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ON THE STABILITY OF SELF p-INVARIANT SETS

BY

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1. Introduction. It is well-known that to consider the Lyapunov stability criteria of a set, it is necessary to know that the set is invariant. However, in many problems of Mathematical applications one has to discuss the stability of sets which are not invariant in the usual sense. Several notions of invariance properties have therefore been used to describe such situations. Thus given a differential system, it is important to check what kind of an invariant set a particular set is. Recently, Lakshmikantham and Leela [8, 9, 10], Ladde and Leela [6, 7], Kayande and Lakshmikantham [3], Ladde and Lakshmikantham [4], Ladde, Lakshmikantham and Leela [5], Leela [11, 12], Brézis [2] and Redheffer [13] among others have done much work regarding the classification of different kinds of invariant sets and the investigations of Lyapunov stability with respect to these different kinds of invariant sets have provided natural frame works for the study of weaker concepts of stability which are useful in applications [5, 8, 9, 11, 12].

In this paper, we introduce a new concept of self p-invariance of a set which is a different kind of an invariant set. This, naturally, gives rise to the introduction of a new notion of q-stability of a self p-invariant set. We then give theorems of very general nature which set out sufficient conditions for the q-stability of a self p-invariant set in terms of Lyapunov like functions and differential inequalities. Our results include as special cases many well-known results, on stability of invariant sets with respect to differential equations.

§2. **Basic concepts and definitions.** Let us consider the differential system

$$\frac{dx}{dt}=f(t, x) \qquad x(t_0)=x_0\,; \tag{1}$$

where $f\in C(R^+\times S(M, \rho), R^n)$. Here R^+ is the nonnegative real line, R^n the euclidean n-space, $S(M, \rho)=\{x\in R^n\,;\ d(x, M)<\rho\}$ with $d(x, M) = \inf_{m\in M} \| x-m \|$, $\|\cdot\|$ being any convenient norm, M a subset of R^n and $C(R^+\times S(M, \rho), R^n)$ the class of continuous functions from $R^+\times S(M, \rho)$ into R^n. Denote by $\overline{S(M, \rho)}$ the set $\{x\in R^n\,;\ d(x, M)\leq \rho\}$. We need the following definitions.

Definition 2.1. *Let $M\subset R^n$. M is said to be a self p-invariant set with respect to the system (1), if for any solution $x(t, t_0, x_0)$ of (1), $x_0\in M$ implies that $x^{(n)}(t, t_0, x_0)\in M$ for $n=0, 1, 2, \ldots, p$ and $t\geq t_0$, where $x^{(n)}(t, t_0, x_0)$ is the n^{th} derivative of $x(t, t_0, x_0)$ with respect to t, and $x^{(0)}(t, t_0, x_0)\equiv x(t, t_0, x_0)$.*

Remark 1. If $p=0$, then a self 0-invariant set reduces to the usual definition of a self invariant set. Hence a self p-invariant set generalizes the notion of a self-invariant set.

Definition 2.2. *Let $M\subset R^n$ and assume that M is a self p-invariant set. Then M is said to be QI_1 : q-equistable with respect to the system (1), if given $\varepsilon<0$, $t_0\in R^+$ and a positive integer q, there exist $\delta(t_0, \varepsilon)$ and a positive number k such that for any solution $x(t, t_0, x_0)$ of the system (1), with $x_0\in\overline{S(M, \delta)}$,*

$$x(t, t_0, x_0)\in S(M, \varepsilon) \text{ and } x^{(m)}(t, t_0, x_0)\in S(M, k\varepsilon)$$

for $m=1, 2, \ldots, q$ and for $t\geq t_0$.

On the basis of definition QI_1, the definitions of QI_2 to QI_8 may be formulated parallel to the definitions S_2 to S_8 of [8]. These definitions are straightforward and to conserve space we omit details.

Remark 2. If $p=0$ and $q=0$, then QI_1—QI_8 reduce to the usual definitions of stability of a self invariant set. If $p>0$ but $q=0$, then QI_1—QI_8 reduce to the stability definitions of a self p-invariant set which generalizes known concepts of stability of self invariant sets. If $q>0$, $p=0$, then QI_1—QI_8 reduce to q-stability definitions of a self invariant set, which again generalize well-known concepts of stability of a self invariant set. If $M=\{O\}$ and it is self-invariant, with $q>0$, and $p=0$, then our definitions QI_1—QI_8 reduce to the q^{th}-order stability definitions of the trivial solution of the system (1) [cf. 1]. If $q=0$, $p=0$ and $M=\{O\}$ and a self-invariant set, then QI_1—QI_8 reduce to the definitions S_1—S_8 of [8].

Example: Consider the differential equation

$$\frac{d^2x}{dt^2}=-c^2x$$

whose solution is $x=Ae^{-ict}+Be^{ict}$ or $x(t)=A'\mathrm{Cos}\, ct+B'\mathrm{Sin}\, ct$. Choosing $|A'|<\frac{\varepsilon}{2}$ and $|B'|<\frac{\varepsilon}{2}$ we have

$$\|x'(t)\|\leq|cA'|+|cB'|\leq|c|\left(\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\right)=c\varepsilon.$$

Similarly $\|x^{(2)}(t)\|\leq|c^2|\varepsilon$ and $\|x^{(n)}(t)\|\leq|c|^n\varepsilon=K\varepsilon$. Let $M=\{$the set $x=0\}$, then M is self p-invariant since $x_0\in M$ implies $x^n(t)=0$. Moreover M is q-equistable, for any positive integer q.

§3. Main Results Definition 3.1. *Denote by $\mathcal{K}$, the set $\{a\in C([0, \rho), R^+)$: $a(0)=0$ and $a(r)$ is increasing in $r\}$. Let $\mathcal{D}=\{b\in C(R^+\times[0, \rho), R^+)$: $b(t, r)$ is decreasing in t for each r, increasing in r for each t and $\lim\limits_{\substack{t\to\infty \\ r\to 0^+}} b(t, r)=0\}$.*

We now obtain sufficient conditions for the various stability properties defined in the last section to hold.

Theorem 3.1. *Assume that*

(i) *$V\in C(R^+\times S(M, \rho), R^+)$, $V(t, x)$ is locally Lipschitzian in x and for $(t, x)\in R^+\times S(M, \rho)$, $b(d(x, M))\leq V(t, x)\leq a(t, d(x, M))$; where $b\in\mathcal{K}$, $a\in C(R^+x[0, \rho), R^+)$ and $a\in\mathcal{K}$ for each $t\in R^+$.*

(ii) *Any solution $x(t, t_0, x_0)$ of (1) is* max $\{p, q\}$ *— times differentiable for $t\geq t_0$ where p, q are non-negative integers and there exist continuous functions $a\in\mathcal{D}$, $b\in\mathcal{K}$ such that if $(t, x^{(n)})\in R^+\times S(M, \rho)$, $n=0, 1, 2, 3, \ldots, k$ where $k=$*max *$\{p, q\}$, then there exist continuous functions a, $b\in\mathcal{K}$ for each $t\in R^+$ such that*

$$\max_{1\leq m\leq k} b(d(x^{(m)}, M))\leq V(t, x)\leq \min_{1\leq m\leq k} a(t, d(x^{(m)}, M)).$$

(iii) $\quad g\in C(R^+\times R^+, R)$ *and* $g(t, 0)\equiv 0$.

(iv) $\quad$ for $(t, x)\in R^+\times S(M, \rho)$

$$D^+V(t, x)=\limsup_{h\to 0}\left[\frac{V(t+h, x+hf(t, x))-V(t, x)}{h}\right]\leq g(t, V(t, x)).$$

Then the equistability of the trivial solution of the scalar differential equation

$$\frac{du}{dt}=g(t, u)\qquad u(t_0)=u_0\geq 0 \tag{2}$$

implies the q-equistability of the self p-invariant set M with respect to the system (1).

Proof: Let $0<\varepsilon<\rho$, $t_0\in R^+$ be given. Assume that the trivial solution of (2) is equistable, then given $b(\varepsilon)>0$, $t_0\in R^+$, $\exists\delta=\delta(t_0, \varepsilon)$ which is continuous in t_0 for each ε such that

$$u(t, t_0, u_0)<b(\varepsilon)\qquad t\geq t_0,$$

provided $u_0\leq\delta$ where $u(t, t_0, u_0)$ is any solution of (2). Choose $u_0=V(t_0, x(t_0))$, then the hypothesis on $a\in D$ implies $\exists\delta_1=\delta_1(t_0, \varepsilon)$ such that the inequalities

$$d(x_0, M)\leq\delta_1 \text{ and } a(t_0, d(x_0, M))\leq\delta$$

hold together. Suppose there exists a solution $x(t, t_0, x_0)$ of (1) such that $x_0\in\overline{S(M, \delta_1)}$ and $t_1>t_0$ such that

$$d(x(t_1), M)=\varepsilon \text{ and } d(x(t, t_0, x_0), M)\leq\varepsilon \text{ for } t\in[t_0, t_1],$$

then $b(\varepsilon)\leq V(t_1, x(t_1, t_0, x_0))$ and $x(t, t_0, x_0)\in S(M, \rho)$ for $t\in[t_0, t_1]$. Using a well-known comparison result [8],

$$V(t, x(t, t_0, x_0))\leq r(t, t_0, u_0),\qquad t\in[t_0, t_1],$$

where $r(t, t_0, u_0)$ is the maximal solution of (2). From the foregoing,

$$b(\varepsilon)\leq V(t_1, x(t_1))\leq r(t_1, t_0, u_0)<b(\varepsilon),$$

which is a contradiction. Hence $x_0\in\overline{S(M, \delta_1)}$ implies $x(t, t_0, x_0)\in S(M, \varepsilon)$.

Now choose A such that $b(\varepsilon)\leqslant b(A\varepsilon)$, then $x_0\in\overline{S(M,\delta_1)}$ implies $x^{(m)}(t,t_0,x_0)\in S(M,A\varepsilon)$, for $m=1,2,3,\dots,q$. Suppose not, then there exists an integer e such that $1\leqslant l\leqslant q$ and a solution $x(t,t_0,x_0)$ such that $x_0\in\overline{S(M,\delta_1)}$ and $t_1>t_0$ with

$$d(x^{(l)}(t_1),M)=A\varepsilon \quad\text{and}\quad d(x^{(l)}(t),M)\leqslant A\varepsilon \quad\text{for } t\in[t_0,t_1].$$

Hence by hypothesis (ii)

$$b(A\varepsilon)\leqslant V(t_1,x(t_1))\leqslant r(t_1,t_0,u_0)<b(\varepsilon)\leqslant b(A\varepsilon),$$

which is a contradiction. Hence the self p-invariant set M is q-equistable with respect to the system (1).

Remark 3. We examine here special cases of our result which also generalize well-known theorems on stability of self invariant sets. If $p>0$ and $q=0$, then the result yield the sufficient conditions for the equistability property of a self p-invariant set which is a new result. On the other hand if $p=0$ and $q>0$ then we have the conditions for the q^{th}-order equistability of a self-invariant set [cf. 1]. If $M=\{0\}$ and self-invariant with $q>0$ and $p=0$, then we have sufficient conditions for the q^{th}-order equistability of the trivial solution of the system (1) [cf. 1]. Moreover if $q=0$ in addition, our result reduces to the equistability property of the trivial solution of (1).

Theorem 3.2. *Assume that all conditions of Theorem* 3.1 *hold where in hypothesis* (*i*)

$$b(d(x,M))\leqslant V(t,x)\leqslant a(d(x,M)),$$

and in hypothesis (*ii*),

$$\max_{1\leqslant m\leqslant k} b(d(x^{(m)},M))\leqslant V(t,x)\leqslant \min_{1<m<k} a(d(x^{(m)},M)),$$

where $a,b\in\mathcal{K}$, and $k=\max\{p,q\}$.

Then the uniform stability of the trivial solution of the scalar differential equation (2) *implies the q-uniform stability of the self p-invariant set M with respect to the system* (1).

Proof: The proof is straightforward.

Theorem 3.3. *Assume that all the hypothesis of Theorem* 3.1 *hold. Then the equi-asymptotic stability of the trivial solution of* (2) *implies the q-equi-asymptotic stability of the self p-invariant set M with respect to the system* (1).

Proof: By theorem 3.1 the self p-invariant set M is q-equistable, hence it suffices to show the q-quasi-equi-asymptotic stability of M. Let $0<\varepsilon<\rho$ and $t_0\in R_0^+$. Given $b(\varepsilon)>0$ there exist two positive numbers $\delta_0=\delta_0(t_0)$ and $T=T(t_0,\varepsilon)$ such that

$$u(t,t_0,u_0)<b(\varepsilon)$$

for $t\geqslant t_0+T$, provided $u_0<\delta$. Choose $u_0=V(t_0,x_0)$, then there exists $\delta_1=\delta_1(t_0)$ such that

$$d(x(t_0),M))\leqslant\delta_1 \quad\text{and}\quad a(t_0,d(x_0,M))\leqslant\delta$$

hold together. Let $\delta^*=\delta_0(t_0,\rho)$ and define $\delta_2=\min\{\delta_1,\delta^*\}$ then $x_0\in\overline{S(M,\delta_2)}$ implies $x(t,t_0,x_0)\in S(M,\varepsilon)$, for $t\geqslant t_0+T$. Suppose there exists

a sequence $\{t_j\}$ $t_j\geqslant t_0+T$ and $t_j\to\infty$ as $j\to\infty$ such that there exist a solution $x(t,t_0,x_0)$ of (1) satisfying

$$d(x(t_j,t_0,x_0),M)\geqslant\varepsilon \quad\text{for } x_0\in\overline{S(M,\delta_2)}\,;$$

then

$$b(\varepsilon)\leqslant b(d(x(t_j,t_0,x_0),M)\leqslant V(t_j,x(t_j,t_0,x_0))\leqslant r(t_j,t_0,r_0)<b(\varepsilon),$$

which is a contradiction. Hence $x(t,t_0,x_0)\in S(M,\varepsilon)$ for $t\geqslant t_0+T$.

Given $b(\varepsilon)$ and $t_0\in R^+$, then $\exists\hat\delta_0=\hat\delta_0(t_0,\varepsilon)$ and $T_2=T_2(t_0,\varepsilon)$ such that

$$u_0\leqslant\hat\delta_0 \text{ implies } u(t,t_0,u_0)<b(\varepsilon) \text{ for } t\geqslant t_0+T.$$

Choose $\hat\delta=\min\{\hat\delta_0,\delta_0\}$, then there exists $\delta_3=\delta_3(t_0,\varepsilon)$ such that

$$d(x(t_0),M)\leqslant\delta_3 \text{ and } a(t_0,d(x(t_0),M)\leqslant\hat\delta$$

hold together. Let $\delta_4=\min\{\delta_2,\delta_3\}$ and $T^*=\max\{T,T_2\}$ and choose constant A such that $b(\varepsilon)\leqslant b(A\varepsilon)$; then $x(t_0)\in\overline{S(M,\delta_4)}$ implies

$$x^{(m)}(t,t_0,x_0)\in S(M,A\varepsilon) \quad\text{for}\quad m=1,2,\dots,q.$$

and $t\geqslant t_0+T^*$.

Suppose not then $\exists$ an integer l, $1\leqslant l\leqslant q$, a solution $x(t,t,x_0)$ of (1) and sequence $\{t_j\}$, $t_j\geqslant t_0+T^*$, $t_j\to\infty$ as $j\to\infty$ such that $x(t_0)\in\overline{S(M,\delta_4)}$ and

$$d(x^{(l)}(t_j),M)\geqslant A\varepsilon\,.$$

By hypothesis

$$V(t,x(t))\leqslant r(t,t_0,u_0) \text{ for } t\geqslant t_0,$$

where $r(t,t_0,u_0)$ is the maximal solution of (2). Hence

$$b(A\varepsilon)\leqslant b(d(x^{(l)}(t_j),M))\leqslant\max_{1<l<q} b(d(x^{(l)}(t_j),M))\leqslant$$

$$\leqslant V(t_j,x(t_j,t_0,x_0))\leqslant r(t_j,t_0,u_0),$$

and so

$$b(A\varepsilon)\leqslant r(t_j,t_0,u_0)<b(\varepsilon)\leqslant b(A\varepsilon)$$

which is a contradiction.

The following theorem becomes obvious from the foregoing:

Theorem 3.4. *Assume that the conditions of Theorem* 3.2. *hold. Then the uniform asymptotic stability of the trivial solution of* (2) *implies the q-uniform asymptotic stability of the self p-invariant set M with respect to the system* (1).

Remark 4. We can make comments parallel to those contained in Remark 3 concerning the special cases of our Theorems 3.2, 3.3 and 3.4. We omit details.

BIBLIOGRAPHY

1. Akinyele O. – *Vector Lyapunov functions and p*th*-order. Conditionl stability and boundedness.* Annali di Mat Pura ed Appl. 119 (1979), 163–173.

2. Brezis H. – *On a characterization of flow-invariant sets.* Comm. Pure Appl. Maths. 223 (1970), 261–263.

3. Kayande A. A. and Lakshminkantham, V. — *Conditionally invariant sets and vector Lyapunov functions*. J. Maths. Analysis and Appl. 13 (1966) 337—347.
4. Ladde G. S. and Lakshminkantham V. — *On flow-invariant sets*. Pacific J. of Maths. 61 (1974) no. 1, 215—220.
5. Ladde G. S., Lakshinkantham V. and Leela S. — *Conditionally asymptotically invariant sets and perturbed systems*. Annali di Mat Pura ed Appl. 94 (1972) 33 — 40.
6. Ladde G. S. and Leela S. — *Analysis of invariant sets*. Annali di Mat Pura ed Appl. 94 (1972) 283—289.
7. Ladde G. S. and Leela S. — *Global results and asymptotically self-invariant sets*. Atti Acad Naz. dei Lincei Rendiconti (8), 54 (1973) fasc 3, 321—327.
8. Lakshminkantham V. and Leela S. — *Differential and Integral inequalities: Theory and Applications*. Vol. I Academic Press, New York 1969.
9. Lakshminkantham V. and Leela S. — *Asymptotically self-invariant sets and conditional stability*: Proc. Intern. Symp. Diff. equations and Dynamical systems. Puerto Rico 1965. Academic Press, New York, 1967, 363 — 373.
10. Lakshminkantham V. and Leela S. — *Global results and stability of motion*. Proc. Camb. Phil. Soc 70 (1971) 95 — 102.
11. Leela S.— *Asymptotically self-invariant sets and perturbed systems*. Annali di Mat Pura ed Appl. 92 (1972) 85 — 94.
12. Leela S. — *Lyapunov stability and conditional invariant sets*. An. şt. Univ. Al. I. Cuza Iaşi (1971) 71—78.
13. Redheffer R. M. — *The Theorems of Bony and Brezis on flow-invariant sets*. Amer. Maths. Monthly 79 (1972) 740—747.

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ON THE CONVERGENCE OF SOLUTIONS OF CERTAIN FOURTH ORDER DIFFERENTIAL EQUATIONS

DE

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1. Introduction. Consider the differential equation

$$x'''' + \alpha x''' + bx'' + cx' + h(x) = p(t, x, x', x'', x'''), \tag{1.1}$$

in which $a > 0$, $b > 0$, $c > 0$ and $h(x)$ is continuous. The function $p(t, x, x', x'', x''')$ is assumed to have the form

$$p = q(t) + r(t, x, x', x'', x'''),$$

with q depending on t only and each of q and r is continuous. Moreover, it will be assumed that $r(t, 0, 0, 0, 0,) = 0$, for if not, we replace q by $q(t) + r(t, 0, 0, 0, 0,)$.

Any two solutions $x_1(t)$, $x_2(t)$ of (1.1) are said to converge if

$$\begin{gathered} x_2(t) - x_1(t) \to 0, \quad x_2'(t) - x_1'(t) \to 0, \\ x_2''(t) - x_1''(t) \to 0 \text{ and } x_2'''(t) - x_1'''(t \to 0 \end{gathered} \tag{1.2}$$

as $t \to \infty$.

The convergence of solutions of various equations of the second and third orders have been studied by Ezeilo in [2] and [3]. While the qualitative behaviour of solutions of fourth order differential equations have been considered by many authors to the authors knowledge none have studied (1.1) with the following restrictions:

$$\frac{h(\xi + \eta) - h(\xi)}{\eta} \in I_0 \quad (\eta \neq 0) \tag{1.3}$$

where I_0 is a closed interval defined by

$$I_0 = \left[\Delta_0,\ K \cdot \frac{(ab - c)c}{a^2} \right]. \tag{1.4}$$

with $K < 1$ and $\Delta_0 > 0$ constants.

For the special case of

$$x'''' + ax''' + bx'' + cx' + dx = 0 \tag{1.5}$$

$d > 0$, a constant, (corresponding to $h(x) = dx$ and $p \equiv 0$ in (1.1)), it is known that all the solutions of (1.5) converge if the Routh-Hurwith conditions