6. Sugiyama S. — *On the stability problems of difference equations*, Bull. Sci. Engr. Research Lab. Waseda Univ. 45 (1969), 140—144.
7. Sugiyama S. — *Stability and boundedness problems on difference and functional difference equations*, Mem. School of Sci. and Engr. Waseda Univ. 33 (1969), 79—88.
8. Willett D. and Wong J. S. W. — *On the discrete analogues of some generalizations of Gronwall's inequality*, Monatsh. Math. 69 (1964), 362—367.

MATSCIENCE
The Institute of Mathematical Science
Madras — 600020, *India*

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ON UNIFORM NÖRLUND SUMMABILITY OF THE CONJUGATE DERIVED FOURIER SERIES

BY

FAZL-UL-HAQ

Saxena (1966) established theorems for uniform Nörlund summability of Fourier series. The present theorem deals with the uniform Nörlund summability of the *conjugate derived fourier series* under very general conditions.

1. Let Σa_n be a given infinite series with the sequence of partial sum $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \ldots + p_n.$$

The sequence — t_n-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k s_{n-k} = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} s_k,\ (P_n \neq 0), \tag{1.1}$$

defines a sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable (N, p_n) to the sum s if $\lim\limits_{n\to\infty} t_n$ exists and equals s.

The conditions for regularity of the method of (N, p_n) summability defined by (1.1) are

$$\lim \frac{p_n}{P_n} = 0, \tag{1.2}$$

and

$$\sum_{k=0}^{n} |p_k| = O(|P_n|),\ \text{as}\ n \to \infty \tag{1.3}$$

In the special case in which $p_n = 1/n+1$, and therefore $P_n \sim \log n$, as $n \to \infty$, the Nörlund mean reduces to the harmonic mean of $\{s_n\}$.

2. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue in the interval $(-\pi, \pi)$. Let $f(t)$ have derivative $f'(t)$ everywhere in E, and let the Fourier series associated with $f(t)$ be

$$\frac{1}{2}a_0 + \sum_{n=1}^{n} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{2.1}$$

Then the series

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} nB_n(t) \tag{2.2}$$

is the first derived series or the derived Fourier series of $f(t)$. The series conjugate to (2.2) is

$$\sum_{n=1}^{\infty} n(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} nA_n(t). \tag{2.3}$$

We shall use the following notations:

$$h(t) = h(t, x) = f(x+t) + f(x-t) - 2f(x),$$

$$H(t) = \int_0^t |dh(u)|,$$

$$H(x) = -\frac{1}{4\pi}\int_0^{\pi} h(t) \operatorname{cosec}^2 \frac{1}{2}t\, dt = \lim_{\epsilon \to 0}\left\{-\frac{1}{4\pi}\int_{\epsilon}^{\pi} h(t) \operatorname{cosec}^2 \frac{t}{2}\, dt\right\}.$$

If we denote

$$H_n(x) = -\frac{1}{4\pi}\int_{1/n}^{r} h(t) \operatorname{cosec}^2 \frac{1}{2} t\, dt,$$

then

$$\lim_{n\to\infty} H_n(x) = H(x).$$

If the function $f(x)$ is of bounded variation, then $H(x)$ exists for almost all values τ of x [Z y g m u n d, 4, p. 146]. $\tau = [1/t]$, where $[\lambda]$ denotes the integral part of λ.

3. S a x e n a (1966) introduced the concept of uniform summability, which is as follows: Let

$$u_0(x) + u_1(x) + u_2(x) + \dots \tag{3.1}$$

be any infinite series, and

$$U_\nu(x) = u_0(x) + u_1(x) + \dots + u_\nu(x).$$

Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + \dots + p_n.$$

If there exists a function $U = U(x)$, such that

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k\{U_{n-k}(x) - U(x)\} = o(1)$$

uniformly is a set E in which $U = U(x)$ is bounded, then the series (3.1) is summable (N, p_n) uniformly is E to the sum U.

In 1966, Saxena established the following theorems for the uniform Nörlund summability of the Fourier series.

Theorem A. *If $\alpha(t)$ denotes a function of t, $\alpha(t)$ and $t/\alpha(t)$ ultimately increase steadily with t,*

$$\log(n) = O(\alpha(P_n)), \text{ as } n \to \alpha,$$

and

$$\Phi(t) = \int_0^t |\Phi(u)|\, du = o\left(\frac{t}{\alpha(P_\tau)}\right)$$

uniformly in a set E in which $s = s(x)$ is bounded, as $t \to +0$, then the series (2.1) is summable (N, p_n) uniformly in E to the sum s.

Theorem B. *If $\alpha(t)$ stands for a function of t and ultimately increases steadily with t,*

$$\int_{1/n}^{n} \frac{P}{\alpha(P_\tau)}\frac{1}{t}\, dt = O(P_n), \text{ as } n \to \infty$$

and

$$\Phi(t) = o\left(\frac{t}{\alpha(P_\tau)}\right)$$

uniformly in a set E in which $s = s(x)$ is bounded as $t \to +0$, then the series (2.1) is summable (N, p_n) uniformly in E to the sum s.

In the present papers, we establish the following theorem for the conjugate derived Fourier series.

Theorem. *If*

$$H(t) = o\left(\frac{\beta(1/t)p_\tau}{\alpha(P_\tau)}\right)$$

uniformly in a set E, as $t \to +0$, then the series (2.3) is summable (N, p_n) uniformly in E to the sum

$$-\frac{1}{4\pi}\int_0^{\pi} h(t) \operatorname{cosec}^2 \frac{1}{2} t\, dt, \tag{3.1}$$

provided that this exists; where $\alpha(t)$ and $\beta(t)$ are functions of t such that $\alpha(t)$, $\beta(t)$ and $\beta(t)\, t/\alpha(t)$ increase monotonically with t as $t \to +0$, and

$$\beta(n) P_n = O(\alpha(P_n)), \text{ as } n \to \infty,$$

and $\{p_n\}$ is a non-negative and non-increasing sequence of constants.

4. In order to prove the theorem, we require the following lemmas:

Lemma 1. (M c F a d d e n, 1942). *If $\{p_n\}$ is a non-negative and non-increasing, then, for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$ and any n,*

$$\left|\sum_{k=1}^{b} p_k e^{i(n-k)t}\right| \leqslant K P_\tau,$$

where K is absolute constant and

$$P_n = p_0 + p_1 + \ldots + p_n$$

Lemma 2. (Hardy & Rogosinski, 1944).

Suppose that f and g have period 2π, that f is L and g is V; that λ and α are real; and that $-\pi \leqslant a \leqslant b \leqslant \pi$, then

$$J(a, b, \alpha, \lambda) = \int_a^b f(\theta+\alpha)\, g(\theta)\, e^{-i\lambda\theta}\, d\theta \to 0$$

uniformly in a, b and α, when $|\lambda| \to \infty$.

Proof of the theorem.

Let the sum of the first n terms of the series (2.3) be denoted by $\bar{\sigma}_n(x)$; then, we have

$$\sigma_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(u) \frac{\partial}{\partial x} \sum_1^n \sin \nu(u-x))du =$$

$$= -\frac{1}{\pi}\int_0^{\pi} \frac{d}{dt}\left[\frac{\cos \frac{1}{2}t - \cos (n+\frac{1}{2})t}{2 \sin \frac{1}{2} t}\right] \{f(x+t)+f(x-t)\}$$

$$= -\frac{1}{\pi}\int_0^{\pi} \frac{\cos \frac{1}{2}t - \cos (n+\frac{1}{2})t}{2 \sin \frac{1}{2} t} dh(t).$$

Therefore,

$$\sigma_k(x) = -\frac{1}{2\pi}\int_0^{\pi} [\cot \tfrac{1}{2} t\,(1-\cos kt) + \sin kt)]dh(t), \quad k < n =$$

$$= -\frac{1}{2r}\left\{\int_0^{1/n} + \int_{1/n}^{\pi}\right\} \cot \tfrac{1}{2} t\,(1-\cos kt)dh(t) - \frac{1}{2\pi}\int_0^{\pi} \sin kt dh(t) =$$

$$= -\frac{1}{2\pi}(L_1 + L_2 + L_3), \text{ say.}$$

But

$$|L_1| \leqslant \int_0^{1/n} \left|\frac{\cos \frac{1}{2} t}{\sin \frac{1}{2} t}\, 2 \sin^2 \tfrac{1}{2} kt\right| \left|dh(t)\right| \leqslant k \int_0^{1/n} |dh(t)| = KH(1/n) = 0\left(\frac{K\beta(n)p_n}{\alpha(P_n)}\right) =$$

$$= o(1), \text{ since } n\, p_n \leqslant P_n$$

and

$$-\frac{1}{2\pi} L_2 = -\frac{1}{2\pi}\int_{1/n}^{\pi} \cot \tfrac{1}{2} t\,(1-\cos kt)dh(t) = -\frac{1}{2\pi}\int_{1/n}^{\pi} \cot \tfrac{1}{2} t\, dh(t) +$$

$$+ \frac{1}{2\pi}\int_{1/n}^{\lambda} \cot \tfrac{1}{2} t \cos kt\, dh(t) = -\frac{1}{2\pi}[\cot \tfrac{1}{2} th(t)] - \frac{1}{2r}\int_{1/n}^{\pi} \tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2} th(t)dt +$$

$$+ \frac{1}{2\pi}\int_{1/n}^{\pi} \cot \tfrac{1}{2} t \cos kt\, dh(t) = o(1) + H_n(x) + \frac{1}{2\pi}\int_{1/n}^{\pi} \cot \tfrac{1}{2} t \cos kt\, dh(t)$$

for $h(t)/t$ tends to zero with t, as $f'(x)$ exists.

Hence

$$\sigma_n(x) - H_n(x) = \frac{1}{2\pi}\int_{1/n}^{\pi} \cot \tfrac{1}{2} t \cos kt\, dh(t) - \frac{1}{2\pi}\int_0^{\pi} \sin kt\, dh(t) + o(1)$$

$$= \frac{1}{2\pi}\int_{1/n}^{\pi} \cot \tfrac{1}{2} t \cos kt\, dh(t) - \frac{1}{2\pi}\int_0^{1/n} \sin kt\, dh(t) - \frac{1}{2\pi}\int_{1/n}^{\lambda} \sin kt\, dh(t) + o(1) =$$

$$= \frac{1}{2\pi}\int_{1/n}^{\pi} \frac{\cos (k+\frac{1}{2})t}{\sin \frac{1}{2} t} dh(t) + o(1);$$

therefore

$$\sigma_{n-k}(x) - H_n(x) = \frac{1}{2\pi}\int_{1/n}^{\pi} \frac{\cos (n-k+\frac{1}{2})t}{\sin \frac{1}{2} t} dh(t) + o(1).$$

Thus, we have

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k \{\sigma_{n-k}(x) - H_n(x)\} = \frac{1}{P_n}\sum_{k=0}^{n} p_k \left\{\frac{1}{2\pi}\int_{1/n}^{\pi} \frac{\cos (n-k+\frac{1}{2})t}{\sin \frac{1}{2} t} dh(t) + o(1)\right\} =$$

$$= \int_{1/n}^{\pi} dh(t) \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_k \frac{\cos (n-k+\frac{1}{2})t}{\sin \frac{1}{2} t} + o(1) = \int_{1/n}^{\pi} dh(t) N_n(t) + o(1).$$

where

$$N_n(t) = \frac{1}{2\pi}\frac{1}{P_n}\sum_{k=0}^{n} p_k \frac{\cos (n-k+\frac{1}{2})t}{\sin \frac{1}{2} t}.$$

In order to prove the theorem, we have to show that

$$\int_{1/n}^{\pi} dh(t)\, N_n(t) = o(1), \quad \text{uniformly in } E,$$

as $n \to \infty$, when (3.1) exists. We set

$$\text{(4.1)} \qquad \int_{1/n}^{\pi} dh(t) N_n(t) = \left(\int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right) dh(t) N_n(t) = I_1 + I_2, \text{ say,}$$

where δ is small but fixed. Now

$$I_2 = \int_{\delta}^{\pi} dh(t) N_n(t) = \int_{\delta}^{\pi} \frac{d}{dt} \{ f(x+t) + f(x-t) - 2f(x) \} N_n(t) dt =$$

$$= o(1) + \int_{\delta}^{\pi} \frac{d}{dt} (f(x-t)) N_n(t) dt, \text{ (by Lemma 2)} = o(1) +$$

$$+ \frac{1}{2\pi} \frac{1}{P_n} \sum_{k=0}^{n} p_k \int_{\delta}^{\pi} \frac{d}{dt} (f(x-t)) \frac{\cos (n - k + \frac{1}{2}) t}{\sin \frac{1}{2} t} dt =$$

$$= o(1) - \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_k \int_{\delta}^{\pi} \frac{d}{dt} (f(x+t)) \frac{\cos (n-k+\frac{1}{2}) t}{\sin \frac{1}{2} t} dt = o(1) +$$

$$+ \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_k \int_{-\pi}^{-\delta} \frac{d}{dt} (f(x+t)) \frac{\cos (n - k + \frac{1}{2} t)}{\sin \frac{1}{2} t} dt =$$

$$= o(1) + o(1), \text{ uniformly in } E \text{ (by Lemma 2)}$$

$$\text{(4.2)} \qquad = o(1) \text{ uniformly in } E.$$

Again, we have

$$N_n(t) = \frac{1}{2\pi P_n} \frac{1}{\sin \frac{1}{2} t} \sum_{k=0}^{n} p_k \cos (n - k + \tfrac{1}{2}) t.$$

Hence, for $n^{-1} \leqslant t \leqslant \delta \leqslant \pi$,

$$N_n(t) = O\left(\frac{1}{P_n \sin \frac{1}{2} t} \left| \sum_{k=0}^{n} p_k \cos (n - k + \tfrac{1}{2}) t \right| \right) = O(P_\tau / P_n t), \text{ (by lemma 1)}$$

Therefore

$$I_1 = \int_{1/n}^{\delta} dh(t) N_n(t) = O\left(\int_{1/n}^{\delta} |dh(t)|\, |N_n(t)| \right) = O\left(\frac{1}{P_n} \int_{1/n}^{\delta} |dh(t)| \frac{P_\tau}{t} \right) =$$

$$= O\left(\frac{1}{P_n} \left| H(t) \frac{P_\tau}{t} \right| \Big|_{1/n}^{\delta} \right) + O\left(\frac{1}{P_n} \int_{1/n}^{\delta} H(t) \frac{P_\tau}{t^2} dt \right) + o(1), \text{ uniformly in } E =$$

$$= I_{1.1} + I_{1.2} + o(1), \text{ say}$$

Now

$$\text{(4.3)} \qquad I_{1.1} = O\left(\frac{1}{P_n} \left[H(t) \frac{P_\tau}{t} \right]_{1/n}^{\delta} \right) = O\left(\frac{1}{P_n} \right) + o\left(\frac{1}{P_n} \frac{\beta(n) p_n}{\alpha(P_n)} n P_n \right) =$$

$$= o(1), \text{ as } n \to \infty, \text{ uniformly in } E, \text{ since } np_n \leqslant P_n$$

and

$$I_{1.2} = O\left(\frac{1}{P_n} \int_{1/n}^{\delta} H(t) \frac{P_\tau}{t^2} dt \right) = o(1) + O\left(\frac{1}{P_n} \sum_{m=1}^{n-1} \int_{m}^{m+1} H(1/v) P_{(v)} dv \right);$$

but

$$\int_{m}^{m+1} H(1/v)\; P_{(v)}\, dv \leqslant H(1/m) P_m = o\left(\frac{P_m \beta(m) p_m}{\alpha(P_m)} \right) = o(p_m), \text{ as } m \to \infty.$$

so

$$\text{(4.4)} \qquad I_{1.2} = o(1) + o\left(\frac{1}{P_n} \sum_{m=1}^{n-1} p_m \right) = o(1), \text{ uniformly in } E.$$

Collecting (4.3) and (4.4), we get

$$\text{(4.5)} \qquad I_1 = o(1), \text{ uniformly in } E.$$

Now from (4.1), (4.2) and (4.5) theorem follows.

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REFERENCES

1. Hardy, G. H. and Rogosinski, W. W. — *Fourier series*, Cambridge Tracts in Mathematics and Mathematical Physics, ed. 1944, No. 38. Tg. 31, p. 24.
2. McFadden, Leonard, — *Absolute Nörlund summability*, Duke Math. Jour. 9 (1942), 168 — 207.
3. Nörlund, N. E. — *Sur une application des fonctions permutables*, Lunds Universitets Arsskrift (2), 16 (1919), No. 3.
4. Zygmund, A. — *Trigonometric series*, Warsaw (1935).
5. Saxena, Ashok — *On Uniform Nörlund summability of Fourier series*, Proc. of the Nat. Inst. of Sci. of India, Post A (32), 1966.

Applied Mathematics Section
Institute of Technology
B.H.U. Varanasi 221005 India

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SOME REMARKS ON NON-RULED SURFACES WHOSE ASYMPTOTES OF BOTH FAMILIES BELONG TO LINEAR COMPLEXES

BY

FROIM MARCUS

1. A. Terracini [1] determined, in finite form, non-ruled surfaces whose asymptotes belong to linear complex. They form there classes, given respectively by :

$$\text{(I)}\qquad \begin{aligned} &x_1=u(V'-U')+2U\ ;\ x_2=v(V'-U')-2V\ ;\\ &x_3=uv(V'-U')-2(uV-vU)\ ;\ x_4=V'-U'\ ; \end{aligned}$$

$$\text{(II)}\qquad x_1=u(U'-V')-2U\ ;\ x_2=v(U'-V')+2V\ ;\ x_3=2uv\ ;\ x_4=-2\ ;$$

$$\text{(III)}\qquad (u-v)\,(U'-V')\ ;\ x_2=U'-V'\ ;\ x_3=u+v\ ;\ x_4=1,$$

where U is an arbitrary function of u alone and V of v alone (u, v being the asymptotic parameters), irreducible to quadratic polynomials. The surfaces of class (I) are obtained by W-transformation of regular quadrics ; those of classes of (II) and (III)—from degenerate quadrics ; all of them are isothermo-asymptotics of Fubini. Terracini observed also that the surfaces of class (III) coincide with those described by Wilczynski in [2].

G. Fubini [3] proved the following :

(1) *If one the focal surfaces of a W-congruence is a regular quadric S, the asymptotes of the second focal surface $\bar{S}$ are contained in linear complexes.*

(2) *The surfaces $\bar{S}$ are in Cayley's metric, of null curvature, and consequently coincident with those studied by Bianchi.*

K. Strubecker [4], starting with Fubini's non-Euclidean interpretation, showed that the surfaces (II) and (III) can also be given by a unitary geometric construction *using* Blaschke's *quasi-elliptic space geometry and the isotrope space studied by Strubecker by analytical representation* :

$$x=x^r(u)\,x^l(v), \tag{1.1}$$

where

$$x^r(u)=\sum_0^3 x_i^r(u)e_i\ ;\ x^l(v)=\sum_0^3 x_i^l(v), \tag{1.2}$$