(4.12) $$W' \leq -D_9 S + D_{13}\, \varphi^\alpha S.$$

This, by (4.3), implies that

$$W' + \{D_{14} - D_{15}\, \varphi^\alpha(t)\} W \leq 0$$

for some constants $D_{14}$ and $D_{15}$.

By integration, we have (4.4). A repeated use of (4.4) gives (2.2), and this completes the proof of Theorem 2.

5. *Proof of Theorem* 1 :

From the estimate (2.2), if

$$\int_{t_1}^{t_2} \varphi^\alpha(\tau)\, d\tau \leq D_3 D_4^{-1} (t_2 - t_1),$$

then the exponetial index remains negative for all $t_2 - t_1 \geq 0$. As $t = (t_2 - t_1) \to \infty$ we have $S(t) \to 0$. this gives

$$x_2 - x_1 \to 0, \quad x_2' - x_1' \to 0, \quad x_2'' - x_1'' \to 0 \ \text{and} \ x_2''' - x_1''' \to 0$$

as $t \to \infty$. Thus Theorem 1 is proved.

6. *Remarks*: (i) Condition (2.1) is a generalised Lipschitz condition on $r(t, x, y, z, \omega)$. In fact, if the assumption that $p(t, x, y, z, \omega)$ be separable into $q(t) + r(t, x, y, z, \omega)$ is removed, (2.1) could, without affecting the results, be replaced by

(6.1) $$| p(t, x, y, z, \omega) - p(t, \bar{x}, \bar{y}, \bar{z}, \bar{\omega}) | \leq \leq \varphi(t) \{ | x - \bar{x} | + | y - \bar{y} | + | z - \bar{z} | + | \omega - \bar{\omega} | \}.$$

Even if $\varphi(t) \equiv D_{16}$ (a constant), our results will still remain valid.

(ii) Throughout this work, no condition had been imposed on $Q(t)$, besides its use in the equivalent system (3.1) (or (4.1)). It is hoped that in later results, some very useful impositions would be made.

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*Received* 15.V.1980

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s. I a, 1981, f. 1

# ON NONLINEAR DISCRETE INEQUALITIES OF GRONWALL TYPE

BY

RAVI P. AGARWAL and E. THANDAPANI

**1. Introduction.** During the last few years there have been number of papers written on the discrete analogue of the Gronwall inequality and its nonlinear version due to B i h a r i. The first discrete version of Bihari's inequality was established by H u l l and L u x e m b u r g [1]. Our purpose is to present here some interesting and useful nonlinear discrete inequalities which can be used in the study of many problems in the theory of difference equations.

**2. Main Results.** Before giving the main results, we shall introduce some notations which are involved in the discussion. Let $N$ be a set of points $\{0, 1, 2, \ldots\}$. The expression $\sum_{s=0}^{t-1} b(s)$ represents the solution of the linear difference equation $\Delta x(t) = b(t)$ for all $t \in N$, under the initial condition $x(0) = 0$, where $\Delta$ is the operator defined by $\Delta x(t) = x(t+1) - x(t)$. It is supposed that $\sum_{s=0}^{-1} b(s) = 0$. The expression $\prod_{s=0}^{t-1} C(s)$ represents a solution of the linear difference equation $x(t+1) = C(t)x(t)$ for all $t \in N$ under the initial condition $x(0) = 1$. It is supposed that $\prod_{s=0}^{-1} C(s) = 1$.

**Definition.** *A function $W : [0, \infty) \to [0, \infty)$ is said to belong to the class $S$, if*

(i) *$W(u)$ is positive, nondecreasing and continuous for $u \geq 0$*

(ii) $$(1/v)\, W(u) \leq W(u/v) \ \text{for all} \ u \geq 0, \ v > 0.$$

**Theorem 1.** *Let the following inequality*

(1) $$u(t) \leq p(t) + q(t) \sum_{r=1}^{n} E_r(t, u) + \sum_{r=1}^{m} \sum_{s=0}^{t-1} h_r(s) W_r(u(s))$$

*be satisfied, where*

(i) *$p(t)$ is positive and nondecreasing,*

(ii) $$W_j \in S\,;\ j = 1, 2, \ldots, m,$$

(iii) $$E_r(t, u) = \sum_{t_1=0}^{t-1} f_{r_1}(t_1) \sum_{t_2=0}^{t_1-1} f_{r_2}(t_2) \ldots \sum_{t_r=0}^{t_{r-1}-1} f_{rr}(t_r) u(t_r),$$

(iv) $q$, $h_r$ and $f_{rs}$ are real-valued, non-negative functions, defined on $N$.

*Then*

$$u(t) \leq p(t)\Phi(t) \prod_{r=1}^{m} E_r(t),$$

*where*

$$\Phi(t) = 1 + q(t) \sum_{s=0}^{t-1} \sum_{r=1}^{n} \Delta E_r(s, 1) \prod_{\tau=s+1}^{t-1} \left[1 + \sum_{r=1}^{n} \Delta E_r(\tau, q)\right],$$

$$E_1(t) = G_1^{-1}\left[G_1(1) + \sum_{s=0}^{t-1} h_1(s)\Phi(s)\right],$$

$$E_k(t) = G_k^{-1}\left[G_k(1) + \sum_{s=0}^{t-1} h_k(s)\Phi(s) \prod_{r=1}^{k-1} E_r(s)\right],$$

$$k = 2, 3, \ldots, m.$$

*and*

$$G_k(u) = \int_{u_0}^{u} \frac{ds}{W_k(s)}, \quad 0 < u_0 \leq u.$$

*as long as*

$$G_k(1) + \sum_{s=0}^{t-1} h_k(s)\Phi(s) \prod_{r=1}^{k-1} E_r(s) \in \operatorname{Dom}(G_k^{-1}),$$

$$k = 1, 2, \ldots, m.$$

*Proof.* Inequality (1) can be written as

$$u(t) \leq p^*(t) + q(t) \sum_{s=1}^{n} E_r(t, u), \tag{2}$$

where $p^*(t) = p(t) + \sum_{r=1}^{n} \sum_{s=0}^{t-1} h_r(s) W_r(u(s))$. Define $R(t) = \sum_{r=1}^{n} E_r(t, u)$; then, we find

$$\Delta R(t) = \sum_{r=1}^{n} \Delta E_r(t, u) \leq \sum_{r=1}^{n} \Delta E_r(t, p^* + qR) \leq \sum_{r=1}^{n} \Delta E_r(t, p^*) + \tag{3}$$

$$+ \sum_{r=1}^{n} \Delta E_r(t, q) R(t)$$

since $R(t)$ is nondecreasing for all $t \in N$.

From (3), we have

$$R(t+1) - \left[1 + \sum_{r=1}^{n} \Delta E_r(t, q)\right] R(t) \leq \sum_{r=1}^{n} \Delta E_r(t, p^*).$$

Multiplying both sides by $\prod_{s=0}^{t-1}\left[1 + \sum_{r=1}^{n} \Delta E_r(s, q)\right]^{-1}$, and summing up from 0 to $t-1$, we get



$$R(t) \leq \sum_{s=0}^{t-1} \sum_{r=1}^{n} \Delta E_r(s, p^*) \prod_{\tau=s+1}^{t-1} \left[1 + \sum_{r=1}^{n} \Delta E_r(\tau, q)\right].$$

Now using the non-decreasing nature of $p^*(t)$ and substituting the above estimate in (2), we get

$$u(t) \leq p^*(t)\Phi(t). \tag{4}$$

Inequality (4) can be written as $V(t) \leq 1 + \sum_{r=1}^{m} \sum_{s=0}^{t-1} h_r(s)\Phi(s) W_r(v(s))$, where

$$v(t) = \frac{u(t)}{p(t)\Phi(t)}.$$

Rest of the proof is by finite induction. For this, let $r=1$ i.e.

$$v(t) \leq 1 + \sum_{s=0}^{t-1} h_1(s)\Phi(s) W_1(v(s)). \tag{5}$$

Define $R_1(t)$ as the right member of (5), then

$$\Delta R_1(t) = h_1(t)\Phi(t) W_1(v(t)) \leq h_1(t)\Phi(t) W_1(R_1(t)), \quad R_1(0) = 1.$$

From the definition of $G_1$, we find

$$G_1(R_1(t+1) - G_1(R_1(t)) = \int_{R_1(t)}^{R_1(t+1)} \frac{ds}{W_1(s)} \leq \frac{\Delta R_1(t)}{W_1(R_1(t))} \leq h_1(t)\Phi(t).$$

Summing up from 0 to $t-1$, we get $R_1(t) \leq E_1(t)$. Now assume that the result is true for some $k$, $1 < k \leq m-1$. Then for $r = k+1$

$$v(t) \leq 1 + \sum_{s=0}^{t-1} h_{k+1}(s)\Phi(s) W_{k+1}(v(s)) + \sum_{r=1}^{k} \sum_{s=0}^{t-1} h_r(s)\Phi(s) W_r(v(s)) \leq \tag{6}$$

$$\leq \left[1 + \sum_{s=0}^{t-1} h_{k+1}(s)\Phi(s) W_{k+1}(v(s))\right] \prod_{r=1}^{k} E_r(t).$$

From (6), we find

$$\frac{V(t)}{\prod_{r=1}^{k} E_r(t)} \leq 1 + \sum_{s=0}^{t-1} h_{k+1}(s)\Phi(s) \prod_{s=1}^{k} E_r(s) W_{k+1}\left(\frac{V(s)}{\prod_{r=1}^{k} E_r(s)}\right)$$

and following the same method as for $r=1$, we obtain

$$V(t) \leq \prod_{r=1}^{k+1} E_r(t)$$

and hence the result follows.

**Theorem 2.** *Let the following inequality be satisfied*

$$u(t) \leq p(t) + q(t) \sum_{r=1}^{n} E_r(t, u) + \sum_{r=1}^{m} E_r(t, W(u)), \tag{7}$$

*where* $W \in S$ ; $p(t)$, $E_r(t, u)$ *are defined in Theorem 1. Then,*

$$u(t) \leqslant p(t)\Phi(t)G^{-1}\left[G(1)+\sum_{s=0}^{t-1}\sum_{r=1}^{m}\Delta E_r(s, \Phi)\right],$$

*where* $\Phi(t)$ *is same as in theorem* 1 *and*

$$G(u)=\int_{u_0}^{u}\frac{ds}{W(s)}, \quad 0<u_0\leqslant u$$

*as long as*

$$G(1)+\sum_{s=0}^{t-1}\sum_{r=1}^{m}\Delta E_r(s, \Phi) \in \mathrm{Dom}\,(G^{-1}).$$

*Proof.* The proof is similar to that of theorem 1.

*Remark 1.* Let $g_r(t) \geqslant 1$ ; $r=1, 2, \ldots, m$ and the inequality

$$u(t) \leqslant p(t)+q(t)\sum_{r=1}^{n}E_r(t, u)+\sum_{r=1}^{m}g_r(t)\sum_{s=0}^{t-1}h_r(s)W_r(u(s))$$

be satisfied then it is easy to find an upper estimate in terms of known functions as in theorem 1.

*Remark 2.* Several particular cases of Theorems 1 and 2 have been considered in [1]—[8].

**Theorem 3.** *Let the inequality* (7) *be satisfied where* $p(t)$ *and* $E_r(t, u)$ *are the same as in Theorem* 2, $W$ *is positive, nondecreasing, continuous and submultiplicative. Then*

$$u(t) \leqslant p(t)\Phi(t)G^{-1}\left[G(1)+\sum_{s=0}^{t-1}\sum_{r=1}^{m}\Delta E_r\left(s, \frac{W(p(s)\Phi(s))}{p(s)}\right)\right],$$

*where* $\Phi$ *and* $G$ *are the same as in Theorem* 2, *and the term inside the bracket* $\in$ Dom $(G^{-1})$.

*Proof.* Following the proof of Theorem 1, we find

$$u(t) \leqslant p^*(t)\Phi(t). \tag{8}$$

where $p^*(t)=p(t)+\sum_{r=1}^{m}E_r(t, W(u))$. From (8), we have

$$\frac{u(t)}{p(t)\Phi(t)} \leqslant 1+\sum_{r=1}^{m}E_r\left(t, \frac{W(u)}{p}\right). \tag{9}$$

Let $R(t)$ be the right member of (9), then

$$\Delta R(t)=\sum_{r=1}^{m}\Delta E_r\left(t, \frac{W(u)}{p}\right) \leqslant \sum_{r=1}^{m}\Delta E_r\left(t, \frac{W(p\Phi R)}{p}\right), \quad R(0)=1.$$

Now, since $W$ is submultiplicative and nondecreasing, we find

$$\Delta R(t) \leqslant \sum_{r=1}^{m}\Delta E_r\left(t, \frac{W(p\Phi)}{p}\right)W(R(t)).$$



Then as in the proof of theorem 1, we obtain

$$R(t) \leqslant G^{-1}\left[G(1)+\sum_{s=0}^{t-1}\sum_{r=1}^{m}\Delta E_r\left(s, \frac{W(p\Phi)}{p}\right)\right].$$

and the result follows from (9).

**3. An Application.** As an application, we consider here the following sum-difference equations of Volterra type

$$x_1(t)=C_1+\sum_{s=0}^{t-1}[F_1(t, s, x_1(s), x_2(s))+H(t, s, x_1(s))] \tag{10}$$

$$x_2(t)=C_2+\sum_{s=0}^{t-1}F_2(t, s, x_1(s)), \tag{11}$$

where the functions $F_1$, $F_2$ and $H$ satisfy

$$|F_1(t, s, x_1(s), x_2(s))| \leqslant a(s)|x_1(s)|+b(s)|x_2(s)| \tag{12}$$

$$|F_2(t, s, x_1(s))| \leqslant C(s)|x_1(s)| \tag{13}$$

$$|H(t, s, x_1(s))| \leqslant h(s)W(|x_1(s)|) \tag{14}$$

for all $t \in N$ and $W$ is same as in theorem 1.

From (11) and (13), we find

$$|x_2(t)| \leqslant C_2+\sum_{s=0}^{t-1}C(s)|x_1(s)|, \tag{15}$$

and from (10), (12), (14) and (15), we obtain

$$|x_1(t)| \leqslant C_1+\sum_{s=0}^{t-1}a(s)|x_1(s)|+\sum_{s=0}^{t-1}b(s)\left[C_2+\sum_{\tau=0}^{s-1}C(\tau)|x_1(\tau)|\right]+$$
$$+\sum_{s=0}^{t-1}h(s)W(|x_1(s)|).$$

The above inequality is exactly of the same form as considered in theorem 1. Thus it is possible to find the estimates for $|x_1(t)|$ and $|x_2(t)|$ in terms of known functions.

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*Received* 13.V.1980

Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXVII, s. I a, 1981, f. 1

# ON UNIFORM NÖRLUND SUMMABILITY OF THE CONJUGATE DERIVED FOURIER SERIES

BY

FAZL-UL-HAQ

Saxena (1966) established theorems for uniform Nörlund summability of Fourier series. The present theorem deals with the uniform Nörlund summability of the *conjugate derived fourier series* under very general conditions.

1. Let $\Sigma a_n$ be a given infinite series with the sequence of partial sum $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \ldots + p_n.$$

The sequence — $t_n$-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k s_{n-k} = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} s_k,\ (P_n \neq 0). \tag{1.1}$$

defines a sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $(N, p_n)$ to the sum $s$ if $\lim\limits_{n\to\infty} t_n$ exists and equals $s$.

The conditions for regularity of the method of $(N, p_n)$ summability defined by (1.1) are

$$\lim \frac{p_n}{P_n} = 0, \tag{1.2}$$

and

$$\sum_{k=0}^{n} |p_k| = O(|p_n|), \text{ as } n \to \infty \tag{1.3}$$

In the special case in which $p_n = 1/n+1$, and therefore $P_n \sim \log n$, as $n \to \infty$, the Nörlund mean reduces to the harmonic mean of $\{s_n\}$.

2. Let $f(t)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue in the interval $(-\pi, \pi)$. Let $f(t)$ have derivative $f'(t)$ everywhere in $E$, and let the Fourier series associated with $f(t)$ be

$$\frac{1}{2}a_0 + \sum_{n=1}^{n} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{2.1}$$