(*iv*) In the time-independent case (with respect to $A(t)$), all the results of this paper hold as long as, in $A1)$, ($A(t)\equiv A$ is supposed to generate a strongly continuous semigroup $\{Y(t),\ t\in[0,\ T]\}$. Throughout the last four sections $Y(t-s)$ will stand then for $S(t,s)$, $\forall(t,s)\in\mathfrak{T}$.

REFERENCES

1. Barbu, V. — *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International Publishing, Leyden, The Netherland — Ed. Acad. Bucureşti, România, 1976.
2. Barbu, V. and Precupanu, T. — *Convexity and optimization in Banach spaces*. Sijthoff & Noordhoff International Publishers — Ed. Acad. R.S.R., Bucureşti, 1978.
3. Batschelet, E., Brand, L. and Steiner, A. — *On the kinetics of lead in the human body*, J. Math. Biology, vol. 8, no. 1, 1979, pp. 15—23.
4. Curtain, R. F. and Pritchard, A. J. — *The infinite dimensional Riccati equation*, J. Math. Analysis and Appl., 47, July, 1974, pp. 43—57.
5. Curtain, R. F. and Pritchard, A. J. — *Infinite dimensional linear systems theory*, Lecture Notes in Control and Information Sciences, vol. 8, Springer—Verlag, Berlin, Heidelberg, New York, 1978.
6. Da Prato G. — *Équations d'évolution dans les algèbres d'opérateurs et application à des équations quasi-linéaires*, J. Math. Pures et Appl., 48, 1969, pp. 59—107.
7. Delfour, M. C. and Mitter, S. K. — *Controllability, observability and optimal feedback control of affine hereditary differential systems*, SIAM J. Control, vol. 10, no. 2, 1972, pp. 298—328.
8. Dunford, N. and Schwartz, T. — *Linear operators (I)*, Interscience Publishers, New York, London, 1958.
9. Lions, J. L. — *Contrôle optimal de sistémes gouvernés par des équations aux dérivées partielles*, Dunod, Gauthier Vilars, Paris, 1968.
10. Lions, J. L. — *Sur quelques questions d'analyse, de mécanique et de contrôle optimal* (ch. 5), Les Presses de l'Université de Montréal, 1976.
11. Noailles, J. — *Contrôle distribué et impulsionnel dans les équations d'évolution : Formulation d'un problème et existence de solutions*, C.R. Acad. Sc. Paris, 290, Série A no. 6, févr. 1980, pp. 283—284.
12. Noailles, J. — *Contrôle distribué et impulsionnel dans les équations d'évolution. Approximation et applications*, C. R. Acad. Sc. Paris, 290, Série A, févr. 1980, pp. 375—378.
13. Pierce, J. G. and Schumitzky, A. — *Optimal impulsive control of compartment models I : Qualitative aspects*, J.O.T.A., vol. 18, no. 4, 1976.
14. Pierce, J. G. and Schumitzky, A. — *Optimal impulsive control of compartment models II : The algorithm*, University of Southern California, Preprint No. 68, Nov. 1975.
15. **Răşcanu, A.** — ***Asupra controlului optimal stochastic pentru sisteme cu un număr finit de stări***, **St. Cerc. Mat., tom 27, Nr. 2, Bucureşti, 1975, pp. 195—217.**

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Received **30.V.1980**

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s. I a, 1981, f. 1

A SINGULAR PERTURBATION METHOD FOR A SYSTEM OF FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

MIHAI TURINICI

1. Throughout this note, for every arbitrary fixed $n\in N$, R^n denotes the usual n-dimensional space endowed with a given usual norm $\|\cdot\|$ ($\|x\|=$ $=(|x_1|^p+\ldots+|x_n|^p)^{1/p}$, $x=(x_1,\ldots,x_n)\in R^n$, $1\leqslant p\leqslant\infty$). We shall denote by $\leqslant$ (respectively, $<$) the order relation in R^n induced by R^n_+ (respectively, $(R^0_+)^n$) and, for every $a,b\in R^n$, $a<b$, put $[a,b]=\{x\in R^n;\ a\leqslant x\leqslant b\}$, $[a,+\infty[=\{x\in R^n;a<x\}$; moreover, for any $x=(x_1,\ldots,x_n)>0$, define $1/x=(1/x_1,\ldots,1/x_n)$ and for every $x=(x_1,\ldots,x_n)\in R^n$, $y=(y_1,\ldots,y_n)\in R^n$, put $xy=(x_1y_1,\ldots,x_ny_n)$. For any linear functional $f\in(R^n)^*$, and every $\lambda\in R$, the notation $(f;\lambda)$ stands for the halfspace $f(x)\geqslant\lambda$. Let I be a finite index set. A family $((f_i;\lambda_i);i\in I)$ of half-spaces is said to be an *admissible family* if its intersection $H=\cap((f_i;\lambda_i);i\in I)$ is not empty, H being called the polygonal domain generated by the family $((f_i;\lambda_i);i\in I)$. In this context, $\psi: R^n\to R^n$ is called a *normal H-mapping*, provided that

(a) $\psi(R^n)\subset H$ and $\psi/H=i_H$ (the identity)
(b) $\|\psi(x)-\psi(y)\|\leqslant\|x-y\|$, for all $x,y\in R^n$
(c) $x\in R^n$, $i\in I$, $f_i(x)<\lambda_i$ imply $f_i(\psi(x))=\lambda_i$.

Concerning the above normality conditions (a)—(c), it should be noted that they are intimately related to „*rectangular*" polygonal domains. For example, if $H=[a,b]$, $a,b\in R^n$, $a<b$, we may write $H=\cap((f_i;\lambda_i);\ i=1,\ldots,2n)$ where $f_i(x)=x_i$, $f_{n+i}(x)=-x_i$, $\lambda_i=a_i$, $\lambda_{n+i}=-b_i$, for all $x=(x_1,\ldots,x_n)\in R^n$, $i=1,\ldots,n$, and a standard normal H-mapping is $\psi(x)=\min(\max(x,a),b)$, $x\in R^n$.

As a somewhat particular case, we may consider $H=[a,+\infty[$; here, obviously, $H=\cap((f_i;\lambda_i);\ i=1,\ldots,n)$ where $f_i(x)=x_i$, $\lambda_i=a_i$, for all $x=(x_1,\ldots,x_n)\in R^n$, $i=1,\ldots,n$, and a standard normal H-mapping is $\psi(x)=$ $=\max(x,a)$, $x\in R^n$

Under these notational conventions, the first aim of the present note is to state and prove a *flow-invariance result* — via *mean value theorems* for a class of functional differential Cauchy problems of the form (CP) (see § 2 below) extending in this way some recent results of this kind obtained by N. Pavel and M. Turinici [10] in the ordinary case. As a second objective, a *singular perturbation method* — via *flow-invariance techniques* — for a vector-parameter functional differential Cauchy problem of the form (CP(μ)) (see §3) is developped, giving thus a „functional" extension of some classical singular perturbation results established by

A. N. Tikhonov [12]. A number of important applications, especially to *enzymatic reactions theory* will be discussed in a forthcoming paper.

2. Let $n \in N$ be a given natural number. Denote by X_n the class of all continuous vector-functions $x: R_+ \to R^n$ and, for every nonempty subset $H \subset R^n$, let $X_n(H)$ denote the set of all elements $x \in X_n$ satisfying $x(t) \in H$, $t \in R_+$. Now, let $x \vdash k(x)$ be a mapping from X_n into itself and $x^0 \in R^n$ a given element. In this case, let us consider the (functional differential) Cauchy problem

$$\text{(CP)} \qquad x'(t) = k(x)(t), \ t \in R_+, \ x(0) = x^0.$$

Furthermore, suppose $H = \cap((f_i; \lambda_i); i \in I)$ is a given polygonal domain generated by an admissible family of half-spaces $((f_i; \lambda_i); i \in I)$. *The considered polygonal domain H is said to possess a flow-invariance property with respect to the Cauchy problem (CP) if every solution $x \in X_n$ of (CP) with $x^0 \in H$ belongs to $X_n(H)$ (or, equivalently, every solution $x \in X_n$ of (CP) with $x^0 \in H$ will belong to H).* Concerning this notion, the following simple result may be stated and proved.

Lemma 2.1. *Under the above notational conventions, suppose that*

$$(2.1) \qquad t \in R_+, \ x \in X_n, \ i \in I, \ f_i(x(t)) < \lambda_i \ \textit{imply} \ f_i(k(x)(t)) \geq 0$$

Then, necessarily, H possesses a flow-invariance property with respect to the Cauchy problem (CP).

Proof. Suppose there exists a solution $x \in X_n$ of (CP) with $x^0 \in H$, that is, not remaining in H. Then, there is a $t_1 > 0$ and an $i \in I$ such that $f_i(x(t_1)) < \lambda_i$. Put $t_2 = \sup(t \in [0, t_1]; \ f_i(x(t)) \geq \lambda_i)$. Clearly, $t_2 \in [0, t_1[$, $f_i(x(t_2)) = \lambda_i$, $f_i(x(t)) < \lambda_i$, for all $t \in]t_2, t_1]$. Now, from Lagrange's theorem we derive (for some $t_3 \in]t_2, t_1[$).$0 > f_i(x(t_1)) - f_i(x(t_2)) = (t_1 - t_2) f_i(x'(t_3)) = (t_1 - t_2) f_i(k(x)(t_3))$ and thus $0 > f_i(k(x)(t_3))$. On the other hand, $t_3 \in]t_2, t_1[$ implies $f_i(x(t_3)) < \lambda_i$ so, by (2.1), $f(k(x)(t_3)) \geq 0$, a contradiction, proving our result. Q.E.D.

A close analysis of the invariance condition (2.1) shows that it is difficult to be handled, because the whole class X_n is involved there. It would be of interest to replace it by an invariance condition in which only the class $X_n(H)$ be involved. To this end, suppose the polygonal domain H defined before possesses at least a normal H-mapping ψ (denoted, by convention, $x \vdash \bar{x}$). In this case, we may construct a mapping $x \vdash \bar{x}$ from X_n into $X_n(H)$ by $\bar{x}(t) = \overline{x(t)}$, $t \in R_+$, $x \in X_n$ and a mapping $x \vdash \bar{k}(x)$ from X_n into itself by $\bar{k}(x) = k(\bar{x})$, $x \in X_n$.

Furthermore, let us associate to the initial Cauchy problem (CP), the following Cauchy problem

$$\overline{\text{(CP)}} \qquad x'(t) = \bar{k}(x)(t), \ t \in R_+, \ x(0) = x^0.$$

In this case, the main flow invariance result of this paragraph is

Theorem 2.1. *Under the general hypotheses stated before, suppose the following condition holds*

$$(2.2) \qquad t \in R_+, \ x \in X_n(H), \ i \in I, \ f_i(x(t)) = \lambda_i \ \textit{imply} \ f_i(k(x)(t)) \geq 0;$$

Then, necessarily, every solution $x \in X_n$ of $\overline{(CP)}$ with $x^0 \in H$ will remain in $X_n(H)$ and hence it is in the same time a solution of (CP) remaining in H.

Proof. Let $t \in R_+$, $x \in X_n$ and $i \in I$ be such that $f_i(x(t)) < \lambda_i$. From the definition of a normal H-mapping, $\bar{x} \in X_n(H)$ and $f_n(\bar{x}(t)) = \lambda_i$, hence by (2.2), $f_i(k(\bar{x})(t)) \geq 0$, that is, $f_i(\bar{k}(x)(t)) \geq 0$, proving (2.1) holds. Therefore, by the preceding lemma, H possesses a flow-invariance property with respect to $\overline{(CP)}$. But, evidently, from the definition of $\bar{k}$, any solution in $X_n(H)$ of $\overline{(CP)}$ is in the same time a solution of (CP) and this completes the proof. Q.E.D.

In what follows, it is convenient to write (2.2) under the symbolic form

$$(2.2)' \qquad f(k(x)(t)/(x \in X_n(H), \ f(x(t)) = \lambda) \geq 0.$$

Concerning the above theorem, it is important to observe that, if $\overline{(CP)}$ has no solution then, (CP) may have no solutions in $X_n(H)$. In this case, it is justified to introduce some useful notations. More exactly, a mapping $x \vdash k(x)$ is said to be *weakly H-admissible* iff, for every $x^0 \in H$, the associated Cauchy problem $\overline{(CP)}$ has at least a solution in X_n and *strongly H-admissible* iff, for every $x^0 \in H$, the Cauchy problem (CP) has at most a solution in X_n. As a simple remark, it is clear that, if $x \vdash k(x)$ is both weakly and strongly H-admissible then, for every $x^0 \in H$, (CP) possesses a unique solution in $X_n(H)$, provided (2.2) holds.

Now, as an application, let us consider some particular cases, largely used in the sequel.

Case 1. $H = [a, b]$, $a, b \in R^n$, $a < b$. Then, the flow-invariance condition (2.2) becomes

$$(2.3)_1 \qquad t \in R_+, \ x \in X_n([a, b]), \ i \in \{1, \ldots, n\}, \ x_i(t) = a_i, \quad \textit{imply} \ k_i(x)(t) \geq 0;$$

$$(2.3)_2 \qquad t \in R_+, \ x \in X_n([a, b]), \ i \in \{1, \ldots, n\}, \ x_i(t) = b_i, \quad \textit{imply} \ k_i(x)(t) \leq 0,$$

or, in our symbolic notations

$$(2.3) \qquad k(x)(t)/(x \in X_n([a, b]), \ x(t) = a) \geq 0; \quad k(x)(t)/(x \in X_n([a, b]), \ x(t) = b) \leq 0;$$

Case 2. $H = [a, +\infty[$, $a \in R^n$. In this case, the flow-invariance condition (2.2) becomes

$$(2.4) \qquad t \in R_+, \ x \in X_n([a, +\infty[), \ i \in \{1, \ldots, n\}, \ x_i(t) = a_i \quad \textit{imply} \ k_i(x)(t) \geq 0,$$

or, in our symbolic form,

$$(2.4)' \qquad k(x)(t)/(x \in X_n([a, +\infty[), \ x(t) = a) \geq 0.$$

It is important to remark that the corresponding statement of theorem 2.1 in these particular cases may be considered as a „functional" extension of a result due to N. Pavel and M. Turinici [10] (see also H. Brézis [2], M. G. Crandall [5]. On the other hand, conditions implying weakly or strongly H-admissibility are to be obtained by a successive approximation method (see, e.g., theorem 3.2 of the author's paper [13]).

3. Let $x \vdash k(x)$ be a mapping from X_n into itself, $\omega \in R^n_+$ a fixed vector and $\mu \vdash x^0(\mu)$ a mapping from the n-dimensional interval $]\omega, +\infty[= \{\mu \in R^n; \ \omega < \mu\}$ into R^n. Let us consider the (vector) parametric Cauchy problem

$$\mu x'(t)=k(x)(t),\ t\in R_+,\ x(0)=x^0(\mu). \tag*{$(CP(\mu))$}$$

It is supposed further that $x^0(\mu)\to x^0=x^0(\omega)$ as $\mu\to\omega$ and, in this case, let us associate to $(CP(\mu))$ the degenerated Cauchy problem

$$\omega x'(t)=k(x)(t),\ t\in R_+,\ x(0)=x^0(\omega). \tag*{$(DCP(\omega))$}$$

An important question concerning these Cauchy problems is that of finding sufficient conditions in order that the (generalized) sequence of solutions of $(CP(\mu))$ be convergent, as $\mu\to\omega$, to a solution of $(DCP(\omega))$, this convergence being understood in the sense of the usually local convex topology of X_n defined by the monotone and sufficient family of seminorms $P=(p_i\,;\, i\in N)$, $p_i(x)=\sup\{\|x(t)\|\,;\ t\in[0,i]\}$ $x\in X_n$, $i\in N$. A first lot of results in this direction was obtained, in case of an ordinary differential system, by A. N. Tikhonov [12] (see also K. O. Friedrichs and W. Wasow [6], N. Levinson [8], and A. B. Vasil'eva [14]) through a parametric stability procedure. It is the main intention of this note to give an useful answer to the question formulated above, extending in this way, to a „functional" situation the results of the authors quoted above and, in this context, it is not without importance to emphasize that our methods in proving our main singular perturbation result are flow-invariance methods (developped in the preceding paragraph) in comparison with the preceding parametric stability methods.

Now, in view of the formulation of the main result, we have to make a number of hypotheses listed below:

(H). *The data satisfy the following conditions*

$$x^0(\mu)\in R^n_+,\ \textit{for all}\ \mu>\omega, \tag{3.1}$$

(3.2) *for every* $\mu>\omega$, *the mapping* $x\vdash(1/\mu)k(x)$ *from* X_n into *itself is both weakly and strongly* R^n_+*-admissible*;

$$k(x)(t)/(x\in X_n(R^n_+),\ x(t)=0)\geqslant 0. \tag{3.3}$$

An important consequence of this fact is that, for every $\mu>\omega$, the Cauchy problem $(CP(\mu))$ has a unique solution in $X_n(R^n_+)$, solution denoted by $x(\mu, x^0(\mu))$ or, simply, by $x(\mu)$. Indeed, it suffices to observe that, from (3.1)—(3.3), all conditions of theorem 2.1 hold.

(K). *A second hypothesis is that expressed by*

$$\omega<\mu\leqslant\nu\ \textit{implies}\ x^0(\mu)\leqslant x^0(\nu), \tag{3.4}$$

(3.5) *for every* $\mu,\nu>\omega$, $\mu\leqslant\nu$, *and every* $y\in X_n(R^n_+)$ *with* $y(0)=x^0(\mu)$, *the mapping* $x\vdash(1/\mu)k(y+x)-(1/\mu)k(y)$ *from* X_n *into itself is both weakly and strongly* R^n_+ *— admissible, and*

$$(1/\nu)k(y+x)(t)-(1/\mu)k(y)(t)/(x\in X_n(R^n_+),\ x(t)=0)\geqslant 0. \tag{3.6}$$

In this case, the mapping $\mu\vdash x(\mu)$ from $]\omega, +\infty[$ into $X_n(R^n_+)$ is *monotone*, i.e.,

$$\omega<\mu\leqslant\nu\ \textit{implies}\ x(\mu)\leqslant x(\nu). \tag*{(m)}$$

Indeed, let $\mu,\nu>\omega$, $\mu\leqslant\nu$, and put $z=x(\nu)-x(\mu)$. It is evident that z is a solution in X_n of the Cauchy problem

$$x'(t)=(1/\nu)k(x(\mu)+x)(t)-(1/\mu)k(x(\mu))(t), t\in R_+;\ x(0)=x^0(\nu)-x^0(\mu). \tag*{$(CP(\mu,\nu))$}$$

From the weakly R^n_+ — admissibility, combined with the invariance condition (3.6) it follows, again by theorem 2.1, that $(CP(\mu,\nu))$ possesses at least a solution in $X_n(R^n_+)\subset X_n$ and, by strongly R^n_+ — admissibility, every solution of $(CP(\mu,\nu))$ must be identical with z; hence, necessarily, $z\in X_n(R^n_+)$, proving the evaluation (m).

It is supposed further that the mapping $x\vdash k(x)$ admits a „*derivative*" in the sense that, there is a mapping $(x, y)\vdash K(x, y)$ from X_n^2 into X_n such that

$$k(x)'(t)=K(x, x')(t),\ t\in R_+\ ('=d/dt). \tag*{(d)}$$

for every $x\in X_n$ having a derivative $x'\in X_n$; note that, in the ordinary case, this hypothesis is equivalent with the existence of *partial derivatives* of the function generating the considered Cauchy problem;

(L). *There exists vectors* $a, b\in R^n_+$, *such that, for every* $\mu>\omega$ *and* $y\in X_n(R^n_+)$ *with* $y(0)=x^0(\mu)$, *the following conditions hold*

$$-\mu a\leqslant k(y)(0)\leqslant\mu b, \tag{3.7}$$

(3.8) *the mapping* $x\vdash(1/\mu)K(y, x)$ *from* X_n *into itself is both weakly and strongly* $[-a, b]$*-admissible and*

$$K(y, x)(t)/(x\in X_n([-a, b]),\ x(t)=-a)\geqslant 0, \tag*{$(3.9)_1$}$$

$$K(y, x)(t)/(x\in X_n([-a, b]),\ x(t)=b)\leqslant 0, \tag*{$(3.9)_2$}$$

As a consequence of this fact, we derive the evaluation, for all $\mu>\omega$;

$$-a\leqslant x(\mu)'(t)\leqslant b,\ t\in R_+. \tag*{(e)}$$

Indeed, let $\mu>\omega$ be arbitrary and put $z=(1/\mu)k(x(\mu))$. It is clear that z satisfies the Cauchy problem

$$x'(t)=(1/\mu)K(x(\mu), x)(t),\ t\in R_+,\quad x(0)=(1/\mu)k(x(\mu))(0), \tag*{$((CP)'(\mu))$}$$

and therefore, by the same argument as that indicated in (K), the evaluation (e) holds.

(M). *Finally, the mapping* $x\vdash k(x)$ *is a continuous mapping from* X_n *into itself, in the sense of the locally convex topology of* X_n *introduced above.*

Let us accept all hypotheses (H)—(M) stated before. Firstly, from the evaluations $(m)+(e)$ combined with the Ascoli-Arzelá theorem and Dini's theorem [7, p. 239] the generalized sequence of functions $(x(\mu)\,;\ \mu>\omega)\subset X_n(R^n_+)\subset X_n$ converges uniformly on every compact of R_+ to a function $x(\omega)\in X_n(R^n_+)\subset X_n$. Secondly, if we put $(CP(\mu))$ under the integral from

$$\mu(x(\mu)(t)-x^0(\mu))=\int_0^t k(x(\mu))(s)ds,\ t\in R_+$$

and take the limit as $\mu\to\omega$ we get (from the last hypothesis (M))

$$\omega(x(\omega)(t)-x^0(\omega))=\int_0^t k(x(\omega))(s)ds,\ t\in R_+$$

and this shows that $x(\omega)$ is a solution in $X_n(R^n_+)\subset X_n$ of the degenerated Cauchy problem $(DCP(\omega))$. Therefore, we proved the following singular perturbation result about $(CP(\mu))$.

Theorem 3.1. *Under the hypotheses (H)—(M), the (generalized) sequence $(x(\mu);\ \mu > \omega) \subset X_n(R_+^n)$ of solutions of the Cauchy problems $((CP(\mu));\ \mu > \omega)$, converges in a decreasing way and uniformly on every compact of R_+ to a solution $x(\omega) \in X_n(R_+^n)$ of the degenerated Cauchy problem $(DCP(\omega))$.*

The practical value of the above singular perturbation result is expressed by the fact that, for values of the (vector) parameter μ sufficiently close to ω, we may approximate, on a given interval $[0, T]$ $T > 0$, the solution $x(\mu)$ of $(CP(\mu))$ by the „asymptotic" solution $x(\omega)$ of $(DCP(\omega))$ (of course, this approximation procedure is effectively practical only in case of a simple $(DCP(\omega))$). From this point of view, our main result could be used in deriving a „functional" type Michaelis-Menten theory of enzymatic reactions (see, e.g., H.T. Banks [1], J. D. Murray [9, ch. 1], S. I. Rubinow [11, ch. 2]) or, in a number of singular perturbation problems arising in mechanics (see, e.g., J. D. Cole [3] and W. Wasow [15].

REFERENCES

1. Banks H. T. — *Modelling and control in the biomedical sciences*, Lecture Notes in Biomathematics, vol. 6, Springer Verlag, Berlin — Heidelberg — New York, 1975.
2. Brezis H. — *On a characterization of flow-invariant sets*, Comm. Pure Appl. Math., 23 (1970), 161—163.
3. Cole J. D. — *Perturbation methods in applied mathematics*, Ginn (Blaisdell) Boston, Mass. 1968.
4. Corduneanu C. — *Sur certaines equations fonctionelles de Volterra*, Funck. Ekv., 9 (1966), 119—127.
5. Crandall M. G. — *A generalization of Peano's existence theorem and flow invariance*, Proc. Amer. Math. Soc., 179 (1973), 399—414.
6. Friedrichs K. O. and Wasow W. — *Singular perturbations of non-linear oscillations*, Duke Math. J., 13 (1946), 367—381.
7. Kelley J. — *General topology*, Van Nostrand, Princeton, N. J., 1955.
8. Levinson N. — *Perturbations of discontinuous solutions of non-linear systems of differential equations*, Acta Math., 82 (1951), 71—106.
9. Murray J. D. — *Lectures on nonlinear-differential equation ; models in biology*, Clarendon, Press, Oxford, 1977.
10. Pavel N. and Turinici M. — *Positive solutions of a system of ordinary differential equations*, An. şt. Univ. Iaşi, s. I a, t. XXIV (1978), 63—70.
11. Rubinow S. I. — *Introduction to mathematical biology*, John Wiley, New York, 1975.
12. Tikhonov A. N. — *Systems of differential equations containing small parameters in the derivatives*, Mat. Sbornik, N.S., 31 (73) (1952), 575—586 (in Russian).
13. Turinici M. — *Invariant polygonal domains for a class of functional differential equations*. Bull. Math. Soc. Sci. Math. Roum., to appear.
14. Vasil'eva A. B. — *Asymptotic behaviour of solutions of certain problems for ordinary non-linear differential equations with a small parameter multiplying the highest derivatives*, Uspehi Mat. Nauk, 18 (1963), 15—86 (in Russian).
15. Wasow W. — *Asymptotic expansions for ordinary differential equations*, Interscience Publ., New York — London — Sydney, 1965.

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Received 12.XII.1979

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s. I a, 1981, f. 1

COMMON FIXED POINT THEOREMS IN UNIFORMLY CONVEX BANACH SPACES

BY

B. G. PACHPATTE

1. Introduction. Recently, Goebel, Kirk and Shimi [3], Bose [1] and Rhoades [8] have proved some interesting fixed point theorems for operators, mapping a uniformly convex Banach space into itself. The main purpose of the present paper is to prove some fixed point theorems for mappings satisfying a new contractive type condition in a uniformly convex Banach space into itself.

Throughout this paper, we denote by R_+ the nonnegative reals. Let Ψ denote the family of all mappings from R_+^4 to R_+ which are upper semi-continuous and nondecreasing in each variable. The modulus of convexity of a Banach space X is a function $\delta : (0,2] \to (0,1]$ defined by

$$\delta(\varepsilon) = \inf \{1 - (1/2)\|x+y\| : x, y \in X, \|x\| = \|y\| = 1, \|x-y\| \geq \}\varepsilon.$$

It is known [5] that if X is uniformly convex then δ is strictly increasing, $\lim\limits_{\varepsilon\to 0} \delta(\varepsilon) = 0$ and $\delta(2) = 1$. Denoting the inverse of δ by η, we note that $\eta(t) < < 2$ for $t < 1$.

2. Main Results. In the proofs of our main results we will make use of the following lemma proved by Goebel, Kirk and Shimi in [3] (see, also [1]).

Lemma 1. *Let X be a uniformly convex Banach space. Let B_R denote the closed ball in X centred at the origin with radius $R > 0$. If $x_1, x_2, x_3 \in B_R$,*

$$\|x_1 - x_2\| \geq \|x_2 - x_3\| \geq d > 0 \ \text{and}\ \|x_2\| \geq (1-(1/2)\delta(d/R))R,$$

$$\text{then}\ \|x_1 - x_3\| \leq \eta(1-(1/2)\delta(d/R))\|x_1 - x_2\|.$$

We are now in a position to prove our theorems.

Theorem 1. *Let X be a uniformly convex Banach space and K a nonempty closed convex subset of X. Let f and g be two mappings of K into K such that for all x, y, in K,*

$$\|f(x) - g(y)\|^2 \leq \Phi(\|x - f(x)\| \cdot \|y - g(y)\|, \|x - g(y)\| \cdot \|y - f(x)\|, \\ \|x - f(x)\| \cdot \|y - f(x)\|, \|x - g(y)\| \cdot \|y - g(y)\|). \tag{1}$$

where $\Phi \in \Psi$ and, for all $t > 0$,

(i) $\Phi(t, 0, 0, \alpha t) \leq \beta t$ and $\Phi(t, 0, \alpha t, 0) \leq \beta t$ where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$;