(*a*) *there exists a unique* $u \in K$ *such that* $f(u)=u$ *for all* $f \in F$;
(*b*) u *is the unique fixed point of any* $f \in F$;
(*c*) *for any pair* $f, g \in F$, *the sequence defined as in Theorem* 1 *converges strongly to* u.

Another interesting and slightly different version of Theorem 1 is established in the following theorem.

Theorem 3. *Let* X *and* K *be defined as in Theorem* 1. *Let* F *be the family of all mappings of* K *into* K, *such that for any pair* $f, g \in F$, *there exist a positive integer* $m=m(f,g)$ *and* $n=n(f,g)$ *and a* $\Phi=\Phi(f,g) \in \Psi$, *such that conditions* (*i*) *and* (*ii*) *of Theorem* 1 *and*

$$\begin{aligned} \|f^m(x)-g^n(y)\| \leq \Phi(&\|x-f^m(x)\| \cdot \|y-g^n(y)\|, \|x-g^n(y)\| \cdot \|y-f^m(x)\|, \\ &\|x-f^m(x)\| \cdot \|y-f^m(x)\|, \|x-g^n(y)\| \cdot \|y-g^n(y)\|) \end{aligned} \tag{8}$$

are satisfied. Then the conclusion of Theorem 2 *hold.*

Proof. Let $f_1=f^m$ and $g_1=g^n$. Then f_1 and g_1 satisfy the conditions of Theorem 1 and hence there is a unique $u \in K$, such that $f^m(u)=g^n(u)=u$ and u is a unique fixed point of each f^m and g^n.

Since $f^m(f(u))=f(f^m(u))=f(u)$, it follows that $f(u)$ is a fixed point of f^m and thus it must coincide with u. Hence $f(u)=u$ and similarly $g(u)=u$. The rest of the proof is obvious in view of the proof of Theorem 1. We omit the details.

In concluding this paper we note that there is no essential difficulty to show that theorems paralled to Theorems 3 and 4 in [4] also hold in the present setting of mappings. We also note that, in [1, 3, 8] the authors have established some fixed point theorems in uniformly convex Banach spaces. However, the mappings considered in this paper are different from those considered in [1, 3, 8]. For several interesting fixed point theorems in the present setup of mappings in metric spaces we refer the interested reader to the authors recent papers [6] and [7].

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JOINTLY-CONTRACTIVE MAPPINGS AND FIXED POINT THEOREMS

BY

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1. Introduction and preliminaries. Various generalizations and extensions of the well-known Banach Contraction Principle have appeared in recent years. E d e l s t e i n extended this famous theorem for contractive mappings.

Definition 1.1. *A mapping* f *of a metric space* (X, d) *into itself is said to be a contractive mapping if*

$$d(f(x), f(y)) < d(x, y), \ x \neq y.$$

A contractive mapping on a complete metric space may not necessarily have a fixed point, as shown by the following simple example.

Example 1.2. The space $X=[1,\infty)$ of the real line is complete. Let f be defined by $f(x)=x+\frac{1}{x}$. Then for $x, y \in X$ and $x<y$

$$d(f(x), f(y))=(y-x)-\left(\frac{1}{x}-\frac{1}{y}\right)<d(x, y).$$

Thus f is contractive but f leaves no point of X fixed.

The next theorem does in fact hold for contractive mappings and was proved by E d e l s t e i n [1].

Theorem 1.3. *If* f *is a contractive mapping on a compact metric space, then there exists a unique fixed point of* f.

Recently I s é k i [3] established the following generalization of Banach Contraction Principle for a pair of self mappings on a complete metric space:

Theorem 1.4. *Let* (X, d) *be a complete metric space and* f, g *two continuous mappings of* X *into itself. If there are positive numbers* a, b *satisfying* $a, b<1$ *such that for all* $x, y \in X$,

$$d(f(g(x)), g(y)) \leq a d(x, g(y)),$$

$$d(g(f(x)), f(y)) \leq b d(x, f(y)),$$

then f *and* g *have a unique common fixed point.*

Taking g as the identity map it is evident that f becomes a contraction map and thus reducing Theorem 1.4 to that of Banach.

We introduce the following definition.

Definition 1.5. *Two self mappings f and g of a metric space (X, d) are said to be jointly-contractive if for all x, y in X*

$$d(f(g(x)), g(y)) < d(x, g(y)), \text{ if } x \neq g(y),$$

and

$$d(g(f(x)), f(y)) < d(x, f(y)), \text{ if } x \neq f(y).$$

It follows that f becomes contractive when g is identity.

The purpose of this note is to show that unlike Theorem 1.3, the Conclusion of Theorem 1.4 may not hold in general for jointly-contractive mappings even if X is a compact metric space.

2. Main results. Theorem 2.1. *Let f and g be two jointly-contractive mappings of a compact metric space (X, d) into itself. Then either f or g has a fixed point, provided at least one of them is continuous.*

Proof. Assume f to be continuous. Consider a function

$$\Phi : X \to R \text{ defined by}$$

$$\Phi(x) = d(x, f(x)), \ x \in X.$$

Then Φ is a real valued lower semicontinuous function on a compact metric space X. Therefore Φ shall attain its minimum value at some point $\xi \in X$. Suppose that $f(\xi) \neq \xi$ and $f(\xi) \neq gf(\xi)$. Then

$$\Phi(gf(\xi)) = d(gf(\xi)/gf(\xi)) < d(f(\xi), gf(\xi)) < d(\xi, f(\xi)) = \Phi(\xi).$$

This contradicts the minimality of ξ. Hence either $f(\xi) = \xi$ or $gf(\xi) = f(\xi)$. In these cases ξ and $\xi' = f(\xi)$ are respectively the fixed points of f and g. This completes the proof.

We give the following simple example to support our result stated in Theorem 2.1.

Example 2.2. Let $X = \{1, 2\}$ with discrete metric. Consider mappings f and g on X defined by

$$f(1) = 1, \ f(2) = 2, \ g(1) = 2, \ g(2) = 1.$$

Then f and g are jointly-contractive. Note that f has two fixed points while g has none.

We also observe the following result which is similar to the case of contractive mappings.

Theorem 2.3. *A common fixed point (if exists) of jointly-contractive mappings is always unique.*

Proof. Suppose ξ and η are two distinct common fixed points of jointly contractive mappings f and g. Then $\eta \neq f(\xi)$ and hence

$$d(\xi, \eta) = d(f(\xi), gf(\eta)) < d(\eta, f(\xi)) = d(\eta, \xi)$$

giving a contradiction. Thus follows the unicity of common fixed points.

3. Applications. Following results are easily derivable from our main theorem.

Proposition 3.1. *Let f be a continuous self mapping of a compact metric space (X, d) such that*

$$d(f(x), f^2(x)) < d(x, f(x)), \ \forall x \in X, \ x \neq f(x).$$

Then f has a fixed point.

Proof. Follows at once from Theorem 2.1 by taking $f = g$.

Proposition 3.2 (Edelstein [1]). *Let f be a self mapping of a compact metric space (X, d) and*

$$d(f(x), f(y)) < d(x, y), \ x \neq y, \ x, y \in X.$$

Then f has a unique fixed point.

Proof. Follows from Proposition 3.1 by replacing y by $f(x)$. Note that a contractive mapping is always continuous and admits a unique fixed point whenever there is one.

Proposition 3.3. (Hardy-Rogers [2]). *Let (X, d) be a compact metric space and f a continuous self mapping of X satisfying the condition*

$$(^*) \quad x \neq y \text{ implies } d(f(x), f(y)) < a_1 d(x, f(X)) + a_2 d(y, f(y)) + a_3 d(x, f(y)) + \\ + a_4 d(y, f(x) + a_5 d(x, y).$$

If $a_1 + a_2 + a_3 + a_4 + a_5 = 1$, $a_i \geq 0$, then f has a unique fixed point.

Proof. Same as Proposition 3.2. Uniqueness follows from the condition $(^*)$.

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