming that, at discrete moments $0 < t_1 < t_2 < \ldots < t_n < T$, the drug is ingested in amounts $d_1, d_2, \ldots, d_n$, the following conditions are imposed

$$
(3.2) \qquad \begin{aligned} &x_1(t_i) = x_1(t_i^-) + d_i, \\ &x_2(t_i) = x_2(t_i^-), \qquad i = 1, 2, \ldots, n. \end{aligned}
$$

For therapeutic reasons, it may be desired, for example, to maintain the amount of drug in one compartment above a certain positive minimum dosage level l_2, while keeping the drug in the other one below a certain maximum or plateau l_1 ; so the constraints (3.3) are involved.

$$0 < x_1(t) \leqslant l_1, \quad l_2 \leqslant x_2(t), \quad t_1 \leqslant t \leqslant T. \tag{3.3}$$

It is also assumed that only nonnegative amounts of drug can be given, that is

$$d_i \geqslant 0, \quad \text{for every } i \in \{1, 2, \dots, n\}. \tag{3.4}$$

Finally, subject to the above considerations, it is desired to minimize both side efects and the total amount of drug to be used, namely the biological cost function

$$J(d_1, d_2, \dots, d_n) = \frac{1}{2} \sum_{i=1}^{n} d_i^2. \tag{3.5}$$

Thus, the problem which arises in this context is to find inf $J(d_1, \dots, d_n)$ subject to (3.1) through (3.4).

Example 3.2. (*with relation to the kinetics of lead in the human body*). One of pollutants with detrimental effect on health is suspected to be lead. That is why some good attempts were recently made (see [3] and its references) to study the lead pollution and the lead content in animals and man, through the use of so-called compartmental models. Such a model for the kinetics of lead in the human body (based on that one given in [3]), involving some delays and impulsive control, is proposed in this example.

It is known that lead enters the body by food, beverages and by inhalation, reaches the digestive tract and lungs, respectively. The digestive tract is not only fed by dietary lead but also by some lead contained in saliva, gastric secretion, and bile whose portion is proceeding from the tissue compartment. The guts will not absorb all this lead ; part of it will be excreted by the faeces. The lungs do not also absorb the total input of lead, since dust particles containing lead are partly exhaled or excreted by mucus. The total amount of absorbtion enters the blood where the lead is mainly attached to the red cells. From the blood, lead is distributed in the tissues and (slowly) in the bones. Lead is first absorbed by the liver and the kidney, but later, with various time lags, by all parts of the body. There is a continuous exchange (by diffusion) of lead between blood and tissues on the one hand and between blood and bones on the other one. Some lead (from blood compartment) leaves the body by urine and some more lead (from tissue compartment) by hairs, nails, and sweat.

Following [3], five compartments are considered here. The lungs and the digestive tract are chosen as compartments 1 and 2, respectively. Compartment 3 includes the skeleton, and compartment 4 the blood and possibly some soft tissues which exchange rapidly with blood. In the compartment 5 are joined primarily soft tissues. The environment is considered to be compartment 0.

If t is the time variable, then let $x_i(t)$ denote the amount of lead stored in compartment $i (i=1, 2, 3, 4, 5)$. Let also $d_{ij} > 0$ be the delay intervening in the exchange of lead between the compartment i and the compartment j. More exactly, there are taken into account the "delays" $\theta_1 = -d_{14}$, $\theta_2 = -d_{21}$, $\theta_3 = -d_{45}$, $\theta_4 = -d_{42}$, $\theta_5 = -d_{52}$, $\theta_6 = -d_{50}$, $\theta_7 = -d_{40}$, $\theta_8 = -d_{20}$, $\theta_9 = -d_{54}$ and $\theta_{10} = -d_{34}$, such that

$$-r = \theta_{10} < \theta_9 < \theta_8 < \dots < \theta_2 < \theta_1 < \theta_0 = 0. \tag{3.6}$$

The constant rate of absorbtion of lead from compartment i to compartment j, with delay θ_k, is denoted by c_{ij}^k. Assuming that diffusion processes remain linear, the dynamical system of this compartment model for the kinetics of lead in the human body is described by the following differential equations

$$\begin{aligned}
x_1'(t) &= -(c_{10}^0 + c_{11}^0)\, x_1(t) - c_{14}^1 x_1(t+\theta_1),\\
x_2'(t) &= -\sum_{i=0}^{2} c_{24}^i x_2(t+\theta_i) - \sum_{i=0}^{8} c_{20}^i x_2(t+\theta_i) + \sum_{i=0}^{5} c_{52}^i x_5(t+\theta_i),\\
x_3'(t) &= -\sum_{i=0}^{10} c_{34}^i x_3(t+\theta_i) + \sum_{i=0}^{4} c_{43}^i x_4(t+\theta_i),\\
x_4'(t) &= c_{14}^0 x_1(t) + c_{14}^1 x_1(t+\theta_1) + \sum_{i=0}^{2} c_{24}^i x_2(t+\theta_i) + \sum_{i=0}^{10} c_{34}^i x_3(t+\theta_i) -\\
&\quad - \sum_{i=0}^{4} c_{43}^i x_4(t+\theta_i) - \sum_{i=0}^{7} c_{40}^i x_4(t+\theta_i) - \sum_{i=0}^{8} c_{45}^i x_4(t+\theta_i) + \sum_{i=0}^{9} c_{54}^i x_5(t+\theta_i),\\
x_5'(t) &= \sum_{i=0}^{8} c_{45}^i x_4(t+\theta_i) - \sum_{i=0}^{9} c_{54}^i x_5(t+\theta_i), \quad 0 < t < T.
\end{aligned} \tag{3.7}$$

The quantities $x_i(t)$ are supposed to be known for $t \in [-r, 0]$, that is

$$x_i(\theta) = h_i(\theta), \quad -r \leqslant \theta \leqslant 0, \; i = 1, 2, 3, 4, 5. \tag{3.8}$$

It may be also assumed that the loading of the system is made by instantaneous lead inputs, at the moments $0 < t_1 < t_2 < \dots < t_n < T$. Let a_i and f_i denote the amount of lead which enters the body at the instant t_i by air and by food, respectively $(i = 1, 2, \dots, n)$. Naturally, the following conditions must be considered then

$$\begin{aligned}
x_1(t_i) &= x_1(t_i^-) + \lambda_i a_i,\\
x_2(t_i) &= x_2(t_i^-) + \mu_i f_i,\\
x_k(t_i) &= x_k(t_i^-), \quad k = 3, 4, 5, \; i = 1, 2, \dots, n,
\end{aligned} \tag{3.9}$$

where

$$a_i \geqslant 0, \quad f_i \geqslant 0, \quad \text{for every } i \in \{1, 2, \dots, n\}, \tag{3.10}$$

and the pair of parameters (λ_i, μ_i) is equal to $(0,1)$, $(1,0)$ or $(1,1)$, according to the inhalation and ingestion circumstance at the instant t_i.

Maintaining the amounts of lead below certain maximum (poissonous) levels l_i, i.e.

$$0 \leqslant x_i(t) \leqslant l_i, \quad 0 \leqslant t \leqslant T, \quad i = 1, 2, 3, 4, 5, \tag{3.11}$$

it is desirable to minimize the pollution effects ; so arr ses the problem : find

$$\inf\left[\frac{1}{2}\sum_{k=1}^{5}\int_0^T x_k^2(t)dt + \frac{1}{2}\sum_{i=1}^{n}(\alpha_i^2 + \beta_i^2)\right]. \tag{3.12}$$

subject to the constraints (3.7) — (3.11).

Example 3.3.(*on fluctuations of capital stock*). As it is mentioned in [5] (Example 1.5), concerning the economic modelling of a trade cycle, there is the K a l e c k i model for the fluctuations of capital stock $x(t)$. This is described by

$$\begin{aligned} &x'(t) = ax(t) + bx(t-r) + cu_0(t), && 0 < t < T, \\ &x(\theta) = h(\theta), && -r \leqslant \theta \leqslant 0, \end{aligned} \tag{3.13}$$

where a, b, c are constants, and the delay r represents the time lag between the moment of decision to invest and the moment of deliveries of capital equipment ; u_0 denotes the "distributed" control and h is a known quantity.

Moreover, it may be assumed that the modifications of the capital stock are instantly yielded by the amounts u_i, at the moments $0 < t_1 < < t_2 < \ldots < t_n < T$, i.e.

$$x(t_i) = x(t_i^-) + u_i, \quad i = 1, 2, \ldots, n. \tag{3.14}$$

Finally, it is desirable to stabilize the system by a feedback control, or to regulate such a impulsive controlled process so that the minimization of the following functional should be achived

$$\int_0^T (\alpha x^2(t) + \beta(t)u_0^2(t))dt + \sum_{i=1}^{n}\gamma_i u_i^2, \tag{3.15}$$

subject to (3.13) and (3.14) (In (3.15), it is considered that $\alpha \geqslant 0$, $\beta(t)$, $\gamma_i > 0$).

Example 3.4. (*relating to some thermic treatments*). In steel making plants several thermic treatments are used for obtaining certain special qualities of metal slabs.

Let be supposed that in such a treatment one can act, at the instant $t_i (0 < t_1 < t_2 < \ldots < t_n < T)$, using a coolant or heating temperature $u_i(\xi)$. Then, a possible model for the temperature distribution $(x(t, \xi))$ in the metal slab $\Omega \subset \mathbf{R}^2$ is given by

$$\begin{aligned} &\frac{\partial x(t, \xi)}{\partial t} = \Delta_\xi x(t, \xi) + c(t)\, u_0(t, \xi), && \xi \in \Omega, \quad 0 < t < T, \\ &x(0, \xi) = x_0(\xi), && \xi \in \bar{\Omega}, \\ &\frac{\partial x(t, y)}{\partial n_y} = 0, && y \in \partial\Omega, \quad 0 \leqslant t \leqslant T \\ &x(t_i, \xi) = x(t_i^-, \xi) + u_i(\xi), && \xi \in \Omega, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{3.16}$$

where Δ is the Laplace operator, $\partial/\partial n$ denotes the normal derivative, u_0 is a distributed control, and $c(t)$, $x_0(\xi)$ are some known quantities.

The problem might be that of maintaining the temperature distribution so that the desired qualities of the slab should be achieved.

As, in general, this is quite impossible, it may be sought the minimization of the cost functional

$$\int_0^T\left(\int_\Omega (x^2(t, \xi) + u_0^2(t, \xi))d\xi\right)dt + \sum_{i=1}^{n}\int_\Omega u_i^2(\xi)d\xi, \tag{3.17}$$

over all $x \in C^-_{reg}(0, T; L^2(\Omega))$, $u_0 \in L^2(0, T; L^2(\Omega))$ and $u_i \in L^2(\Omega)$, subject to (3.16).

§ 4. Basic assumptions and statement of the problem. This section presents the hypotheses of a control problem which will be investigated. Let H and U be the Hilbert spaces introduced in §2. Here are, then, the basic notations and assumptions relating to the operators which will be used in the sequel.

(A1) The family of closed and densely defined linear operators in H $\{-A(t)\,;\ 0 \leqslant t \leqslant T\}$ generates a strong evolution operator $S : \mathcal{T} = \{(t, s)\,;\ 0 \leqslant s \leqslant t \leqslant T\} \to \mathfrak{L}(H)$ satisfying

$$\| S(t, s) \| \leqslant M, \quad \forall (t, s) \in \mathcal{T}, \tag{4.1}$$

where M is a positive constant.

At the same time, $S^*(t, s)$ is also assumed to be a strong evolution operator on H which is generated by $-A^*(t)$ (see [4]) and satisfies (4.1).

(A2) The operator B_i is in $\mathfrak{L}(U, H)$, for each $i = 1, 2, \ldots, n$ and $C : [0, T] \to \mathfrak{L}(U, H)$ is such that, for every $v \in U$, the function $t \to C(t)v$ is strongly measurable over $]0, T[$ and

$$\|C(t)\| \leqslant k_1 \quad \text{a.e.} \quad 0 < t < T, \tag{4.2}$$

with some positive constant k_1.

The adjoint $C^*(t)$ is assumed to have the same properties as well.

(A3) The self-adjoint operator $N_i \in \mathfrak{L}(U)$ satisfies

$$(N_i v, v) \geqslant \gamma \mid v \mid_U^2, \quad \text{for every} \quad v \in U, \tag{4.3}$$

where $\gamma > 0$ is a constant and $i = 1, 2, \ldots, n$.

(A4) $\{Q(t)\,;\ 0\leqslant t\leqslant T\}$ is a family included in $\mathcal{K}(H)$, such that, for every $h\in H$, the map $t\to Q(t)h$ is strongly measurable over $]0, T[$, and

$$(4.4)\qquad \|Q(t)\|\leqslant k_2 \quad \text{a.e.} \quad 0<t<T,$$

with some positive constant k_2.

(A5) The operator $R(t)$ belongs to $\mathcal{K}(U)$ for every $t\in[0, T]$. Moreover, for any $v\in U$, the function $t\to R(t)v$ is strongly measurable over $]0, T[$, and

$$(4.5)\qquad \|R(t)\|\leqslant k_3 \quad \text{a.e.} \quad 0<t<T,$$

$$(4.6)\qquad (R(t)v, v)\geqslant \rho\,|v|^2_U, \quad \forall v\in U, \quad \text{a.e.} \quad 0<t<T,$$

where k_3 and ρ are some positive constants.

For some fixed positive integer n, let $\{t_i\}_{i=0}^{n+1}$ be a given strict partition of the real finite closed interval $[0, T]$ (so as (2.6)).

In order to make easy references to some notations already used in §2, let $\mathfrak{X}$ and $\mathfrak{U}$ stand for $C^-_{res}(0, T\,; H)$ and $L^2(0, T\,; U)\times U^n$, respectively.

The evolutionary differential system that will be considered in this paper is

$$(4.7)\qquad x'(t)+A(t)x(t)=C(t)\,u_0(t)+\sum_{i=1}^{n} B_i u_i\otimes\delta(t-t_i),\ 0<t\leqslant T,\quad x(0)=x_0,$$

where $'=d/dt$ means the derivative in the sense of H-valued distributions over $]0, T[$, δ is the Dirac-delta $(u_0, u_1, \dots, u_n)$ belongs to $\mathfrak{U}$ and x_0 is given in H.

Taking into acocunt the above mentioned assumptions and the recalled results on abstract evolution equations, given in §2, it is easily seen that (4.7) has a unique solution, which, according to (2.15), can be expressed as

$$(4.8)\qquad x(t)=S(t, 0)\,x_0+\int_0^t S(t, s)C(s)u_0(s)ds+\sum_{i=1}^{n} S(t, t_i)B_i u_i\chi(t-t_i),\quad 0\leqslant t\leqslant T,$$

where χ denotes the Heaviside unit function.

Finally, let the following impulsive regulator problem—called $\mathfrak{P}$ in the sequel—be formulated here:

Minimize the functional

$$(4.9)\qquad \int_0^T(<Q(t)x(t), x(t)>+(R(t)u_0(t), u_0(t)))\,dt+\sum_{i=1}^{n}(N_i u_i, u_i)$$

over all $x\in\mathfrak{X}$ *and* $u=(u_0, u_1, \dots, u_n)\in\mathfrak{U}$, *subject to the state constraint* (4.8).

Relying on (4.8), it may be considered that (4.9) depends only on $u=(u_0, u_1, \dots, u_n)\in\mathfrak{U}$ and x_0. For x_0 fixed in H, let $J(u)$ denote this cost functional.

The main result that will be established here (in §5—§7) relating to the problem $\mathfrak{P}$ may be summarized as follows: *Under the basic hypotheses (A1) ~ (A5), the cost functional (4.9) can be minimized, and the unique minimizing control can be expressed in feedback form in terms of an operator which satisfies a Ricatti equation.*

§ 5. **Optimality theorem.** Concerning the optimality result for the above probleme, a suitable theorem is given in this section.

To begin with, let be noticed that $J:\mathfrak{U}\to[0, +\infty]$ is obviously a strict convex and continuous function. By virtue of (4.3) and (4.6) J is coercive. Consequently, there exists a unique $\tilde{u}\in\mathfrak{U}$ which minimizes $J(u)$ over all $u\in\mathfrak{U}$ (see [9], ch. I). As usual, $\tilde{u}$ is the optimal control in $\mathfrak{P}$, while the function that corresponds to $\tilde{u}$ in (4.8), denoted by $\tilde{x}$, is the optimal state.

The optimal elements of $\mathfrak{P}$ can be characterized by:

Theorem 5.1. *Let the hypotheses (A1), (A2), (A3), (A4) and (A5) be satisfied. Then the pair* $(\tilde{x}, \tilde{u})$ *is optimal in* $\mathfrak{P}$ *if and only if the following relations hold*

$$(5.1)\qquad \tilde{x}(t)=S(t, 0)x_0+\int_0^t S(t, s)\,C(s)\tilde{u}_0(s)ds+\sum_{i=1}^{n} S(t, t_i)B_i\tilde{u}_i\chi(t-t_i),\quad 0\leqslant t\leqslant T,$$

$$(5.2)\qquad \tilde{p}(t)=\int_t^T S^*(s, t)Q(s)\,\tilde{x}(s)\,ds,\quad 0\leqslant t\leqslant T,$$

$$(5.3)\qquad \tilde{u}_0(t)=-R^{-1}(t)C^*(t)\tilde{p}(t),\quad \text{a.e.}\quad 0<t<T,$$

$$(5.4)\qquad \tilde{u}_i=-N_i^{-1}B_i^*\tilde{p}(t_i),\qquad i=1, 2, \dots, n.$$

Proof: *Necessity.* If $(\tilde{x}, \tilde{u})$ is the optimal pair of the problem $\mathfrak{P}$, then $J(\tilde{u})\leqslant J(\tilde{u}+\lambda u)$ for every $u\in\mathfrak{U}$ and any real number λ. It follows that

$$(5.5)\qquad \lim_{\lambda\to 0}\lambda^{-1}(J(\tilde{u}+\lambda u)-J(\tilde{u}))=0,$$

whenever the limit exists. In fact, after some standard calculation, the existence of the above limit can be clearly seen, for every $u\in\mathfrak{U}$. Moreover

$$(5.6)\qquad \lim_{\lambda\to 0}\lambda^{-1}(J(\tilde{u}+\lambda u)-J(\tilde{u}))=2\Big[\int_0^T(C^*(t)\int_t^T S^*(s, t)\,Q(s)\tilde{x}(s)ds+R(t)\tilde{u}_0(t),\ u_0(t))dt+\sum_{i=1}^{n}(B_i^*\int_{t_i} S^*(s, t_i)\,Q(s)\tilde{x}(s)\,ds+N_i\tilde{u}_i,\ u_i)\Big],$$

for each $u=(u_0, u_1, \dots, u_n)\in\mathfrak{U}$.

Setting $\tilde{p}(t)=\int_t^T S^*(s,t)\,Q(s)\tilde{x}(s)ds$ in (5.6) (as (5.2) claimes), and taking into account (5.5) it readily follows that (5.3) and (5.4) hold. To conclude, it has merely to be noticed that (5.1) means, in this context, (4.8).

Sufficiency. Under the suppositions that $(\tilde{x},\tilde{u})\in\mathfrak{X}\times\mathfrak{U}$ represents a pair for which relations (5.1) — (5.4) are fulfilled, and $(x,u)\in\mathfrak{X}\times\mathfrak{U}$ (with an arbitrary u in $\mathfrak{U}$) satisfies (4.8), due to the positivity of $Q(t)$, $R(t)$ (for every $t\in[0,T]$) and $N_i(i=1,2,\ldots,n)$, it comes out

$$J(u)-J(\tilde{u})\geq 2\Big[\int_0^T (u_0(t)-\tilde{u}_0(t),\ R(t)\tilde{u}_0(t)+C^*(t)\tilde{p}(t))dt+ \tag{5.7}$$
$$+\sum_{i=1}^{n}(u_i-\tilde{u}_i,\ N_i\tilde{u}_i+B_i\tilde{p}(t_i))\Big],$$

where $\tilde{p}$ is given by (5.2). According to the relations (5.3) and (5.4), from (5.7) it clearly follows that $J(\tilde{u})\geq J(u)$, for every $u\in\mathfrak{U}$. This concludes the proof.

Remark 5.1. Obviously, the optimal adjoint state $\tilde{p}$ belongs to $C(0,T;H)$ (see [5]).

Remark 5.2. (*i*) Let the integer $m\in\{1,2,\ldots,n+1\}$ and the real $\tau\in\in[t_{m-1},t_m[$, respectively. Furthermore, let $\mathfrak{U}_\tau$ denote the Hilbert space $L^2(\tau,T;U)\times U^{n-m+1}$, $\mathfrak{X}_\tau$ stand for the set $C^-_{reg}(\tau,T;H)$, and h be an arbitrary element in H.

Then the following problem may be considered:

Minimize

$$\int_\tau^T (<Q(t)y(t),\,y(t)>+(R(t)v_0(t),\,v_0(t)))\,dt+\sum_{i=m}^{n}(N_iv_i,\,v_i) \tag{5.8}$$

over all $y\in\mathfrak{X}_\tau$ *and* $v=(v_0,v_m,\ldots,v_n)\in\mathfrak{U}_\tau$ *subject to*

$$y(t)=S(t,\tau)h+\int_\tau^t S(t,s)C(s)v_0(s)ds+\sum_{i=m}^{n}S(t,t_i)\,B_i\,v_i\chi(t-t_i),\quad \tau\leq t\leq T. \tag{5.9}$$

This problem will be called $\mathfrak{P}_\tau$ in the sequel.

When $m=1$, $\tau=t_0=0$ and $h=x_0$, $\mathfrak{P}_\tau$ is obvious $\mathfrak{P}$; when $m=n+1$, the sum in (5.8), 5.9) will vanish.

Denoting by $(\tilde{y}_\tau,\tilde{v})$ the optimal pair of $\mathfrak{P}_\tau$, similarly with Theorem 5.1, the corresponding optimality conditions will be

$$\tilde{y}_\tau(t)=S(t,\tau)h+\int_\tau^t S(t,s)\,C(s)\tilde{v}_0(s)ds+\sum_{i=m}^{n}S(t,t_i)B_i\tilde{v}_i\chi(t-t_i),\quad \tau\leq t\leq T, \tag{5.10}$$

$$\tilde{q}_\tau(t)=\int_t^T S^*(s,t)Q(s)\tilde{y}_\tau(s)ds,\quad \tau\leq t\leq T, \tag{5.11}$$

$$\tilde{v}_0(t)=-R^{-1}(t)\,C^*(t)\tilde{q}_\tau(t)\quad\text{a.e.}\quad \tau<t<T, \tag{5.12}$$

$$\tilde{v}_i=-N_i^{-1}B_i^*\tilde{q}_\tau(t_i),\qquad i=m,\,m+1,\ldots,n. \tag{5.13}$$

(*ii*) If $\tilde{x}$ is kept to denote the optimal state in $\mathfrak{P}$, and h is choosen in $\mathfrak{P}_\tau$ such that $h=\tilde{x}(\tau)$, then, as it can be easily verified, $\tilde{y}_\tau$ coincides with the restriction of $\tilde{x}$ to $[\tau,T]$, and so does $\tilde{q}_\tau$ with respect to $\tilde{p}$.

As in [9], the problem $\mathfrak{P}_\tau$ is connected with the question of the control synthesis for $\mathfrak{P}$. Such a question is discussed next.

§6. Synthesis of optimal control. As it is known, the control synthesis means the determination of the optimal control $\tilde{u}$ as a function of t and of the state variable $\tilde{x}$, along the optimal trajectory, such that (at each time $t\in[0,T]$) no memory involving post states or prediction of future states be required.

To obtain the feedback control for $\mathfrak{P}$, an appropriate operator will be introduced and investigated in this section (as in [9]).

To begin with, let τ, h, $\tilde{y}_\tau$ and $\tilde{q}_\tau$ be as in Remark 5.2. Then, let us consider the map $P(\tau):H\to H$, defined by

$$P(\tau)h=\tilde{q}_\tau(\tau),\qquad 0\leq\tau<T,\qquad h\in H. \tag{6.1}$$

Beside this, the following relation is supposed by definition

$$P(T)h=0,\qquad\text{for every}\qquad h\in H. \tag{6.2}$$

Remark 6.1. According to the part (*ii*) of Remark 5.2, by (6.1), together with (6.2), the equality

$$P(\tau)\tilde{x}(\tau)=\tilde{p}(\tau),\qquad 0\leq\tau\leq T, \tag{6.3}$$

can be deduced.

$P(\tau)$ realizes the optimal synthesis function for the feedback control law, as (6.3) suggests and as it will be shown in this section.

For the time being, some properties of $P(\tau)$ are pointed out in what follows; throughout this and the next sections, $G(t)$ and D_i will stand for $C(t)R^{-1}(t)C^*(t)$ and $B_iN_i^{-1}B_i^*$, respectively.

Proposition 6.1. *Under the basic assumptions* (*A*1) — (*A*5), *the operator* $P(\tau)$ *(given by* (6.1.) and (6.2)) *belongs to the class* $\mathfrak{R}(H)$, *for every* τ *in* $[0,T]$.

Proof: The case when $\tau=T$ is trivial. For the other one, in which $0\leq\tau<T$, let $\tilde{y}^1_\tau$ and $\tilde{y}^2_\tau$ be optimal states in the problem $\mathfrak{P}_\tau$ that correspond to some arbitrary elements in H, h_1 and h_2, respectively. Let also $\tilde{q}^1_\tau$ and $\tilde{q}^2_\tau$ be the corresponding optimal adjoint states in $\mathfrak{P}_\tau$. Using the relations (5.10) through (5.13), it is not difficult to see that $P(\tau)$ is a linear operator in H, for each $\tau\in[0,T[$.

Furthermore, forming the inner product of $\tilde{q}^1_\tau(\tau)$ with h_2 and taking into account (5.10) ~ (5.13), one obtains

$$(6.4)\qquad \langle \tilde{q}^1_\tau(\tau), h_2\rangle = \int_\tau^T (\langle Q(t)\tilde{y}^1_\tau(t), \tilde{y}^2_\tau(t)\rangle + \langle \tilde{q}^1_\tau(t), G(t)\tilde{q}^2_\tau(t)\rangle)dt + \sum_{i=m}^{n} \langle \tilde{q}^1_\tau(t_i), D_i\tilde{q}^2_\tau(t_i)\rangle .$$

Since $Q(t)$, $G(t)$ and D_i are positive and self-adjoint operators, from (6.4), taking into account (6.1), it follows that so is $P(\tau)$ too. By the same reasons, when $h_1=h_2=h$ and, implicitly, $\tilde{y}^1_\tau=\tilde{y}^2_\tau=\tilde{y}_\tau$, together with $\tilde{q}^1_\tau= =\tilde{q}^2_\tau=\tilde{q}_\tau$, from (6.4) it also comes easily out

$$(6.5)\qquad \langle \tilde{q}_\tau(\tau), h\rangle \geq \int_\tau^T \langle Q(t)\tilde{y}_\tau(t), \tilde{y}_\tau(t)\rangle\, dt, \qquad \forall h\in H.$$

On the other hand, using (4.1) and (4.4), by (5.11) one has

$$(6.6)\qquad |\tilde{q}_\tau(\tau)|^2_H \leq k_4 \int_\tau^T \langle Q(t)\tilde{y}_\tau(t), \tilde{y}_\tau(t)\rangle\, dt,$$

where $k_4=TM^2k_2$. Finally, combining (6.5) with (6.6), and taking into account (6.1), one obtains

$$(6.7)\qquad |P(\tau)h|_H \leq k_4 |h|_H, \qquad \forall h\in H, \qquad \forall \tau\in[0, T].$$

The proof is thereby complete.

Proposition 6.2. *If the hypotheses (A1) — (A5) still hold, then $P(.)$ is strongly continuous on $[t_{m-1}, t_m[$, for each $m\in\{1, 2, \ldots, n+1\}$.*

Proof: The assertion in the present proposition leans on some estimations in which optimal elements of $\mathfrak{A}_\tau$ are involved. Therefore, the first step of the proof is to establish them.

Thus, letting $h_1=h_2=h$ in (6.4), employing (6.7) and recalling the positivity of $Q(t)$, $G(t)$ and D_i, the following inequalities can be firstly obtained:

$$(6.8)\qquad \int_\tau^T \langle \tilde{q}_\tau(t), G(t)\tilde{q}_\tau(t)\rangle dt \leq k_4 |h|^2_H, \qquad \forall h\in H,$$

$$(6.9)\qquad \langle \tilde{q}_\tau(t_i), D_i\tilde{q}_\tau(t_i)\rangle \leq k_4 |h|^2_H, \qquad \forall h\in H, \qquad i=m, \ldots, n,$$

for any $\tau\in[0, T]$.

Next, using (4.1) — (4.6) together with (6.8) and (6.9), from (5.10), in which (5.11) — (5.13) are inserted, one gets:

$$(6.10)\qquad |\tilde{y}_\tau(t)|_H \leq k_5 |h|_H, \quad 0\leq\tau\leq t\leq T, \quad \forall h\in H,$$

where
$$k_5=M\left(1+k_1\sqrt{k_4\rho T}+\sqrt{k_4\gamma}\cdot\sum_{i=1}^{n}\|B_i\|\right).$$

Accordingly, from (5.11) it follows

$$(6.11)\qquad |\tilde{q}_\tau(t)|_H \leq k_6 |h|_H, \quad 0\leq\tau\leq t\leq T, \quad \forall h\in H,$$

with $k_6=k_2k_5TM$.

Furthermore, for any fixed integer $m\in\{1, 2, \ldots, n+1\}$ and some τ, η such that $t_{m-1}\leq\tau<\eta<t_m$, by an easy computation on the inner product of $\tilde{y}_\tau(\eta)-\tilde{y}_\eta(\eta)$ with $\tilde{q}_\tau(\eta)-\tilde{q}_\eta(\eta)$ one has

$$(6.12)\qquad \langle \tilde{y}_\tau(\eta)-\tilde{y}_\eta(\eta), \tilde{q}_\tau(\eta)-\tilde{q}_\eta(\eta)\rangle = \sum_{i=m}^{n} \langle D_i(\tilde{q}_\tau(t_i)-\tilde{q}_\eta(t_i)), \tilde{q}_\tau(t_i)-\tilde{q}_\eta(t_i)\rangle + \int_\eta^T (\langle Q(t)(\tilde{y}_\tau(t)-\tilde{y}_\eta(t)), \tilde{y}_\tau(t)-\tilde{y}_\eta(t)\rangle + \langle G(t)(\tilde{q}_\tau(t)-\tilde{q}_\eta(t)), \tilde{q}_\tau(t)-\tilde{q}_\eta(t)\rangle)dt.$$

If the positiveness of $Q(t)$ and D_i is used again, and also (6.1), (6.7), (6.11), then (6.12) implies

$$\int_\eta^T \langle G(t)(\tilde{q}_\tau(t)-\tilde{q}_\eta(t)), \tilde{q}_\tau(t)-\tilde{q}_\eta(t)\rangle\, dt \leq k_7 |h|_H |\tilde{y}_\tau(\eta)-\tilde{y}_\eta(\eta)|_H = k_7 |h|_H \left| S(\eta, \tau)h - h - \int_\tau^\eta S(\tau, t)\, G(t)\tilde{q}_\tau(t)dt\right|_H ,$$

with $k_7=k_4+k_6$.

Hence, in wiew of (4.1), (4.2), (4.6) and (6.11), it follows:

$$(6.13)\qquad \int_\eta^T \langle G(t)(\tilde{q}_\tau(t)-\tilde{q}_\eta(t)), \tilde{q}_\tau(t)-\tilde{q}_\eta(t)\rangle dt \leq w(\tau, \eta, h),$$

where $w(\tau, \eta, h)=k_7 |S(\eta, \tau)h-h|_H |h|_H + k_1^2k_6\rho M(\eta-\tau)|h|^2_H$.

Similarly, one obtains

$$(6.14)\qquad \langle D_i(\tilde{q}_\tau(t_i)-\tilde{q}_\eta(t_i)), \tilde{q}_\tau(t_i)-\tilde{q}_\eta(t_i)\rangle \leq w(\tau, \eta, h), \qquad \forall i\in\{m, \ldots, n\}.$$

Then, in as much as

$$\tilde{y}_\tau(t)-\tilde{y}_\eta(t)=S(t,\tau)h-S(t, \eta)h-\int_\tau^\eta S(t, s)G(s)\tilde{q}_\tau(s)ds - \int_\eta^t S(t, s)G(s)(\tilde{q}_\tau(s)-\tilde{q}_\eta(s))\, ds - \sum_{i=m}^{n} S(t, t_i)D_i(\tilde{q}_\tau(t_i)-\tilde{q}_\eta(t_i))\chi(t-t_i), \quad \forall t\in[\eta, T],$$

by (4.1), (4.2), (4.6), (6.11), (6.13) and (6.14) it results

(6.15) $$\|\tilde{y}_\tau(t)-\tilde{y}_\eta(t)\|_H \leqslant \omega(\tau,\eta,h), \quad \tau<\eta\leqslant t\leqslant T,$$

where $\omega(\tau,\eta,h)=M\,\|S(\eta,\tau)h-h\|_H+k_1^2k_6\rho M(\eta-\tau)+Mw(\tau,\eta,h)\,(k_1\sqrt{\rho}+\sqrt{\gamma}\sum_{i=1}^{n}\|B_i\|)$.

Next, for $t_{m-1}\leqslant\tau<\eta<t_m$, noticing that

$$P(\tau)h-P(\eta)h=\tilde{q}_\tau(\tau)-\tilde{q}_\eta(\eta)=\int_\tau^\eta S^*(t,\tau)Q(t)\tilde{y}_\tau(t)dt+$$
$$+\int_\eta^T S^*(t,\tau)Q(t)\,(\tilde{y}_\tau(t)-\tilde{y}_\eta(t))dt+(S^*(\eta,\tau)-I)\int_\eta^T S^*(t,\eta)Q(t)\tilde{y}_\eta(t)dt,$$

and taking into account (4.1), (4.4), (6.10) and (6.15), we get:

(6.16) $$\|P(\tau)h-P(\eta)h\|_H\leqslant Mk_2k_5(\eta-\tau)\,\|h\|_H+\|S^*(\eta,\tau)\tilde{q}_\eta(\eta)-\tilde{q}_\eta(\eta)\|_H+TMk_2\omega(\tau,\eta,h)$$

for any h in H.

Similarly, when $t_{m-1}\leqslant\eta<\tau<t_m$, it comes out

(6.17) $$\|P(\tau)h-P(\eta)h\|_H\leqslant Mk_2k_5(\tau-\eta)\|h\|_H+\|S^*(\tau,\eta)\tilde{q}_\eta(\eta)-\tilde{q}_\eta(\eta)\|_H+$$
$$+TMk_2\omega(\eta,\tau,h)+(M+1)\|q_\eta(\tau)-q_\eta(\eta)\|_H,$$

for every $h\in H$.

Once the above estimations established, let η be arbitrarily fixed in $[t_{m-1},t_m[$ and let τ be in the same interval, tending to η. Then, according to the strong continuity of $S(t,s)$, $S^*(t,s)$ and $\tilde{q}_\eta(t)$, one can notice that $w(\tau,\eta,h)$ vanishes and so does $\omega(\tau,\eta,h)$ too, for every h in H. Consequently, by (6.16) and (6.17) the proof is over now.

Remark 6.2. As a consequence of Proposition 6.2, it may be easily seen ([1]) that the map $t\to P(t)h$ is strongly measurable over $[0,T[$, for every $h\in H$. Accordingly, invoking Proposition 2.2, by assumptions $(A1)$, $(A2)$ and $(A5)$, in conjunction with (6.7), it follows that the operator $-A(t)-G(t)P(t)$ generates a mild evolution operator which, in the sequel, will be called $V(t,s)$ and satisfies

(6.18) $$\operatorname*{ess\,sup}_{(t,s)\in\mathcal{T}}\|V(t,s)\|\leqslant Me^{M\rho k_1^2k_4T}.$$

***Remark* 6.3.** The linearity of the state-costate system (5.1) ~ (5.4) suggests that an appropriate Riccati equation might be applicable in this context, namely

(6.19) $$P(t)h=S^*(t_m,t)\,P(t_m^-)\,S(t_m,t)h+\int_t^{t_m}S^*(s,t)\,Q(s)\,S(s,t)hds-$$
$$-\int_t^{t_m}S^*(s,t)P(s)G(s)P(s)S(s,t)hds,\quad t_{m-1}\leqslant t<t_m,\quad \forall h\in H,$$

(6.20) $$P(t_m^-)h=P(t_m)\,(I+D_m\,P(t_m))^{-1}h,\quad \forall h\in H,\ m\in\{1,\dots,n+1\},$$

where $P(t_m^-)h$ denotes the left-hand limit of $P(\,.\,)h$ in t_m (for $m\in\{1,\dots,n\}$, $h\in H$) and implicitly, when $m=n+1$, (6.20) includes (6.2).

The equation (6.19) with boundary condition (6.20) can be obtained by some straightforward considerations on $\bar{x}$ and $\bar{p}$, as it will be shown in the next section.

Relying on this remark, one can now use a standard procedure to express $\bar{u}(t)$ in terms of $\bar{x}(t)$, as follows.

Theorem 6.1. *Let the assumptions $(A1)$ ~ $(A5)$ be yet kept up. If there exists a map $\tilde{P}:[0,T]\to\mathcal{K}(H)$ which has strong continuous restriction to $[t_{m-1},t_m[$ as well as strong left-hand limit in t_m and satisfies* (6.19) *and* (6.20) *for every $h\in H$ and each $m\in\{1,\dots,n+1\}$, then the feedback control $\bar{u}$ in $\mathfrak{L}$ can be expressed as*

(6.21) $$\bar{u}_0(t)=-R^{-1}(t)C^*(t)\tilde{P}(t)\bar{x}(t)\quad \text{a.e.}\quad 0\leqslant t\leqslant T,$$

(6.22) $$\bar{u}_i=-N_i^{-1}B_i^*\tilde{P}(t_i)\bar{x}(t_i),\qquad i=1,2,\dots,n.$$

Moreover, the minimum cost in $\mathfrak{L}$ is given by

(6.23) $$J(\bar{u})=\langle x_0,\tilde{P}(0)x_0\rangle$$

for any initial state $x_0\in H$.

Proof: To obtain (6.21) and (6.22) it suffices to observe that these two relations derive from (5.3) and (5.4), respectively, as long as the optimal adjoint state $\bar{p}(t)$ is expressed in terms of $\bar{x}(t)$ by $\bar{p}(t)=\tilde{P}(t)\bar{x}(t)$. In other words, it should be shown that if the optimal state $\bar{x}(t)$ is the solution of the closed loop equation

(6.24) $$\bar{x}(t)=S(t,0)x_0-\int_0^t S(t,s)G(s)\tilde{P}(s)\bar{x}(s)ds-$$
$$-\sum_{i=1}^{n}S(t,t_i)\,D_i\tilde{P}(t_i)\bar{x}(t_i)\chi(t-t_i),\qquad 0\leqslant t\leqslant T,$$

then the following relation hold

(6.25) $$\tilde{P}(t)\bar{x}(t)=\int_t^T S^*(s,t)\,Q(s)\bar{x}(s)ds,\qquad \forall t\in[0,T].$$

In order to prove this, it is useful to be noticed firstly that (6.24) is equivalent to

(6.26) $$\bar{x}(s)=S(s,t)\bar{x}(t)-\int_t^s S(s,\sigma)G(\sigma)\tilde{P}(\sigma)\bar{x}(\sigma)d\sigma,\quad t_{m-1}\leqslant t\leqslant s<t_m,$$

$m=1,2,\dots,n+1$, and

$$(6.27)\qquad \tilde{x}(t_m)=\tilde{x}(t_m^-)-D_m\tilde{P}(t_m)\tilde{x}(t_m),\qquad m=1,2,\dots,n.$$

Next, as it is easily seen, setting $h=\tilde{x}(t_m)$ in (6.20) and using (6.27), one has

$$(6.28)\qquad \tilde{P}(t_m^-)\,\tilde{x}(t_m^-)=\tilde{P}(t_m)\tilde{x}(t_m),\qquad \forall m\in\{1,2,\dots,n\}.$$

Furthermore, for $t_{m-1}\leqslant t<t_m$, with any $m\in\{1,2,\dots,n+1\}$, by (6.26) and (6.19), after a direct calculation, one obtains

$$\tilde{P}(t)\tilde{x}(t)=S^*(t_m,t)\tilde{P}(t_m^-)\tilde{x}(t_m^-)+\int_t^{t_m}S^*(s,t)\,Q(s)\tilde{x}(s)ds.$$

Clearly, the latest relation, in conjunction with (6.28), yields (6.25).

As regards (6.23), the proof leans on a tedious but routine computation using (6.19), (6.21), (6.22), (6.26) and (6.27). Thus, using the strong evolution operator $\tilde{V}(t,s)$ generated by $-A(t)-\tilde{G}(t)P(t)$ (according to Remark 6.2), it is not difficult to see that

$$J(\tilde{u})=<x_0,\ \tilde{P}(0)x_0-\tilde{V}^*(t_1,0)\tilde{P}(t_1^-)\ \tilde{V}(t_1,0)x_0>+$$

$$+\sum_{i=1}^{n}(<\tilde{x}(t_i),\ \tilde{P}(t_i)D_i\tilde{P}(t_i)\,\tilde{x}(t_i)>+<\tilde{x}(t_i),\ \tilde{P}(t_i)\tilde{x}(t_i)-$$

$$-\tilde{V}^*(t_{i+1},t_i)\tilde{P}(t_{i+1}^-)\ \tilde{V}(t_{i+1},t_i)\tilde{x}(t_i)>).$$

From this, by some more calculation, (6.23) can be easily reached. So Theorem 6.1 has been completely proved.

The existence question which arises from the statement of the latest theorem (with regard to $\tilde{P}$) will be treated in the next section.

§ 7. Riccati operator equation (mild version). As it has been said in Remark 6.3, the motivation of (6.19) and (6.20) will be given here. The existence of a solution of (6.19) and (6,20), including (6.2), will be discussed next.

Let t be in $[t_{m-1},t_m[$, for some $m\in\{1,2,\dots,n+1\}$, and $P(t)$ the operator given by (6.1) together with (6.2). Due to the linearity of $P(t)$, invoking (6.3), it may be written then

$$(7.1)\qquad P(t)\tilde{x}(t_m^-)=p(t)-P(t)\ \ (\tilde{x}(t)-x(t_m^-)).$$

In view of Remark 5.1 and relation (6.7), by (7.1) it infers that $P(t)\tilde{x}(t_m^-)$ has left-hand limit in t_m, denoted by $P(t_m^-)\tilde{x}(t_m^-)$, and moreover

$$(7.2)\qquad P(t_m^-)\ \tilde{x}(t_m^-)=P(t_m)\ \tilde{x}(t_m),\qquad \text{for each}\quad m\in\{1,2,\dots,n\}.$$

Accordingly, since $\tilde{x}(t_m^-)=\tilde{x}(t_m)+D_m\tilde{\ }(t_m)$, it follows

$$(7.3)\qquad [P(t_m^-)-P(t_m)+P(t_m^-)D_mP(t_m)]\tilde{x}(t_m)=0,\quad \forall m\in\{1,2,\dots,n\}.$$

Formally, (7.3) yields the relation

$$P(t_m^-)\ [I+D_mP(t_m)]=P(t_m),\qquad m=1,2,\dots,n,$$

from which, according to Propositions 6.1 and 2.1, (6.20) is precisely derived, for every $m\in\{1,2,\dots,n\}$.

To reach (6.19), one can start from writing (5.1) and (5.2), by virtue of (5.3), as

$$(7.4)\qquad \tilde{x}(t)=S(t,s)\ \tilde{x}(s)-\int_s^t S(t,\sigma)\,G(\sigma)\ \tilde{P}(\sigma)d\sigma,\quad t_{m-1}\leqslant s\leqslant t<t_m$$

and

$$(7.5)\qquad \tilde{\ }(t)=S^*(t_m,t)\ \tilde{P}(t_m)+\int_t^{t_m}S^*(s,t)\,Q(s)\ \tilde{x}(s)ds,\qquad t_{m-1}\leqslant t\leqslant t_m,$$

respectively, with m arbitrarily fixed in $\{1,2,\dots,n+1\}$.

Then, using (7.4), for $s=t_{m-1}$, in (7.5), by a direct calculation one obtains

$$(7.6)\qquad \tilde{\ }(t)=S^*(t_m,t)\tilde{\ }(t_m)+L(t)\tilde{x}(t)-\int_t^{t_m}S^*(s,t)\ L(s)G(s)\tilde{\ }(s)ds,$$

where

$$L(t)h=\int_t^{t_m}S^*(s,t)\ Q(s)S(s,t)h\ ds,\qquad t_{m-1}\leqslant t<t_m,\qquad \forall h\in H.$$

Further on, using (6.3), (7.2) and (7.4) again, (7.6) gives

$$(7.7)\qquad P(t)\tilde{x}(t)=W(t)\tilde{x}(t)-\int_t^{t_m}S^*(s,t)W(s)G(s)P(s)\tilde{x}(s)ds,$$

where

$$(7.8)\qquad W(t)h=L(t)h+S^*(t_m,t)P(t_m^-)S(t_m,t)h,\qquad \forall h\in H.$$

Finally, considering

$$F(t)h=\int_t^{t_m}S^*(s,t)P(s)G(s)P(s)S(s,t)h\ ds,\qquad \forall h\in H,$$

and setting $Z(t)=P(t)-W(t)+F(t)$ in (7.7), one has

$$(7.9)\qquad Z(t)\tilde{x}(t)=\int_t^{t_m}S^*(s,t)Z(s)G(s)P(s))\tilde{x}(s)ds,\qquad t_{m-1}\leqslant t<t_m.$$

Clearly,

$$Z(t)h=P(t)h-S^*(t_m,t)\,P(t_m^-)\,S(t_m,t)h-\int_t^{t_m}S^*(s,t)Q(s)S(s,t)h\,ds+$$

$$(7.10)\qquad +\int_t^{t_m}S^*(s,t)\,P(s)G(s)P(s)S(s,t)h\,ds,\qquad t_{m-1}\leqslant t<t_m,\quad \forall h\in H.$$

Next, according to Remark 6.2, it can be noticed that (7.4) means also
$\bar{x}(t)=V(t,s)\bar{x}(s)$, $t_{m-1}\leqslant s\leqslant t<t_m$, $m\in\{1,2,\dots,n+1\}$.
Consequently, from (7.9) it follows that

$$Z(t)\bar{x}(t)-\int_t^{t_m}S^*(s,t)Z(s)G(s)P(s)V(s,t)\bar{x}(t)ds=0,\quad t_{m-1}\leqslant t<t_m.$$

Therefore, formally, the following relation may be reached:

$$Z(t)h=\int_t^{t_m}S^*(s,t)Z(s)G(s)P(s)V(s,t)h\,ds,\qquad \forall h\in H.$$

So, by (4.1), (4.2), (4.6), (6.7) and (6.18), the relation

$$\|Z(t)\|\leqslant k_8\int_t^{t_m}\|Z(s)\|\,ds,\qquad t_{m-1}\leqslant t\leqslant t_m,$$

where $k_8=M^2\rho k_1^2k_4e^{M\rho k_1k_4T}$, can be deduced.

Then, by Gronwall's inequality, $\|Z(t)\|=0$ on H, for every $t\in[t_{m-1},t_m[$. Hence, in view of (7.10), (6.19) is justified now.

As to solution of the equation (6.19), (6.20) and (6.2), the following existence result can be stated:

Theorem 7.1. *If the assumptions (A1) ~ (A5) hold, then there exists a unique operator $\tilde{P}:[0,T]\to\mathcal{K}(H)$ which is strongly continuous on $[t_{m-1},t_m[$ for every $m=1,2,\dots,n+1$, has strong left-hand limit in t_m, $\forall m\in\{1,2,\dots,n\}$, and satisfies* (6.19), *along with* (6.20), *and* (6.2), *inclusively.*

Proof: First we will show the existence and uniqueness of $\tilde{P}$ on $[t_n,T]$, hence, for $m=n+1$ setting $\tilde{P}(t_{n+1}^-)=\tilde{P}(T)=0$ in (6.19), in order to satisfy (6.2). Then, knowing already that $\tilde{P}(t_n)\in\mathcal{K}(H)$, as D_n does too, by virtue of Proposition 2.1, (6.20) would be easily deduced with $m=n$). Once $\tilde{P}(t_n^-)$ determined, we shall be able to treat the next case ($m=n$), by the same manner and so on.

When m is arbitrarily fixed in $\{1,2,\dots,n+1\}$, the proof of existence of a unique claimed operator $\tilde{P}(\,.\,)$ on $[t_{m-1},t_m[$ can be generally made using results from [6] or [5]. For the sake of a computational point of view, the utilization of succesive approximations method (in this context) is briefly described here.

Thus, letting $\tau=t_m-t$ in (6.19), $t\in[t_{m-1},t_m[$, and denoting $P(t_m-\tau)$, $Q(t_m-\tau)$, $G(t_m-\tau)$, $S(t_m-\sigma,\,t_m-\tau)$ (where $\sigma\in[0,\tau]$) by $\check{P}(\tau)$, $\check{Q}(\tau)$, $\check{G}(\tau)$ and $\check{S}(\sigma,\tau)$, respectively, one can consider

$$(7.11)\qquad \check{P}_0(\tau)h=\check{S}^*(0,\tau)\tilde{P}(t_m^-)\check{S}(0,\tau)h+\int_0^{\tau}\check{S}^*(\sigma,\tau)\check{Q}(\sigma)\check{S}(\sigma,\tau)h\,d\sigma$$

and

$$(7.12)\qquad \check{P}_\nu(\tau)h=\check{P}_0(\tau)h-\int_0^{\tau}\check{S}^*(\sigma,\tau)\,\check{P}_{\nu-1}(\sigma)\,\check{G}(\sigma)\check{P}_{\nu-1}(\sigma)\check{S}(\sigma,\tau)h\,d\sigma,$$

for $\tau\in[0,t_m-t_{m-1}]$, any $h\in H$ and $\nu=1,2,\dots$.

Obviously, under the basic assumptions (A1) ~ (A5), $\check{P}_0(\tau)$ is a self-adjoint strong continuous operator in $\mathfrak{L}(H)$, and, by induction, so is $\check{P}_\nu(\tau)$ too, for every $\nu=1,2,\dots$. It may be also noticed that $\check{P}_\nu(\tau)$ is strongly measurable over $]0,t_m-t_{m-1}[$. Taking into account all these, by a standard argument one deduces that $\check{P}_\nu(\tau)$ is strongly convergent on H, for $\tau\in[0,\theta]$, with θ adequately settled in $]0,t_m-t_{m-1}]$. Moreover $\check{\bar{P}}(\tau)$ — the unique strong limit of $\check{P}_\nu(\tau)$ — is self-adjoint, strongly continuous and satisfies the equation

$$(7.13)\qquad \hat{\bar{P}}(\tau)h=\check{P}_0(\tau)h\int_0^{\tau}\check{S}^*(\sigma,\tau)\,\check{\bar{P}}(\sigma)\check{G}(\sigma)\check{\bar{P}}(\sigma)\check{S}(\sigma,\tau)\,hd\sigma,\quad 0\leqslant\tau\leqslant\theta.$$

On the other hand, from (7.12) and (7.11) one can easily obtain the estimation

$$|\check{P}(\tau)h|_H\leqslant|k_9|\,h\,|_H,$$

with $k_9=M^2(k_4+k_2T)$, for every $h\in H$ and $\tau\in[0,t_m-t_{m-1}]$.

Consequently, there exists a unique extension of $\check{\bar{P}}$, denoted by $\check{\tilde{P}}$, which verifies (7.13) on the whole interval $[0,t_m-t_{m-1}]$. It is also clear that $\check{\tilde{P}}(\tau)$ is a self-adjoint strong continuous operator in $\mathfrak{L}(H)$, for every $\tau\in]0,t_m-t_{m-1}]$, and $\check{\tilde{P}}(\tau)$ has strong right-hand limit when τ vanishes. Then $\tilde{P}(t)$ $\check{\tilde{P}}(t_m-t)$, for each $t\in[t_{m-1},t_m[$.

To complete the proof, the positivity of $P(t)$ needs to be shown. In view of this, Remark 6.2 is called on for $\tilde{P}$ and (as in [5]) the corresponding mild evolution operator $\tilde{V}(t,s)$ is used in (7.4). After some tedious calculation on (6.19), it follows that $\tilde{P}$ satisfies also

$$(7.14)\qquad \tilde{P}(t)h=\tilde{\Gamma}^*(t_m, t)\tilde{P}(t_m^-)\tilde{\Gamma}(t_m, t)h+\int_t^{t_m}\tilde{\Gamma}^*(s, t)\,(Q(s)+$$

$$+\tilde{P}(s)G(s)\tilde{P}(s))\,\tilde{\Gamma}(s, t)h\,ds,\qquad t_{m-1}\leqslant t<t_m,\qquad \forall h\in H.$$

Obviously, (7.14) implies that $\tilde{P}(t)$ is a positive operator, $\forall t\in[t_{m-1}, t_m[$, and the proof is thereby closed up.

§ 8. Linear hereditary systems. The result established in Theorems 5.1, 6.1 and 7.1 can be immediately applied to the optimal impulsive control of linear differential- delay systems with quadratic performance index. To illustrate this, a certain case is treated in the present section.

Thus, let the positive integers n, k, l, d and the real numbers $0<r<<T<\infty$, together with $-r=\theta_k<\theta_{k-1}<\ldots<\theta_1<\theta_0=0=t_0<t_1<\ldots<t_n<<t_{n+1}=T$ be given.

With these, we consider the following class of linear hereditary differential system

$$x'(t)=A_{00}(t)x(t)+\sum_{j=1}^{k}A_j(t)x(t+\theta_j)+\int_{-r}^{0}A_{01}(t, \theta)x(t+\theta)d\theta+C(t)u_0(t)$$

$$(8.1)\qquad \text{a.e.}\quad t_i<t<t_{i+1},\quad i=0, 1, \ldots, n,$$

$$x(t_i)=x(t_i^-)+B_iu_i,\quad i=1, 2, \ldots, n,$$

$$x(0)=h^0,\ x(\theta)=h^1(\theta)\quad \text{a.e.}\quad -r<\theta<0,$$

where $h=[h^0, h^1]\in M^2(-r, 0;\mathbf{R}^d)$, A_{00} and $A_j(j=1, 2, \ldots, k)$ are in $C(0, T;\mathfrak{L}_{dd})$, $A_{01}:[0, T]\times[-r, 0]\to\mathfrak{L}_{dd}$ is continuous in $t\in[0, T]$ and essentially bounded in $\theta\in[-r, 0]$, $C\in L^\infty(0, T;\mathfrak{L}_{ld})$, $u_0\in L^2(0, T;\mathbf{R}^l)$, $B_i\in\mathfrak{L}_{ld}$ and $u_i(i=1, 2, \ldots, n)$ are in $\mathbf{R}^l$.

When A_{00} together with A_i belong to $L^\infty(0, T;\mathfrak{L}_{dd})$ and $A_{01}\in L^\infty(]0, T[\times]-r, 0[;\mathfrak{L}_{dd})$ it is known that (8.1) has a unique solution $x\in\in AC^2(t_{m-1}, t_m;\mathbf{R}^d)$, for every $m\in\{1, 2, \ldots, n+1\}$. The symbol $AC^2(a, b;\mathbf{R}^d)$ (written shortly AC^2) is used to denote the Hilbert space of absolutely continuous functions on $[a, b]\to\mathbf{R}^d$, whose derivatives are square integrable, endowed with the inner product

$$(f, g)_{AC^2}=\int_a^b\left(\frac{df(t)}{dt}, \frac{dg(t)}{dt}\right)_{\mathbf{R}^d}dt.$$

So that, in the present case, the hypotheses taken into account (near by (8.1)) are enough to place the solution of (8.1) in AC^2 on $[t_{m-1}, t_m[$, $m==1, 2, \ldots, n+1$.

Let X denote the class of all functions $x:[0, T]\to\mathbf{R}^d$ which are in AC^2 on every $[t_{m-1}, t_m[(m=1, 2, \ldots, n+1)$ and have left-hand limit in t_m; for each $m=1, 2, \ldots, n$.

Moreover, let N_i be in $\mathfrak{L}_{ll}$ such that $N_i=N_i^*$ and

$$(8.2)\qquad (N_iw, w)_{\mathbf{R}^l}\geqslant\gamma\,|w|^2_{\mathbf{R}^l},\qquad \forall w\in R^l,\quad i=1, 2, \ldots, n,$$

with some real $\gamma>0$.

Finally, let Q and R belong to $L^\infty(0, T;\mathfrak{L}_{dd})$ such that

$$(8.3)\qquad Q(t)=Q^*(t),\quad R(t)=R^*(t),\quad \forall t\in[0, T]$$

and

$$(8.4)\quad (Q(t)v, v)_{\mathbf{R}^d}\geqslant 0,\ \forall v\in R^d,\ (R(t)w, w)_{\mathbf{R}^l}\geqslant\rho\,|w|^2_{\mathbf{R}^l},\quad \forall w\in R^l,\quad \forall t\in[0, T],$$

where ρ is some positive constant.

Then, the control problem which will be examined in this section is denoted by Π and has the following statement:

Minimize the functional

$$(8.5)\qquad \int_0^T[(Q(t)x(t), x(t))_{\mathbf{R}^d}+(R(t)u_0(t), u_0(t))_{\mathbf{R}^l}]dt+\sum_{i=1}^{n}(N_iu_i, u_i)_{\mathbf{R}^l}$$

over all x in X and $u=(u_0, u_1, \ldots, u_n)\in L^2(0, T;\mathbf{R}^l)\times(\mathbf{R}^l)^n$ subject to the contraint (8.1).

According to [7], a closed linear operator $\hat{A}(t)$, $t\in[0, T]$, can be defined on $M^2(-r, 0;\mathbf{R}^d)$, by

$$(8.6)\qquad [\hat{A}(t)h](\theta)=\begin{cases}-A_{00}(t)h^0-\sum_{j=1}^{k}A_j(t)h^1(\theta_j)-\int_{-r}^{0}A_{01}(t, \theta)h^1(\theta)d\theta, & \text{for } \theta=0\\ -\dfrac{dh^1(\theta)}{d\theta}, & \text{otherwise,}\end{cases}$$

for $h=[h^0, h^1]$ in the domain $\mathfrak{D}$ of $\hat{A}(t)$, where $\mathfrak{D}=AC^2(-r, 0;\mathbf{R}^d)\cup\cap M^2(-r, 0;\mathbf{R}^d)$.

If the symbol x_t is used to denote the function defined by

$$(8.7)\qquad x_t(\theta)=x(t+\theta),\qquad \theta\in[-r, 0],$$

then $\hat{x}:[0, T]\to M^2(-r, 0;\mathbf{R}^d)$ will stand for the map expressed as

$$(8.8)\qquad \hat{x}(t)=[x(t), x_t],\qquad t\in[0, T].$$

Now let us introduce the operators $\hat{C}(t):\mathbf{R}^l\to M^2(-r, 0;\mathbf{R}^d)$, $\hat{Q}(t):M^2(-r, 0;\mathbf{R}^d)\to M^2(-r, 0;\mathbf{R}^d)$, $t\in[0, T]$, and $\hat{B}_i:\mathbf{R}^l\to M^2(-r, 0;\mathbf{R}^d)$, $i=1, 2, \ldots, n$, respectively defined as

$$(8.9)\qquad [\hat{C}(t)w](\theta)=\begin{cases}C(t)w, & \text{for } \theta=0\\ 0, & \text{otherwise}\end{cases},\quad \forall w\in Rl,\ t\in[0, T],$$

$$(8.10)\qquad [\hat{Q}(t)h](\theta)=\begin{cases}Q(t)h^0, & \text{for } \theta=0\\ 0, & \text{otherwise}\end{cases},\quad \forall h=[h^0, h^1]\in M^2(-r, 0;\mathbf{R}^d),$$

$$(8.11)\qquad [\hat{B}_iu_i](\theta)=\begin{cases}B_ju_j, & \text{for } \theta=t_j-t_i \text{ and } j\in I_i\\ 0, & \text{otherwise}\end{cases},\quad i=1, 2, \ldots, n,$$

where I_i means the class of all positive integers $j \leqslant i$ for which $t_j - t_i$ belongs to $[-r, 0]$.

In terms of $\hat{x}, \hat{C}, \hat{Q}, \hat{B}_i$ and $\hat{A}$, the problem Π can be expressed as:

Minimize

$$\int_0^T [(\hat{Q}(t)\hat{x}(t), \hat{x}(t))_{M^2} + (R(t)u_0(t), u_0(t))_{\mathbf{R}^l}]dt + \sum_{i=1}^n (N_i u_i, u_i)_{\mathbf{R}^l}, \tag{8.12}$$

over all $\hat{x}$ *in* $C^-_{\text{res}}(0, T; M^2(-r, 0; \mathbf{R}^d))$ *and* $u \in L^2(0, T; \mathbf{R}^l) \times (\mathbf{R}^l)^n$, *subject to*

$$\begin{aligned} &\hat{x}'(t) + \hat{A}(t)\hat{x}(t) = \hat{C}(t)u_0(t) + \sum_{i=1}^n \hat{B}_i u_i \otimes \delta(t - t_i), \quad 0 < t < T, \\ &\hat{x}(0) = [h^0, h^1]. \end{aligned} \tag{8.13}$$

Furthermore, it may be noticed that (as it is proved in [7]) $\hat{A}(t)$ generates a strong evolution operator $S(t, s)$ defined by the agency of the matrices $\Phi_0(t, s)$ and $\Phi_1(t, s)$ given as follows:

$$\Phi_0 : \mathfrak{T} = \{(t, s) \; ; \; 0 \leqslant s \leqslant t \leqslant T\} \to \mathfrak{L}_{dd}$$

is the unique solution in $AC^2(s, T; \mathbf{R}^d)$ of the equation

$$\begin{aligned} \frac{\partial \Phi_0(t, s)}{\partial t} &= A_{00}(t)\,\Phi_0(t, s) + \sum_{j=1}^k A_j(t) \begin{Bmatrix} \Phi_0(t+\theta_j, s), & t+\theta_j \geqslant s \\ 0, & \text{otherwise} \end{Bmatrix} + \\ &+ \int_{-r}^0 A_{01}(t, \theta) \begin{Bmatrix} \Phi_0(t+\theta, s), & t+\theta \geqslant s \\ 0, & \text{otherwise} \end{Bmatrix} d\theta, \; a.e. \text{ on }]s, T[, \quad \Phi_0(s, s) = I; \end{aligned} \tag{8.14}$$

$$\begin{aligned} \Phi_1(t, s, \sigma) &= \sum_{j=1}^k \begin{Bmatrix} \Phi_0(t, s+\sigma-\theta_j)A_j(s+\sigma-\theta_j), & \sigma+s-t \leqslant \theta_j \leqslant \sigma \\ 0, & \text{otherwise} \end{Bmatrix} + \\ &+ \int_{\beta_t(\sigma+s)}^{\sigma} \Phi_0(t, s+\sigma-\theta)A_{01}(s+\sigma-\theta, \theta)d\theta, \end{aligned} \tag{8.15}$$

where $\sigma \in [-r, 0]$ and $\beta_y(z)$ is given by

$$\beta_y(z) = \max\{z - y, -r\}, \qquad \forall z, y \in \mathbf{R}. \tag{8.16}$$

So the evolution operator $S : \mathfrak{T} \to \mathfrak{L}(M^2(-r, 0; \mathbf{R}^d))$ is defined as

$$S(t, s)h = S_0(t, s)h^0 + S_1(t, s)h^1, \qquad \forall h = [h^0, h^1] \in M^2(-r, 0; \mathbf{R}^d), \tag{8.17}$$

where $S_0(t, s) \in \mathfrak{L}(\mathbf{R}^d, M^2(-r, 0; \mathbf{R}^d))$ and $S_1(t, s) \in \mathfrak{L}(L^2(-r, 0; \mathbf{R}^d); M^2(-r, 0; \mathbf{R}^d))$ are respectively given by

$$[S_0(t, s)h^0](\theta) = \begin{cases} \Phi_0(t+\theta, s)h^0, & \text{for } t+\theta \geqslant s \\ 0, & \text{for } t+\theta < s \end{cases}, \quad (t, s) \in \mathfrak{T} \tag{8.18}$$

and

$$[S_1(t, s)h^1](\theta) = \begin{cases} \int_{-r}^0 \Phi_1(t+\theta, s, \eta)h^1(\eta)d\eta, & \text{for } t+\theta \geqslant s \\ h^1(t+\theta-s), & \text{for } t+\theta < s \end{cases}, \quad (t, s) \in \mathfrak{T}. \tag{8.19}$$

Then the solution of (8.13) can be expressed as

$$\hat{x}(t) = S(t, 0)h + \int_0^t S(t, s)\,\hat{C}(s)u_0(s)ds + \sum_{i=1}^n S(t, t_i)\hat{B}_i u_i \chi(t - t_i), \quad t \in [0, T]. \tag{8.20}$$

At the same time, the unique solution of the original equation (8.1) is

$$\begin{aligned} x(t) &= [\hat{x}(t)](\theta = 0) = \Phi_0(t, 0)h^0 + \int_{-r}^0 \Phi_1(t, 0, \eta)h^1(\eta)d\eta + \\ &+ \int_0^t \Phi_0(t, s)\,C(s)\,u_0(s)ds + \sum_{i=1}^n \Phi_0(t, t_i)B_i u_i \chi(t - t_i), \quad 0 \leqslant t \leqslant T. \end{aligned} \tag{8.21}$$

Now the problem of minimization of (8.12) over all $\hat{x}$ in $\mathfrak{X} = C^-_{\text{res}}(0, T; M^2(-r, 0; \mathbf{R}^d))$ and u in $\mathfrak{U} = L^2(0, T; \mathbf{R}^l) \times (\mathbf{R}^l)^n$, subject to (8.20), is of type $\mathfrak{P}$, where $U = \mathbf{R}^l$, $H = M^2(-r, 0; \mathbf{R}^d)$, $Q = \hat{Q}$, $B_i = \hat{B}_i$ and $C = \hat{C}$. Moreover, as it is easily verified, by (8.2), (8.3), (8.4) and (8.17), the assumptions (A1)–(A5) are clearly satisfied.

Then the results obtained in the previous sections can be readily applied in the present case, noticing that from (8.17), (8.18) and (8.19) we get:

$$\begin{aligned} S^*(t, s)h = [S_0^*(t, s)\,h, S_1^*(t, s)h], \quad \forall h = [h^0, h^1] \in M^2(-r, 0; \mathbf{R}^d), \\ (t, s) \in \mathfrak{T}, \end{aligned} \tag{8.22}$$

where

$$S_0^*(t, s)h = \Phi_0^*(t, s)h^0 + \int_{\beta_t(s)}^0 \Phi_0^*(t+\theta, s)\,h^1(\theta)d\theta, \tag{8.23}$$

and

$$\begin{aligned} [S_1^*(t, s)h](\eta) &= \Phi_1^*(t, s, \eta)h^0 + \int_{\beta_t(s)}^0 \Phi_1^*(t+\theta, s, \eta)h^1(\theta)d\theta + \\ &+ I_{t,s}(\eta) - h^1(\eta + s - t), \qquad \eta \in [-r, 0[. \end{aligned} \tag{8.24}$$

Here $\beta_t(s)$ is given by (8.16) and the symbol $I_{t,s}$ is used to denote the indicator function of the set $\{\eta \in [-r, 0[\ ;\ t-r-s \leqslant \eta < t+\beta_t(s)-s\}$, i.e.

$$(8.25)\qquad I_{t,s}(\eta)=\begin{cases}1, & \text{for } t-s<r \text{ and } t-s-r\leqslant\eta<0\\ 0, & \text{for } t-s\geqslant r \text{ or } -r\leqslant\eta<t-s-r<0.\end{cases}$$

Thus, according to Theorem 5.1, the optimal elements $(\tilde{x}, \tilde{u}, \tilde{p})$ of problem II are characterized as follows:

$$(8.26)\qquad \begin{aligned}\tilde{x}(t)&=\Phi_0(t,0)h^0+\int_{-r}^{0}\Phi_1(t,0,\eta)h^1(\eta)d\eta+\int_0^t\Phi_0(t,s)\,C(s)\tilde{u}_0(s)ds+\\&+\sum_{i=1}^{n}\Phi_0(t,t_i)\,B_i\tilde{u}_i\chi(t-t_i),\qquad 0\leqslant t\leqslant T,\end{aligned}$$

$$(8.27)\qquad \tilde{p}(t)=\int_t^T\Phi_0^*(s,t)\,Q(s)\tilde{x}(s)ds,\quad 0\leqslant t\leqslant T,$$

while $\tilde{u}_0$ and $\tilde{u}_i$ are expressed just as in (5.3) and (5.4), respectively.

Concerning the feedback operator $P(t)$, $t\in[0,T]$, that is fitted to this case, by virtue of decomposition (2.5) one can consider $P_{00}(t)\in\mathfrak{L}_{dd}$, $P_{01}(t)\in\mathfrak{L}(L^2(-r,0;\mathbf{R}^d),\mathbf{R}^d)$, $P_{10}(t)\in\mathfrak{L}(\mathbf{R}^d, L^2(-r,0;\mathbf{R}^d))$ and $P_{11}(t)\in\mathfrak{L}(L^2(-r,0;\mathbf{R}^d))$. Relating to the latest two operators, the matrices $P_{10}(t,\theta)$ and $P_{11}(t,\theta,\eta)$ in $\mathfrak{L}_{dd}$ may be considered, by relations

$$(8.28)\qquad P_{10}(t,\theta)w=(P_{10}(t)w)(\theta),\ t\in[0,T],\ \theta\in[-r,0],\ \forall w\in\mathbf{R}^d,$$

$$(8.29)\qquad (P_{11}(t)f)(\theta)=\int_{-r}^{0}P_{11}(t,\theta,\eta)f(\eta)d\eta,\ t\in[0,T],\ \theta\in[-r,0],\quad \forall f\in L^2(-r,0;\mathbf{R}^d).$$

Since $P(t)$ is self-adjoint, it is naturally imposed

$$(8.30)\qquad P_{01}(t)f=\int_{-r}^{0}P_{10}^*(t,\theta)\,f(\theta)d\theta,\ t\in[0,T],\ \forall f\in L^2(-r,0;\mathbf{R}^d).$$

Substituing these into (6.19), a tedious calculation leads to the following equations:

$$(8.31)\qquad \begin{aligned}P_{00}(t)w&=\Phi_0^*(t_m,t)P_{00}(t_m^-)\Phi_0(t_m,t)w+\int_{\beta_{t_m}(t)}^{0}\int_{\beta_{t_m}(t)}^{0}\Phi_0^*(t_m+\theta,t)P_{11}(t_m^-,\theta,\eta)\Phi_0(t_m+\eta,t)w\,d\eta d\theta+\\&+\int_{\beta_{t_m}(t)}^{0}[\Phi_0^*(t_m,t)P_{10}^*(t_m^-,\theta)\Phi_0(t_m+\theta,t)w+\Phi_0^*(t_m+\theta,t)P_{10}(t_m^-,\theta)\Phi_0(t_m,t)w]d\theta+\\&+\int_t^{t_m}[\Phi_0^*(s,t)Q(s)\Phi_0(s,t)w-\Phi_0^*(s,t)P_{00}(s)G(s)P_{00}(s)\Phi_0(s,t)w]ds-\\&-\int_t^{t_m}\int_{\beta_s(t)}^{0}[\Phi_0^*(s,t)P_{00}(s)G(s)P_{10}^*(s,\theta)\Phi_0(s+\theta,t)w+\Phi_0^*(s+\theta,t)P_{10}(s,\theta)G(s)P_{00}(s)\Phi_0(s,t)w]d\theta ds-\\&-\int_t^{t_m}\int_{\beta_s(t)}^{0}\int_{\beta_s(t)}^{0}\Phi_0^*(s+\theta,t)P_{10}(s,\theta)G(s)P_{10}^*(s,\eta)\Phi_0(s+\eta,t)w\,d\eta d\theta ds,\quad t\in]t_{m-1},t_m],\ \forall w\in\mathbf{R}^d;\end{aligned}$$

$$(8.32)\qquad \begin{aligned}P_{10}(t,\theta)w&=\Phi_1^*(t_m,t,\theta)P_{00}(t_m^-)\Phi_0(t_m,t)w+\int_{\beta_{t_m}(t)}^{0}\int_{\beta_{t_m}(t)}^{0}\Phi_1^*(t_m+\tau,t,\theta)P_{11}(t_m^-,\tau,\eta)\Phi_0(t_m+\eta,t)w\,d\eta d\tau+\\&+\int_{\beta_{t_m}(t)}^{0}[\Phi_1^*(t_m,t,\theta)P_{10}^*(t_m^-,\tau)\Phi_0(t_m+\tau,t)w+\Phi_1^*(t_m+\tau,t,\theta)P_{10}(t_m^-,\tau)\Phi_0(t_m,t)w]d\tau+\\&+I_{t_m,t}(\theta)[P_{10}(t_m^-,\theta+t-t_m)\Phi_0(t_m,t)w+\int_{\beta_{t_m}(t)}^{0}P_{11}(t_m^-,\theta+t-t_m,\tau)\Phi_0(t_m+\tau,t)w\,d\tau]+\\&+\int_t^{t_m}[\Phi_1^*(s,t,\theta)Q(s)\Phi_0(s,t)w-\Phi_1^*(s,t,\theta)P_{00}(s)G(s)P_{00}(s)\Phi_0(s,t)w]ds-\\&-\int_t^{t_m}\int_{\beta_s(t)}[\Phi_1^*(s,t,\theta)P_{00}(s)G(s)P_{10}^*(s,\tau)\Phi_0(s+\tau,t)w+\\&+\Phi_1^*(s+\tau,t,\theta)P_{10}(s,\tau)G(s)P_{00}(s)\Phi_0(s,t)w]d\tau ds-\\&-\int_t^{t_m}\Big\{I_{s,t}(\theta)[P_{10}(s,\theta+t-s)G(s)P_{00}(s)\Phi_0(s,t)w+\\&+\int_{\beta_s(t)}^{0}P_{10}(s,\theta+t-s)G(s)P_{10}^*(s,\tau)\Phi_0(s+\tau,t)w\,d\tau]\Big\}ds-\\&-\int_t^{t_m}\int_{\beta_s(t)}^{0}\int_{\beta_s(t)}^{0}\Phi_1^*(s+\tau,t,\theta)P_{10}(s,\tau)G(s)P_{10}^*(s,\eta)\Phi_0(s+\eta,t)w\,d\eta d\tau ds,\ t_{m-1}<t\leqslant t_m,\ \theta\in[-r,0],\ \forall w\in\mathbf{R}^d;\end{aligned}$$

$$
\begin{aligned}
P_{11}(t,\theta,\eta)w = {} & \Phi_1^*(t_m,t,\theta)P_{00}(t_m^-)\Phi_1(t_m,t,\eta)w + \int\limits_{\beta_{t_m}(t)}^{0}\int\limits_{\beta_{t_m}(t)}^{0}\Phi_1^*(t_m+\tau,t,\theta)P_{11}(t_m^-,\tau,\sigma)\Phi_1(t_m+\sigma,t,\eta)w\,d\sigma d\tau + \\
& + \int\limits_{\beta_{t_m}(t)}^{0}[\Phi_1^*(t_m,t,\theta)P_{10}^*(t_m^-,\tau)\Phi_1(t_m+\tau,t,\eta)w + \Phi_1^*(t_m+\tau,t,\theta)P_{10}(t_m^-,\tau)\Phi_1(t_m,t,\eta)w]d\tau + \\
& + I_{t_m,t}(\eta)\Big[\Phi_1^*(t_m,t,\theta)P_{10}^*(t_m^-,\eta+t-t_m)w + \int\limits_{\beta_{t_m}(t)}^{0}\Phi_1^*(t_m+\tau,t,\theta)P_{11}(t_m^-\tau,\eta+t-t_m)w\,d\tau\Big] + \\
& + I_{t_m,t}(\theta)\Big[P_{10}(t_m^-,\theta+t-t_m)\Phi_1(t_m,t,\eta)w + \int\limits_{\beta_{t_m}(t)}^{0}P_{11}(t_m^-,\theta+t-t_m,\tau)\Phi_1(t_m+\tau,t,\eta)w\,d\tau + \\
& + I_{t_m,t}(\eta)P_{11}(t_m^-,\theta+t-t_m,\eta+t-t_m)w\Big] + \int\limits_{t}^{t_m}\Phi_1^*(s,t,\theta)Q(s)\Phi_1(s,t,\eta)w\,ds - \\
(8.33)\quad & - \int\limits_{t}^{t_m}\int\limits_{\beta_s(t)}^{0}[\Phi_1^*(s,t,\theta)P_{00}(s)G(s)P_{10}^*(s,\tau)\Phi_1(s+\tau,t,\eta)w + \Phi_1^*(s+\tau,t,\theta)P_{10}(s,\tau)G(s)P_{00}(s)\Phi_1(s,t,\eta)w]d\tau ds - \\
& - \int\limits_{t}^{t_m}\Phi_1^*(s,t,\theta)P_{00}(s)G(s)P_{00}(s)\Phi_1(s,t,\eta)w\,ds - \int\limits_{t}^{t_m}\int\limits_{\beta_s(t)}^{0}\int\limits_{\beta_s(t)}^{0}\Phi_1^*(s+\tau,t,\theta)P_{10}(s,\tau)G(s)P_{10}^*(s,\sigma)\Phi_1(s+ \\
& +\sigma,t,\eta)w\,d\sigma d\tau ds - I_{t_m,t}(\theta)\int\limits_{t}^{t_m}\Big[P_{10}(s,\theta+t-s)G(s)P_{00}(s)\Phi_1(s,t,\eta)w + \\
& + \int\limits_{\beta_s(t)}^{0}P_{10}(s,\theta+t-s)G(s)P_{10}^*(s,\tau)\Phi_1(s+\tau,t,\eta)w\,d\tau\Big]ds - \\
& - \int\limits_{t}^{\alpha_{t_m,t}(\eta)}[\Phi_1^*(s,t,\theta)P_{00}(s)G(s)P_{10}^*(s,\eta+t-s)w + I_{t_m,t}(\theta)P_{10}(s,\theta+t-s)G(s)P_{10}^*(s,\eta+t-s)w]ds - \\
& - \int\limits_{t}^{\alpha_{t_m,t}(\eta)}\int\limits_{\beta_s(t)}^{0}\Phi_1^*(s+\tau,t,\theta)P_{10}(s,\tau)G(s)P_{10}^*s,\eta+t-s)w\,d\tau ds,\quad t_{m-1}\leqslant t<t_m, \\
& \theta\in[-r,0[\,,\ \eta\in[-r,0[\,w\in\mathbf{R}^d,
\end{aligned}
$$

where $I_{s,t}$ is given by (8.25) and $\alpha_{t_m,t}(\eta)=t_m\cdot I_{t_m,t}(\eta)+(\eta+r+t)\cdot[1-I_{t_m,t}(\eta)]$.
Similarly, from (6.20), taking into account (8.28) and (8.29), it follows

$$(8.34)\qquad P_{00}(t_m^-)w=P_{00}(t_m)\,[I+D_mP_{00}(t_m)]^{-1}w,$$

$$(8.35)\qquad P_{10}(t_m^-,\theta)w=P_{10}(t_m,\theta)\,[I+D_mP_{00}(t_m)]^{-1}w,\quad \forall\theta\in[-r,0[,$$

and

$$
(8.36)\qquad
\begin{aligned}
& P_{11}(t_m^-,\theta,\eta)w=P_{11}(t_m,\theta,\eta)w-P_{10}(t_m^-,\theta)D_mP_{10}^*(t_m,\eta)w,\\
& \theta\in[-r,0[,\ \eta\in[-r,0[,
\end{aligned}
$$

for every $w\in\mathbf{R}^d$ and each $m\in\{1,2,\dots,n\}$.

If $(\tilde P_{00}(t),\ \tilde P_{10}(t,\theta),\ \tilde P_{11}(t,\theta,\eta))$ represents the unique solution of (8.31) – (8.36) (including (6.2), sui able transposed in this context), then, according to Theorem 6.1, the feedback control of problem II is expressed as

$$
(8.37)\qquad
\begin{aligned}
\tilde u_0(t)= & -R^{-1}(t)C^*(t)\tilde P_{00}(t)\tilde x(t)- \\
& -\int\limits_{-r}^{0}R^{-1}(t)C^*(t)\tilde P_{10}^*(t,\theta)\tilde x(t+\theta)d\theta,\ a.e.\ t\in]0,T],
\end{aligned}
$$

$$(8.38)\qquad \tilde u_i=-N_i^{-1}B_i^*\tilde P_{00}(t_i)\tilde x(t_i)-\int\limits_{-r}^{0}N_i^{-1}B_i^*\tilde P_{10}^*(t_i,\theta)\tilde x(t_i+\theta)d\theta,\quad i=1,2,\dots,n.$$

Finally, the minimum cost is given (in this case) by

$$
(8.39)\qquad
\begin{aligned}
& (h^0,\tilde P_{00}(0)h^0)_{\mathbf{R}^d}+2\int\limits_{-r}^{0}(\tilde P_{10}(0,\eta)\,h^0,\ h^1(\eta))_{\mathbf{R}^d}\,d\eta+ \\
& +\int\limits_{-r}^{0}\int\limits_{-r}^{0}(\tilde P_{11}(0,\eta,\sigma)h^1(\sigma),\ h^1(\eta))_{\mathbf{R}^d}\,d\sigma\,d\eta,
\end{aligned}
$$

where $[h^0,h^1]=[\tilde x(0),\tilde x_0]$.

General remarks and conclusions. (*i*) The pure impulsive control case is that in which u_0 does not act at all, i.e. $C(t)\equiv 0$, $t\in[0,T]$. Obvionsly the results established here hold also in this case. Moreover, it should be noticed, then, that the feedback operator $\tilde P(t)$, and all optimal elements of $\mathfrak{P}$, can be obtained without difficulty.

(*ii*) On account of the assumption (*A*1), it could be considered not only an integral Riccati equation formulation for the feedback law, as in the present paper, but also the differential version of (6.19). The study of that appropriate differential operator equation and, implicitly, the adequate analytical aspects of $\mathfrak{P}$, can be made without any great difficulty, following [5], for instance. In order to maintain the form of the results, also in the case when, in (*A*1), $S(t,s)$ is assumed to be only a mild evolution operator, such a study has been avoided in this work.

(*iii*) It seems that the question of optimization of instants t_i, simultaneously with controls $u_i (i=1,2,\dots,n)$ does not make sense as a part of problem $\mathfrak{P}$. Certainly, something like this will be possible when other constraints on control and state will intervene in $\mathfrak{P}$.

(*iv*) In the time-independent case (with respect to $A(t)$), all the results of this paper hold as long as, in $A1$), $(A(t)\equiv A$ is supposed to generate a strongly continuous semigroup $\{Y(t),\ t\in[0,\ T]\}$. Throughout the last four sections $Y(t-s)$ will stand then for $S(t, s)$, $\forall (t, s)\in\mathfrak{T}$.

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A SINGULAR PERTURBATION METHOD FOR A SYSTEM OF FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

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1. Throughout this note, for every arbitrary fixed $n\in N$, R^n denotes the usual n-dimensional space endowed with a given usual norm $\|\cdot\|$ ($\|x\| = (|x_1|^p+\ldots+|x_n|^p)^{1/p}$, $x=(x_1,\ldots,x_n)\in R^n$, $1\leqslant p\leqslant\infty$). We shall denote by $\leqslant$ (respectively, $<$) the order relation in R^n induced by R^n_+ (respectively, $(R^0_+)^n$) and, for every $a, b\in R^n$, $a<b$, put $[a, b]=\{x\in R^n;\ a\leqslant x\leqslant b\}$, $[a, +\infty[=\{x\in R^n; a<x\}$; moreover, for any $x=(x_1,\ldots,x_n)>0$, define $1/x=(1/x_1,\ldots,1/x_n)$ and for every $x=(x_1,\ldots,x_n)\in R^n$, $y=(y_1,\ldots,y_n)\in R^n$, put $xy=(x_1y_1,\ldots,x_ny_n)$. For any linear functional $f\in(R^n)^*$, and every $\lambda\in R$, the notation $(f;\lambda)$ stands for the halfspace $f(x)\geqslant\lambda$. Let I be a finite index set. A family $((f_i;\lambda_i);i\in I)$ of half-spaces is said to be an *admissible family* if its intersection $H=\cap((f_i;\lambda_i);i\in I)$ is not empty, H being called the polygonal domain generated by the family $((f_i;\lambda_i);i\in I)$. In this context, $\psi: R^n\to R^n$ is called a *normal H-mapping*, provided that

(a) $\psi(R^n)\subset H$ and $\psi/H=i_H$ (the identity)
(b) $\|\psi(x)-\psi(y)\|\leqslant\|x-y\|$, for all $x, y\in R^n$
(c) $x\in R^n$, $i\in I$, $f_i(x)<\lambda_i$ imply $f_i(\psi(x))=\lambda_i$.

Concerning the above normality conditions (a)—(c), it should be noted that they are intimately related to „*rectangular*" polygonal domains. For example, if $H=[a, b]$, $a, b\in R^n$, $a<b$, we may write $H=\cap((f_i;\lambda_i);\ i=1,\ldots,2n)$ where $f_i(x)=x_i$, $f_{n+i}(x)=-x_i$, $\lambda_i=a_i$, $\lambda_{n+i}=-b_i$, for all $x=(x_1,\ldots,x_n)\in R^n$, $i=1,\ldots,n$, and a standard normal H-mapping is $\psi(x)=\min(\max(x, a), b)$, $x\in R^n$.

As a somewhat particular case, we may consider $H=[a, +\infty[$; here, obviously, $H=\cap((f_i;\lambda_i);\ i=1,\ldots,n)$ where $f_i(x)=x_i$, $\lambda_i=a_i$, for all $x=(x_1,\ldots,x_n)\in R^n$, $i=1,\ldots,n$, and a standard normal H-mapping is $\psi(x)=\max(x, a)$, $x\in R^n$

Under these notational conventions, the first aim of the present note is to state and prove a *flow-invariance result* — via *mean value theorems* for a class of functional differential Cauchy problems of the form (CP) (see § 2 below) extending in this way some recent results of this kind obtained by N. Pavel and M. Turinici [10] in the ordinary case. As a second objective, a *singular perturbation method* — via *flow-invariance techniques* — for a vector-parameter functional differential Cauchy problem of the form (CP(μ)) (see §3) is developped, giving thus a „functional" extension of some classical singular perturbation results established by