Theorem 3.1. *Under the hypotheses (H)—(M), the (generalized) sequence* $(x(\mu);\ \mu > \omega) \subset X_n(R^n_+)$ *of solutions of the Cauchy problems* $((CP(\mu));\ \mu > \omega)$, *converges in a decreasing way and uniformly on every compact of* R_+ *to a solution* $x(\omega) \in X_n(R^n_+)$ *of the degenerated Cauchy problem* $(DCP(\omega))$.

The practical value of the above singular perturbation result is expressed by the fact that, for values of the (vector) parameter μ sufficiently close to ω, we may approximate, on a given interval $[0, T]$ $T > 0$, the solution $x(\mu)$ of $(CP(\mu))$ by the „asymptotic" solution $x(\omega)$ of $(DCP(\omega))$ (of course, this approximation procedure is effectively practical only in case of a simple $(DCP(\omega))$). From this point of view, our main result could be used in deriving a „functional" type Michaelis-Menten theory of enzymatic reactions (see, e.g., H.T. Banks [1], J. D. Murray [9, ch. 1], S. I. Rubinow [11, ch. 2]) or, in a number of singular perturbation problems arising in mechanics (see, e.g., J. D. Cole [3] and W. Wasow [15]).

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COMMON FIXED POINT THEOREMS IN UNIFORMLY CONVEX BANACH SPACES

BY

B. G. PACHPATTE

1. Introduction. Recently, Goebel, Kirk and Shimi [3], Bose [1] and Rhoades [8] have proved some interesting fixed point theorems for operators, mapping a uniformly convex Banach space into itself. The main purpose of the present paper is to prove some fixed point theorems for mappings satisfying a new contractive type condition in a uniformly convex Banach space into itself.

Throughout this paper, we denote by R_+ the nonnegative reals. Let Ψ denote the family of all mappings from R^4_+ to R_+ which are upper semi-continuous and nondecreasing in each variable. The modulus of convexity of a Banach space X is a function $\delta : (0,2] \to (0,1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - (1/2)\|x+y\| : x, y \in X, \|x\| = \|y\| = 1, \|x-y\| \geq\}\varepsilon.$$

It is known [5] that if X is uniformly convex then δ is strictly increasing, $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ and $\delta(2) = 1$. Denoting the inverse of δ by η, we note that $\eta(t) < 2$ for $t < 1$.

2. Main Results. In the proofs of our main results we will make use of the following lemma proved by Goebel, Kirk and Shimi in [3] (see, also [1]).

Lemma 1. *Let X be a uniformly convex Banach space. Let B_R denote the closed ball in X centred at the origin with radius $R > 0$. If $x_1, x_2, x_3 \in B_R$,*

$$\|x_1 - x_2\| \geq \|x_2 - x_3\| \geq d > 0 \ \text{ and } \ \|x_2\| \geq (1-(1/2)\delta(d/R))R,$$

$$\text{then } \|x_1 - x_3\| \leq \eta(1-(1/2)\delta(d/R))\|x_1 - x_2\|.$$

We are now in a position to prove our theorems.

Theorem 1. *Let X be a uniformly convex Banach space and K a nonempty closed convex subset of X. Let f and g be two mappings of K into K such that for all x, y, in K,*

$$(1) \quad \|f(x) - g(y)\|^2 \leq \Phi(\|x - f(x)\|\cdot\|y - g(y)\|, \|x - g(y)\|\cdot\|y - f(x)\|, \|x - f(x)\|\cdot\|y - f(x)\|, \|x - g(y)\|\cdot\|y - g(y)\|).$$

where $\Phi \in \Psi$ and, for all $t > 0$,

(i) $\Phi(t, 0, 0, \alpha t) \leq \beta t$ *and* $\Phi(t, 0, \alpha t, 0) \leq \beta t$ *where* $\beta = 1$ *for* $\alpha = 2$ *and* $\beta < 1$ *for* $\alpha < 2$;

(*ii*) $\Phi(0, t, 0, \alpha t) < t$ *for* $\alpha < 2$.
Then there exists a point $u \in K$ *such that*
(*a*) $f(u) = g(u) = u$;
(*b*) u *is the unique fixed point of each* f *and* g;
(*c*) *for any* $x_0 \in K$, *the sequence* $\{x_n\}$ *defined recursively by* $x_1 = f(x_0)$, $x_2 = g(x_1)$, $x_3 = f(x_2), \ldots$ *converges strongly to* u.

Proof. Choose $x_0 \in K$ and let $\{x_n\}$ be the sequence defined as in (*c*). Let $d_n = \| x_n - x_{n+1} \|$. First we show that

$$d_n \leqslant d_{n-1}, \quad n = 1, 2, \ldots \tag{2}$$

Using (1), we have for odd n,

$$d_n^2 = \| f(x_{n-1}) - g(x_n) \|^2 \leqslant \Phi(d_{n-1} \cdot d_n, 0, 0, \| x_{n-1} - x_{n+1} \| \cdot d_n), \tag{3}$$

and for even n

$$d_n^2 = \| g(x_{n-1}) - f(x_n) \|^2 \leqslant \Phi(d_n \cdot d_{n-1}, 0, d_n \cdot \| x_{n-1} - x_{n+1} \|, 0). \tag{4}$$

Since $\| x_{n-1} - x_{n+1} \| \leqslant d_{n-1} + d_n$, we have

$$d_n^2 \leqslant \begin{cases} \Phi(d_{n-1} \cdot d_n, 0, 0, [d_{n-1} + d_n] \cdot d_n) & \text{for odd } n \\ \Phi(d_n \cdot d_{n-1}, 0, d_n \cdot [d_{n-1} + d_n], 0) & \text{for even } n. \end{cases}$$

Suppose for some n, $d_n > d_{n-1}$. Then $d_{n-1} + d_n = \alpha d_n$ with some $\alpha < 2$ and Φ being nondecreasing in each variable

$$d_n^2 \leqslant \begin{cases} \Phi(d_n^2, 0, 0, \alpha\, d_n^2) & \text{if } n \text{ is odd}, \\ \Phi(d_n^2, 0, \alpha\, d_n^2, 0) & \text{if } n \text{ is even}. \end{cases}$$

In either case, (i) gives the contradiction: $d_n^2 \leqslant \beta d_n^2$ for some $\beta < 1$. This proves (2) and therefore $d_n \to d$ (say) $\geqslant 0$.

Now we show that $d = 0$. If not, suppose $d > 0$. Assume without loss of generality that $0 \in K$. Let $R' = \limsup_n \| x_n \|$.

We can assume $R' \neq 0$. Let $R > R'$ be chosen so that $R[1 - (1/2)\delta(d/R)] < R'$. Clearly there is a sequence $\{n_i\}$, $i = 0, 1, \ldots$ of positive integers such that for any $j \in \{n_i\}$, $\| x_j \| \geqslant (1 - (1/2)\delta(d/R))$, while for all $n \geqslant n_0$, $\| x_n \| \leqslant R$. It therefore follows from Lemma 1 that for any $j \in \{n_i\}$,

$$\| x_{j-1} - x_{j+1} \| \leqslant \eta_j (1 - (1/2)\delta(d/R)) \| x_{j-1} - x_j \| = \alpha_1 \| x_{j-1} - x_j \|, \tag{5}$$

where $\alpha_1 = \eta_j (1 - (1/2)\delta(d/R)) < 2$ because of uniform convexity. Since $d_j \leqslant d_{j-1}$, it follows from (3), (4) and (5) that

$$d_j^2 \leqslant \begin{cases} \Phi(d_{j-1}^2, 0, 0, \alpha_1 d_{j-1}^2) & \text{for odd } j \in \{n_i\}, \\ \Phi(d_{j-1}^2, 0, \alpha_1 d_{j-1}^2, 0) & \text{for even } j \in \{n_i\}. \end{cases}$$

In either case, (*i*) yields

$$d_j^2 \leqslant \beta_1 d_{j-1}^2 \quad \text{for some } \beta_1 < 1. \tag{6}$$

Note that this β_1 is independent of j. If $\varepsilon > 0$ is small enough so that $\beta_1(d^2 + \varepsilon) < d^2$ and $j \in \{n_i\}$ is chosen so large that $d_{j-1}^2 \leqslant d^2 + \varepsilon$, then (6) gives $d_j^2 < d^2$, which contradicts $d_n \downarrow d$ and proves $d = 0$.

Now let us consider the sequence $\{x_{2n}\}$. For convenience, set $y_n = x_{2n}$.
We have $\| y_n - f(y_n) \| \to 0$. For $y \in K$, define $r(y) = \limsup_n \| y - y_n \|$. It is
easy to show that r is continuous and convex on K and so is weakly lower
semicontinuous. Therefore it attains its infimum at a point u say. By uniform
convexity, one can easily show that u is unique (u is called the asympto-
tic centre of $\{y_n\}$ un K, see [2]).

Now we show that $g(u) = u$. Since

$$\| g(u) - y_n \|^2 \leqslant [\| g(u) - f(y_n) \| + \| y_n - f(y_n) \|]^2 = \| g(u) - f(y_n) \|^2 + 2 \| g(u) - f(y_n) \| \cdot \| y_n - f(y_n) \| + \| y_n - f(y_n) \|^2$$

we have $r^2(g(u)) \leqslant \limsup_n \| g(u) - f(y_n) \|^2$. Now

$$\| g(u) - f(y_n) \|^2 \leqslant \Phi(\| y_n - f(y_n) \| \cdot \| u - g(u) \|, \| y_n - g(u) \| \cdot \| u - f(y_n) \|, \| y_n - f(y_n) \| \cdot \| u - f(y_n) \|, \| y_n - g(u) \| \cdot \| u - g(u) \|).$$

Thus,

$$r^2(g(u)) \leqslant \limsup_n \Phi(\| y_n - f(y_n) \| \cdot \| u - g(u) \|, \| y_n - g(u) \| \cdot \| u - f(y_n) \|, \| y_n - f(y_n) \| \cdot \| u - f(y_n) \|, \| y_n - g(u) \| \cdot \| u - g(u) \|).$$

Passing to subsequences, it follows from the upper semicontinuity of Φ that

$$r^2(g(u)) \leqslant \Phi(0, r(g(u)) \cdot r(u), 0, r(g(u)) [r(u) + r(g(u))]). \tag{7}$$

If $g(u) \neq u$, then $r(g(u)) > r(u)$ and in that case (7) gives $r^2(g(u)) \leqslant$
$\leqslant \Phi(0, r^2(g(u)), 0, \alpha_2 r^2(g(u))$ for some $\alpha_2 < 2$. But then (ii) yields the contra-
diction $r^2(g(u)) < r^2(g(u))$ and this proves $g(u) = u$. The fact that $f(u) = u$
follows from: $\| f(u) - u \|^2 = \| f(u) - g(u) \|^2 \leqslant \Phi(0, 0, \| u - f(u) \|^2, 0) < \| u -$
$- f(u) \|^2$ by (1) unless $f(u) = u$. To complete the proof of (*a*) and (*b*) we need
only to show that if there exists $v \in K$ such that $v = f(v)$ or $v = g(v)$, then v
must coincide with u. Let us assume, for instance, that $v = g(v)$ and $v \neq u$.
Then

$$\| u - v \|^2 = \| f(u) - g(v) \|^2 \leqslant \Phi(0, \| u - v \|^2, 0, 0) < \| u - v \|^2$$

by (*ii*) which is a contradiction. The proof of (*a*) and (*b*) is thus complete.

For the proof of (*c*), it suffices to show that $s(u) = \limsup_n \| u - x_n \| = 0$.
For odd n we have

$$\| u - x_{n\pm1} \|^2 = \| f(u) - g(x_n) \|^2 \leqslant \Phi(0, \| u - x_{n\pm1} \| \cdot \| x_n - u \|, 0, \| u - x_{n\pm1} \| \cdot d_n),$$

and for even n

$$\| u - x_{n\pm1} \|^2 = \| g(u) - f(x_n) \|^2 \leqslant \Phi(0, \| x_n - u \| \cdot \| u - x_{n+1} \|, d_n \cdot \| u - x_{n\pm1} \|, 0).$$

Passing to subsequences, it follows from the upper semi-continuity of Φ that $s^2(u) \leqslant \Phi(0, s^2(u), 0, 0)$. If $s(u) \neq 0$, then (*ii*) gives the contradiction $s(u) < s(u)$ proving that $s(u) = 0$. This completes the proof of the theorem.

The following theorem is an immediate consequence of Theorem 1.

Theorem 2. *Let* X *and* K *be as in Theorem* 1. *Let* F *be the family of all mappings from* K *into* K, *such that for any pair* $f, g \in F$, *there exists a* $\Phi = \Phi(f, g) \in \Psi$ *such that* (1), (*i*) *and* (*ii*) *of Theorem* 1 *are satisfied. Then*

(a) *there exists a unique* $u \in K$ *such that* $f(u)=u$ *for all* $f \in F$;
(b) u *is the unique fixed point of any* $f \in F$;
(c) *for any pair* $f, g \in F$, *the sequence defined as in Theorem 1 converges strongly to* u.

Another interesting and slightly different version of Theorem 1 is established in the following theorem.

Theorem 3. *Let* X *and* K *be defined as in Theorem 1. Let* F *be the family of all mappings of* K *into* K, *such that for any pair* $f, g \in F$, *there exist a positive integer* $m=m(f, g)$ *and* $n=n(f, g)$ *and a* $\Phi=\Phi(f, g) \in \Psi$, *such that conditions (i) and (ii) of Theorem 1 and*

$$\|f^m(x)-g^n(y)\| \leq \Phi(\|x-f^m(x)\| \cdot \|y-g^n(y)\|, \|x-g^n(y)\| \cdot \|y-f^m(x)\|, \|x-f^m(x)\| \cdot \|y-f^m(x)\|, \|x-g^n(y)\| \cdot \|y-g^n(y)\|) \tag{8}$$

are satisfied. Then the conclusion of Theorem 2 hold.

Proof. Let $f_1=f^m$ and $g_1=g^n$. Then f_1 and g_1 satisfy the conditions of Theorem 1 and hence there is a unique $u \in K$, such that $f^m(u)=g^n(u)=u$ and u is a unique fixed point of each f^m and g^n.

Since $f^m(f(u))=f(f^m(u))=f(u)$, it follows that $f(u)$ is a fixed point of f^m and thus it must coincide with u. Hence $f(u)=u$ and similarly $g(u)=u$. The rest of the proof is obvious in view of the proof of Theorem 1. We omit the details.

In concluding this paper we note that there is no essential difficulty to show that theorems paralled to Theorems 3 and 4 in [4] also hold in the present setting of mappings. We also note that, in [1, 3, 8] the authors have established some fixed point theorems in uniformly convex Banach spaces. However, the mappings considered in this paper are different from those considered in [1, 3, 8]. For several interesting fixed point theorems in the present setup of mappings in metric spaces we refer the interested reader to the authors recent papers [6] and [7].

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JOINTLY-CONTRACTIVE MAPPINGS AND FIXED POINT THEOREMS

BY

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1. Introduction and preliminaries. Various generalizations and extensions of the well-known Banach Contraction Principle have appeared in recent years. Edelstein extended this famous theorem for contractive mappings.

Definition 1.1. *A mapping* f *of a metric space* (X, d) *into itself is said to be a contractive mapping if*

$$d(f(x), f(y)) < d(x, y),\ x \neq y.$$

A contractive mapping on a complete metric space may not necessarily have a fixed point, as shown by the following simple example.

Example 1.2. The space $X=[1,\infty)$ of the real line is complete. Let f be defined by $f(x)=x+\dfrac{1}{x}$. Then for $x, y \in X$ and $x<y$

$$d(f(x), f(y))=(y-x)-\left(\frac{1}{x}-\frac{1}{y}\right)<d(x, y).$$

Thus f is contractive but f leaves no point of X fixed.

The next theorem does in fact hold for contractive mappings and was proved by Edelstein [1].

Theorem 1.3. *If* f *is a contractive mapping on a compact metric space, then there exists a unique fixed point of* f.

Recently Iséki [3] established the following generalization of Banach Contraction Principle for a pair of self mappings on a complete metric space:

Theorem 1.4. *Let* (X, d) *be a complete metric space and* f, g *two continuous mappings of* X *into itself. If there are positive numbers* a, b *satisfying* $a, b<1$ *such that for all* $x, y \in X$,

$$d(f(g(x)),\ g(y)) \leq a d(x, g(y)),$$

$$d(g(f(x)),\ f(y)) \leq b d(x, f(y)),$$

then f *and* g *have a unique common fixed point.*

Taking g as the identity map it is evident that f becomes a contraction map and thus reducing Theorem 1.4 to that of Banach.

We introduce the following definition.