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*Received* 15.III.1980

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s. I a, 1981, f. 1

# ON THE CONVERGENCE OF SOLUTIONS OF CERTAIN FOURTH ORDER DIFFERENTIAL EQUATIONS

DE

ANTHONY UYI AFUWAPE

**1. Introduction.** Consider the differential equation

$$x'''' + \alpha x''' + bx'' + cx' + h(x) = p(t, x, x', x'', x'''), \tag{1.1}$$

in which $a>0$, $b>0$, $c>0$ and $h(x)$ is continuous. The function $p(t, x, x', x'', x''')$ is assumed to have the form

$$p = q(t) + r(t, x, x', x'', x''''),$$

with $q$ depending on $t$ only and each of $q$ and $r$ is continuous. Moreover, it will be assumed that $r(t, 0, 0, 0, 0,) = 0$, for if not, we replace $q$ by $q(t) + r(t, 0, 0, 0, 0, )$.

Any two solutions $x_1(t)$, $x_2(t)$ of (1.1) are said to converge if

$$\begin{gathered} x_2(t) - x_1(t) \to 0, \quad x_2'(t) - x_1'(t) \to 0, \\ x_2''(t) - x_1''(t) \to 0 \text{ and } x_2'''(t) - x_1'''(t \to 0 \end{gathered} \tag{1.2}$$

as $t \to \infty$.

The convergence of solutions of various equations of the second and third orders have been studied by Ezeilo in [2] and [3]. While the qualitative behaviour of solutions of fourth order differential equations have been considered by many authors to the authors knowledge none have studied (1.1) with the following restrictions :

$$\frac{h(\xi + n) - h(\xi)}{\eta} \in I_0 \quad (\eta \neq 0) \tag{1.3}$$

where $I_0$ is a closed interval defined by

$$I_0 = \left[ \Delta_0, \ K \cdot \frac{(ab - c)c}{a^2} \right]. \tag{1.4}$$

with $K < 1$ and $\Delta_0 > 0$ constants.

For the special case of

$$x'''' + ax''' + bx'' + cx' + dx = 0 \tag{1.5}$$

$d > 0$, a constant, (corresponding to $h(x) = dx$ and $p \equiv 0$ in (1.1)), it is known that all the solutions of (1.5) converge if the Routh-Hurwith conditions

$$ab-c>0,\ (ab-c)c-a^2d>0$$

hold.

These conditions have been generalised for non-linear equations by many authors (cf. [4]), with the assumption that $h(x)$ is differentiable and even in some cases, with $h(x)$ twice differentiable. These, we find too weak to be too strong condition. We shall therefore assume in addition to (1.3) that $h(x)$ is not differentiable.

**2. Main results.** The following results may be regarded as a fourth order analogue of [2] and [3].

**Theorem 1.** *Suppose that $h(0)=0$ and that* (i) (1.3) *holds for arbitrary* $\xi, \eta (\neq 0)$, (ii) *there is a continuous function $\varphi(t)$ such that*

$$\begin{aligned} &|r(t, x, y, z, \omega)-r(t, \bar{x}, \bar{y}, \bar{z}, \bar{\omega})| \leqslant \\ &\leqslant \varphi(t)\,(|x-\bar{x}|+|y-\bar{y}|+|z-\bar{z}|+|\omega-\bar{\omega}|) \end{aligned} \tag{2.1}$$

*holds for arbitrary $t, x, y, z, \omega, \bar{x}, \bar{y}, \bar{z}$, and $\bar{\omega}$. Then there exists a constant $D_1$ such that if $\int_0^t \varphi^\alpha(\tau)d\tau \leqslant D_1 t$ for some $\alpha$, $1\leqslant\alpha\leqslant 2$, then all solutions of* (1.1) *converge.*

The following estimates for any two distinct solutions of (1.1) will be very useful in the proof of Theorem 1.

**Theorem 2.** *Assume that the conditions of Theorem 1 hold. Let $x_1(t)$, $x_2(t)$ be any two distinct solutions of* (1.1). *Then for each fixed $\alpha$, $1\leqslant\alpha\leqslant 2$, there are constants $D_2$, $D_3$, and $D_4$ such that*

$$S(t_2)\leqslant D_2 S(t_1)\exp\left\{-D_3(t_2-t_1)+D_4\int_{t_1}^{t_2}\varphi^\alpha(\tau)d\tau\right\}\ (t_2\geqslant t_1) \text{ where} \tag{2.2}$$

$$S(t)\equiv\{[x_2(t)-x_1(t)]^2+[x_2'(t)-x_1'(t)]^2+[x_2''(t)-x_1''(t)]^2+[x_2'''(t)-x_1'''(t)]^2\}.$$

On putting $x_1(t)\equiv 0$ and $t_1\equiv 0$, we immediately have

**Corollary :** *If $p\equiv 0$ and the hypothesis* (i) *of Theorem 1 holds, then the trivial solution of* (1.1) *is exponentially asimptotically stable in the large.*

**3. Preliminaries to the Proof of Theorem 2.** Let $Q(t)=\int_0^t q(\tau)d\tau$. Replace (1.1) by the equivalent system

$$\begin{aligned} &x'=y,\ y'=z,\ z'=\omega+Q(t), \text{ and} \\ &\omega'=-a\omega-bz-cy-h(x)+r(t, x, y, z, \omega+Q)-aQ(t). \end{aligned} \tag{3.1}$$

Our main tool in the proofs will be the following quadratic form $V=V(x, y, z, \omega)$ defined by

$$\begin{aligned} 2V&=c^2(1-\varepsilon)x^2+[ac(1-\varepsilon)\,(D-1)+b^2]y^2+[a^2(1-\varepsilon)D+ \\ &+b\delta/(ab-c-\delta)]z^2+D\omega^2+2bc(1-\varepsilon)xy+2ac(1-\varepsilon)xz+ \\ &+2c(1-\varepsilon)x\omega+2(ab-\delta)\,Dyz+2by\omega+2a(1-\varepsilon)Dz\omega, \end{aligned} \tag{3.2}$$



where

$$D=(ab-c+c\varepsilon)/(ab-c-\delta),$$

with $\varepsilon$ and $\delta$ satisfying

$$(ab-c)>\delta>0\ ;\ 0<\varepsilon<1\ ;\ \text{and } 2\varepsilon-\varepsilon^2=\delta/ab.$$

The following lemmas will be needed :

**Lemma 1 :** *$V$ defined above is positive definite. Moreover, there exist constants $D_5$, $D_6$ such that*

$$D_5(x^2+y^2+z^2+\omega^2)\leqslant V\leqslant D_6(x^2+y^2+z^2+\omega^2) \tag{3.3}$$

*hold.*

*Proof of Lemma 1 :* On rearranging (3.2) we have

$$2V\equiv 2V_1+2V_2 \tag{3.4}$$

where

$$\begin{aligned} 2V_1&=[c(1-\varepsilon)x+by+az+\omega]^2+[(1-\varepsilon)\,D-1]\,(az+\omega)^2+ \\ &+c^2\varepsilon(1-\varepsilon)x^2+ac\varepsilon(1-\varepsilon)\,Dy^2+\varepsilon D\omega^2, \end{aligned} \tag{3.5}$$

$$2V_2=(ab-c-\delta)^{-1}\{ac(1-\varepsilon)^2\,(c\varepsilon+\delta)y^2+2c(ab\varepsilon+\delta(1-\varepsilon))yz+b\delta z^2\}. \tag{3.6}$$

Since $0<\varepsilon<1$, $V_1$ is positive definite.

Also, $2V_2$ regarded as a quadratic form in $y$ and $z$ is always positive. Hence $V$ is positive definite. We can therefore find a constant $D_5>0$ such that

$$D_5(x^2+y^2+z^2+\omega^2)\leqslant V. \tag{3.7}$$

Furthermore, as we always have $|y|\,|z|\leqslant 1/2\,(y^2+z^2)$, then $|2V_2|\leqslant$ $\leqslant\bar{D}(y^2+z^2)$ for some $\bar{D}=\bar{D}(a, b, c, \varepsilon)>0$. Thus, there exists $D_6>0$ such that :

$$V\leqslant D_6(x^2+y^2+z^2+\omega^2). \tag{3.8}$$

Combining (3.7) and (3.8) we have (3.2) as required.

**Lemma 2 :** *Let* (1.3) *hold for $\xi=0$ and*

$$K=[2(ab-c)]^{-1}\min\{a^2c\varepsilon(1-\varepsilon)/b\ ;\ ab\varepsilon D\ ;\ a^3\varepsilon(1-\varepsilon)D\}<1, \tag{3.9}$$

*then*

$$\frac{dV}{dt}\leqslant -U_1-U_2+U_3+U_4,$$

*where $U_1$ is a positive definite quadratic form in $x$, $y$, $z$, and $\omega$, while*

$$U_2=h(0)\,\{c(1-\varepsilon)\,x+by+a(1-\varepsilon)Dz+D\omega\}\ ;$$

$$U_3=\{c(1-\varepsilon)x+by+a(1-\varepsilon)Dz+D\omega\}r(t, x, y, z, \omega+Q)\ ;$$

*and* $$U_4=\{[(ab-\delta)D-ab]y+[b\delta/(ab-c-\delta)]z-a\varepsilon D\omega\}Q(t).$$

*Proof of Lemma 2 :*

Let $H_0=\dfrac{h(x)-h(0)}{x}\ (x\neq 0)$.

Using (3.1) and the fact that
$$\frac{dV}{dt}=\frac{\partial V}{\partial x}x'+\frac{\partial V}{\partial y}y'+\frac{\partial V}{\partial z}z'+\frac{\partial V}{\partial \omega}\omega'$$
we have, on rearranging,
$$\frac{dV}{dt}=-U_1-U_{12}-U_{13}-U_{14}-U_2+U_3+U_4,$$
where
$$U_1=\frac{1}{2}\left\{\frac{1}{2}c(1-\varepsilon)H_0x^2+bc\,\varepsilon\,y^2+ab\varepsilon(1-\varepsilon)\,Dz^2+a\varepsilon\,D\omega^2\right\},$$
$$U_{12}=\frac{1}{2}\left\{\frac{1}{2}c(1-\varepsilon)H_0x^2+2bH_0xy+bc\varepsilon\,y^2\right\},$$
$$U_{13}=\frac{1}{2}\left\{\frac{1}{2}c(1-\varepsilon)H_0x^2+2a(1-\varepsilon)DH_0xz+ab\varepsilon(1-\varepsilon)Dz^2\right\},$$
$$U_{14}=\frac{1}{2}\left\{\frac{1}{2}c(1-\varepsilon)H_0x^2+2DH_0x\omega+a\varepsilon\,D\omega^2\right\};$$
and $U_2$, $U_3$, $U_4$ as given in lemma 2.

Regarding $U_{12}$, $U_{13}$ and $U_{14}$ as quadratic forms in $x, y$; $x, z$; and $x, \omega$ respectively, and provided we have
$$H_0<K\left[\frac{(ab-c)c}{a^2}\right],$$
then $U_{12}\geqslant 0$, $U_{13}\geqslant 0$ and $U_{14}\geqslant 0$.

Since $0<\varepsilon<1$, $U_1$ is positive definite and the proof of lemma 2 is complete.

4. *Proof of Theorem 2*: Let $x_1(t)$, $x_2(t)$ be any two distinct solutions of (1.1).

Let

$$(4.1)\qquad \begin{gathered} x_i'=y_i,\quad y_i'=z_i,\quad z_i'=\omega_i+Q,\\ \omega_i'=-a\omega_i-bz_i-cy_i-h(x_i)+r(t,x_i,y_i,z_i\,\omega_i+Q)-aQ \qquad (i=1,2). \end{gathered}$$

Define $W=W(t)$ by
$$(4.2)\qquad W=W(x_2-x_1,\ y_2-y_1,\ z_2-z_1,\ \omega_2-\omega_1)$$
where $W\equiv V$ is the function given in (3.2).

Using Lemma 1, there exist constants $D_7$ and $D_8$ such that
$$(4.3)\qquad \begin{gathered} D_7(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(\omega_2-\omega_1)^2)\leqslant\\ \leqslant W\leqslant D_8((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(\omega_2-\omega_1)^2). \end{gathered}$$

To prove (2.2), it will be enough to show that
$$(4.4)\qquad W(t_2)\leqslant W(t_1)\exp\left\{-D_8(t_2-t_1)+D_4\int_{t_1}^{t_2}\varphi^\alpha(\tau)d\tau\right\},\quad (t_2\geqslant t_1)$$
holds.



Since $(x_1, y_1, z_1, \omega_1)$ and $(x_2, y_2, z_2, \omega_2)$ are two distinct solutions of (4.1), by Lemma 2, we immediately have, as $h(0)=0$, that
$$(4.5)\qquad \frac{dW}{dt}\leqslant -W_1+W_2,$$
where
$$(4.6)\qquad \begin{gathered} 2W_1=\left\{\frac{1}{2}c(1-\varepsilon)H(x_2-x_1)^2+bc\,\varepsilon\,(y_2-y_1)^2+\right.\\ \left.+ab\varepsilon(1-\varepsilon)D(z_2-z_1)^2+a\varepsilon\,D(\omega_2-\omega_1)^2\right\} \end{gathered}$$
with $H=(h(x_2)-h(x_1))/(x_2-x_1)$,

and
$$(4.7)\qquad \begin{gathered} W_2=\{c(1-\varepsilon)\,(x_2-x_1)+b(y_2-y_1)+a(1-\varepsilon)D(z_2-z_1)+\\ +D(\omega_2-\omega_1)\}\,\theta(t), \end{gathered}$$
with
$$\theta(t)\equiv r(t,x_2,y_2,z_2,\omega_2+Q)-r(t,x_1,y_1,z_1,\omega_1+Q).$$

Because (1.3) holds for arbitrary $\xi,\eta(\neq 0)$,
$$\begin{gathered} 2W_1\geqslant\{1/2c(1-\varepsilon)\Delta_0(x_2-x_1)^2+bc\,\varepsilon\,(y_2-y_1)^2+\\ +ab\varepsilon(1-\varepsilon)D(z_2-z_1)^2+a\varepsilon\,D(\omega_2-\omega_1)^2\}. \end{gathered}$$

In fact,
$$(4.8)\qquad W_1\geqslant 2D_9\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(\omega_2-\omega_1)^2\}\equiv D_9S,$$
for some $D_9>0$.

Using (2.1), we can find a $D_{10}>0$ small enough such that
$$(4.9)\qquad W_2\leqslant D_{10}S^{\frac{1}{2}}\,|\,\theta\,|.$$

Combining (4.8) and (4.9) in (4.5), we have
$$(4.10)\qquad \frac{dW}{dt}\leqslant -2D_9S+D_{10}S^{\frac{1}{2}}\,|\,\theta\,|.$$

The remaining part of the proof follows directly from [3]. Let $\alpha$ be any constant such that $1\leqslant\alpha\leqslant 2$ and set $\mu=1-1/2\alpha$, so that $0\leqslant\mu\leqslant 1/2$.

Consider (4.10) in the form
$$W'+D_9S\leqslant D_{10}\,S^{\frac{1}{2}}\,|\,\theta\,|-D_9S=D_{10}S^\mu W^*.$$
where $W^*=(|\,\theta|-D_9D_{10}^{-1}S^{\frac{1}{2}})$. It is immediate from $W^*$ that for some $D_{12}>0$ small enough, we always have
$$W^*\leqslant D_{12}\,|\,\theta\,|^{2(1-\mu)}.$$

Therefore
$$W'+D_9S\leqslant D_{10}D_{12}S^\mu|\,\theta\,|^{2(1-\mu)}\leqslant D_{13}\,S^\mu\varphi^{2(1-\mu)}S^{1-\mu}.$$

Furthermore, it follows from (4.11) and $2-\alpha=2\mu$ that

$$W' \leqslant -D_9 S + D_{13}\, \varphi^\alpha S. \tag{4.12}$$

This, by (4.3), implies that

$$W' + \{D_{14} - D_{15}\, \varphi^\alpha(t)\} W \leqslant 0$$

for some constants $D_{14}$ and $D_{15}$.

By integration, we have (4.4). A repeated use of (4.4) gives (2.2), and this completes the proof of Theorem 2.

5. *Proof of Theorem* 1:

From the estimate (2.2), if

$$\int_{t_1}^{t_2} \varphi^\alpha(\tau) d\tau \leqslant D_3 D_4^{-1} (t_2 - t_1),$$

then the exponetial index remains negative for all $t_2 - t_1 \geqslant 0$. As $t = (t_2 - t_1) \to \infty$ we have $S(t) \to 0$. this gives

$$x_2 - x_1 \to 0,\ x_2' - x_1' \to 0,\ x_2'' - x_1'' \to 0 \text{ and } x_2''' - x_1''' \to 0$$

as $t \to \infty$. Thus Theorem 1 is proved.

6. *Remarks*: (i) Condition (2.1) is a generalised Lipschitz condition on $r(t, x, y, z, \omega)$. In fact, if the assumption that $p(t, x, y, z, \omega)$ be separable into $q(t) + r(t, x, y, z, \omega)$ is removed, (2.1) could, without affecting the results, be replaced by

$$\begin{aligned} &| p(t, x, y, z, \omega) - p(t, \bar{x}, \bar{y}, \bar{z}, \bar{\omega}) | \leqslant \\ &\leqslant \varphi(t) \{ | x - \bar{x} | + | y - \bar{y} | + | z - \bar{z} | + | \omega - \bar{\omega} | \}. \end{aligned} \tag{6.1}$$

Even if $\varphi(t) \equiv D_{16}$ (a constant), our results will still remain valid.

(ii) Throughout this work, no condition had been imposed on $Q(t)$, besides its use in the equivalent system (3.1) (or (4.1)). It is hoped that in later results, some very useful impositions would be made.

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*Received* 15.V.1980

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s. I a, 1981, f. 1

# ON NONLINEAR DISCRETE INEQUALITIES OF GRONWALL TYPE

BY

RAVI P. AGARWAL and E. THANDAPANI

**1. Introduction.** During the last few years there have been number of papers written on the discrete analogue of the Gronwall inequality and its nonlinear version due to Bihari. The first discrete version of Bihari's inequality was established by Hull and Luxemburg [1]. Our purpose is to present here some interesting and useful nonlinear discrete inequalities which can be used in the study of many problems in the theory of difference equations.

**2. Main Results.** Before giving the main results, we shall introduce some notations which are involved in the discussion. Let $N$ be a set of points $\{0, 1, 2, \ldots\}$. The expression $\sum_{s=0}^{t-1} b(s)$ represents the solution of the linear difference equation $\Delta x(t) = b(t)$ for all $t \in N$, under the initial condition $x(0) = 0$, where $\Delta$ is the operator defined by $\Delta x(t) = x(t+1) - x(t)$. It is supposed that $\sum_{s=0}^{-1} b(s) = 0$. The expression $\prod_{s=0}^{t-1} C(s)$ represents a solution of the linear difference equation $x(t+1) = C(t)x(t)$ for all $t \in N$ under the initial condition $x(0) = 1$. It is supposed that $\prod_{s=0}^{-1} C(s) = 1$.

**Definition.** *A function $W: [0, \infty) \to [0, \infty)$ is said to belong to the class $S$, if*

(i) *$W(u)$ is positive, nondecreasing and continuous for $u \geqslant 0$*

(ii) $$(1/v)\, W(u) \leqslant W(u/v) \text{ for all } u \geqslant 0,\ v > 0.$$

**Theorem 1.** *Let the following inequality*

$$u(t) \leqslant p(t) + q(t) \sum_{r=1}^{n} E_r(t, u) + \sum_{r=1}^{m} \sum_{s=0}^{t-1} h_r(s) W_r(u(s)) \tag{1}$$

*be satisfied, where*

(i) *$p(t)$ is positive and nondecreasing,*

(ii) $$W_j \in S;\ j = 1, 2, \ldots, m,$$

(iii) $$E_r(t, u) = \sum_{t_1=0}^{t-1} f_{r_1}(t_1) \sum_{t_2=0}^{t_1-1} f_{r_2}(t_2) \ldots \sum_{t_r=0}^{t_{r-1}-1} f_{rr}(t_r) u(t_r),$$