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ON THE CONSTRUCTION OF BOL-LOOPS

BY

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1. **Introduction.** A loop $G(.)$ is a Bol-loop if and only if

$$(xy \,.\, z)y = x(yz \,.\, y) \tag{1}$$

for all $x, y, z \in G$. Bol-loops had their origin in the work B o l [3] and their basic algebraic properties were discussed by R o b i n s o n [4]. Recently B u r n [2] has proved that, for any prime p, a Bol-loop of order $2p$ or of order p^2 is necessarily a group. Also S h a r m a [5] has constructed Bol loops of order $p^n (p \geq 2, n \geq 3)$, which are not groups. A natural question arises: are Bol-loops of order $4p$ or of order $2p^2$, p being prime also groups. The object of this paper is to answer the preceeding question. First of all we prove a theorem on the construction of Bol-loops whose application will show the existence of a Bol-loop of order $2p^2$.

Further we construct Bol-loops of order $2n^2 (n > 2)$ or of order $4n (n > 2)$, which fail to be M o u f a n g. For M o u f a n g-loops, see [1].

2. **Theorem 1.** *Let H be a non abelian subgroup of a group G and let $A = H \times G$. For (h_1, g_1) and (h_2, g_2) in A, define*

$$(h_1, g_1) \circ (h_2, g_2) = (h_1 h_2, h_2 g_1 h_2^{-1} g_2) \tag{2}$$

then $A(\circ)$ is a Bol-loop.

Proof. Let $x = (h_1, g_1)$, $y = (h_2, g_2)$ and $z = (h_3, g_3)$ be elements of A. Then

$$\begin{aligned} x \circ (y \circ z) &= (h_1, g_1) \circ [(h_2, g_2) \circ (h_3, g_3)] = (h_1, g_1) \circ (h_2 h_3, h_3 g_2 h_3^{-1} g_3) = \\ &= (h_1 h_2 h_3, h_2 h_3 g_1 h_3^{-1} h_2^{-1} h_3 g_2 h_3^{-1} g_3). \end{aligned} \tag{3}$$

Also

$$\begin{aligned} (x \circ y) \circ z &= [(h_1, g_1) \circ (h_2, g_2)] \circ (h_3, g_3) = (h_1 h_2, h_2 g_1 h_2^{-1} g_2) \circ (h_3, g_3) = \\ &= (h_1 h_2 h_3, h_3 h_2 g_1 h_2^{-1} g_2 h_3^{-1} g_3). \end{aligned} \tag{4}$$

From (3) and (4), we have

$$x \circ (y \circ z) \neq (x \circ y) \circ z. \tag{5}$$

Thus $A(\circ)$ is a loop with identity (e, e), where e is the identity of G. Now we show that $A(\circ)$ satisfies the identity. Let $x = (h_1, g_1)$, $y = (h_2, g_2)$ and $z = (h_3, g_3)$.

$$(xy\circ z)y=[(h_1,g_1)(h_2,g_2)\circ(h_3,g_3)]\circ(h_2,g_2)=$$

$$(6)\quad =[(h_1h_2,\ h_2g_1h_2^{-1}g_2)\circ(h_3,g_3)]\circ(h_2,g_2)=(h_1h_2h_3,\ h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3)\circ(h_2,g_2)=$$

$$=(h_1h_2h_3h_2,\ h_2h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3hg_3h_2^{-1}g_2).$$

$$x(yz\circ y)=(h_1,g_1)\circ[(h_2,g_2)(h_3,g_3)\circ(h_2,g_2)]=$$

$$(7)\quad =(h_1,g_1)\circ[(h_2h_3,\ h_3g_2h_3^{-1}g_3)\circ(h_2,g_2)]=$$

$$=(h_1,g_1)\circ(g_2h_3h_2,\ h_2h_3,\ g_2h_3^{-1}g_3h_2^{-1}g_2)=$$

$$=(h_1h_2h_3h_2,\ h_2h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3h_2^{-1}g_2).$$

From (6) and (7), it follows that $A(\circ)$ is a Bol loop. Further $(h^{-1}, h^{-1}g^{-1}h)$ is the inverse of the element (h,g); we consider

$$(8)\quad \begin{aligned}(h,g)\circ(h^{-1},\ h^{-1}g^{-1}h)(h_1g_1)&=(h,g)\circ(h^{-1},\ h_1h^{-1}g^{-1}h\,h_1^{-1}g_1)=\\ &(e,\ h^{-1}ghh_1h^{-1}g^{-1}hh_1^{-1}g_1).\end{aligned}$$

It follows from (8) that $A(\circ)$ does not satisfy the left inverse property. Hence $A(\circ)$ is not a Moufang-loop.

Corollary 1. *Given a non-abelian group G, we can always construct a Bol-loop.*

Proof. It easily follows from the theorem I.

Corollary 2. *Let G be a non-abelian group of order m; then we can construct a Bol-loop of order m^2.*

Proof. It easily follows from the theorem I.

3. **Application.** (i) Let D_3 the dihedral group of order 6, be the non-abelian group G in theorem I, then $A(\circ)$ is a Bol-loop of order 36. A careful study of this Bol loop revealed that it contains a sub-loop of order 18 (see table I). Further by replacing D_3 by D_5 in theorem I, we

TABLE I

(Bol-loop of order 18)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	3	4	5	6	1	11	16	15	12	14	10	18	17	9	8	13	7
3	4	5	6	1	2	13	15	10	16	18	8	14	7	12	9	11	17
4	5	6	1	2	3	18	9	12	8	7	16	17	11	10	15	14	13
5	6	1	2	3	4	14	12	16	9	17	15	7	13	8	11	18	11
6	1	2	3	4	5	17	10	8	15	13	9	11	18	16	12	7	14
7	10	13	16	14	9	8	1	11	18	6	5	15	12	3	17	4	2
8	18	15	17	12	11	1	7	6	2	9	14	3	5	13	4	16	10
9	7	10	13	16	14	4	18	1	3	15	11	6	2	17	5	8	12
10	13	16	14	9	7	6	17	3	5	12	18	2	4	11	1	15	8
11	8	18	15	17	12	16	2	7	13	3	6	9	10	4	14	1	5
12	11	8	18	15	17	5	14	4	6	16	13	1	3	7	2	11	9
13	16	14	9	7	10	15	3	18	17	2	1	12	8	5	11	6	4
14	9	7	10	13	16	12	5	17	11	4	3	8	15	1	18	2	6
15	17	12	11	8	18	3	13	2	4	10	7	5	1	14	6	9	16
16	14	9	7	10	13	2	11	5	1	8	17	4	6	18	3	12	15
17	12	11	8	18	15	10	6	14	7	1	4	16	9	2	13	5	3
18	15	17	12	11	8	9	4	13	14	5	2	10	16	6	7	3	1

found that $A(\circ)$ is a Bol-loop of order 100 and it contains a sub-loop of order 50. These examples disprove the conjecture that a Bol-loop of order $2p^2$, p being odd prime, is necessarily a group.

Now we give a general construction for Bol-loops of order $2n^2(n>2)$.

4. **Theorem 2.** *Let L be defined as all formal symbols y^ix^j, $i=0,1,2,\ldots,n-1$ $j=0,1,2,\ldots,2n-1$ where $x^{2n}=y^n=e$ and*

$$(9)\qquad y^{\alpha_1}x^{\beta_1}\circ y^{\alpha_2}x^{\beta_2}=y^{\mu}x^{\nu}\quad where$$

$$\mu=\begin{cases}(\alpha_1+\alpha_2)\bmod n \text{ if } \beta_1\equiv 0 \pmod 2\\ (\alpha_1+p-\alpha_2)\bmod n \text{ if } \beta_1\equiv 1 \pmod 2\end{cases}$$

$$\nu=\begin{cases}(\beta_1+\beta_2)\bmod 2n \text{ if } \beta_1\equiv 0 \pmod 2\\ (\beta_1+\beta_2+2p-2\alpha_2)\bmod 2n \text{ if } \beta_1\equiv 1 \pmod 2,\ \beta_2\equiv 0 \pmod 2\\ (\beta_1+\beta_2+2\alpha_2)\bmod 2n \text{ if } \beta_1\equiv 1 \pmod 2,\ \beta_2\equiv 1 \pmod 2.\end{cases}$$

then $L(\circ)$ is a is a Bol-loop of order $2n^2$, where n is any positive integer greater than 2.

Proof: Let $A=y^{n-1}x$; $B=y^{n-1}x^n$ $C=x^n$, then

$$(10)\qquad (A\circ B)\circ C=x^{n-1}\circ x^n=x^{2n-1},$$

$$(11)\qquad A\circ(B\circ C)=y^{n-1}x\circ y^{n-1}=x^3.$$

From (10) and (11), since $2n-1\neq 3$ for n any positive integer greater than 2. We have that

$$(A\circ B)\circ C\neq A\circ(B\circ C).$$

Thus $L(\circ)$ is a loop of order $2n^2$ $(n>2)$ with identity e. We show that it satisfies the identity

$$(1)\qquad (xy,z)y=x(yz,y).$$

All possible combinations of values of β_1 and β_2 in the multiplication of the elements of L are displayed in the table II below. In table II $\beta_i=$ even or odd means β_i is an even or odd integer.

Let $A=y^{\alpha_1}x^{\beta_1}$, $B=y^{\alpha_2}x^{\beta_2}$, $C=y^{\alpha_3}x^{\beta_3}$.

Since $A(BC\circ B)=(AB\circ C)B$ in all possible cases of the elements of L. Thus $L(\circ)$ is a Bol-loop or order $2n^2$ where n is any positive integer greater than 2, with identity e.

Further

$$x^{2^n-1}(x\circ y)=x^{2^n-1}\circ y^{n-1}x^{2^n-1}=yx^{2^n-4},\quad \text{since } n\neq 2.$$

Thus $L(\circ)$ does not satisfy left inverse property. Hence $L(\circ)$ is not Moufang.

In (3) we have obtained Bol-loops of order 36, 100, and $4n^2$ from D_n, this shows the existence of Bol-loops of order $4m$ for some positive integer m. Below we give a general construction for Bol-loops of order $4n(n>2)$.

Table II

Table of $(A\ B\circ C)\ B$	$A(BC\circ B)$ for all values of $B_1,\ \beta_2,\ \beta_3$
(1) When β_1 = even β_2 = even, β_3 = even	$(A\,B\circ C)B = y^{(\alpha_1+2\alpha_2+\alpha_3) \bmod n}\, x^{(\beta_1+\beta_2+\beta_3) \bmod 2n}$ $A(BC\circ B) = y^{(\alpha_1+2\alpha_2+\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3) \bmod 2n}$
(2) when β_1 = even, β_2 = even, β_3 = odd	$(A\,B\circ C)B = y^{(\alpha_1+\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3-2\alpha_2) \bmod 2n}$ $A(BC\circ B) = y^{(\alpha_1+\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3-2\alpha_2) \bmod 2n}$
(3) when β_1 = even, β_2 = odd, β_3 = even	$(A\,B\circ C)B = y^{(\alpha_1-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3+2\alpha_3-2\alpha_2) \bmod 2n}$ $A(BC\circ B) = y^{(\alpha_1-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3+2\alpha_3-2\alpha_2) \bmod 2n}$
(4) when β_1 = even, β_2 = odd, β_3 = even	$(A\,B\circ C)B = y^{(\alpha_1+2\alpha_2-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3+2\alpha_3) \bmod 2n}$ $A(BC\circ B) = y^{(\alpha_1+2\alpha_2-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3+2\alpha_3) \bmod 2n}$
(5) when β_1 = odd, β_2 = even, β_3 = even	$(A\,B\circ C)B = y^{(\alpha_1-2\alpha_2-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3-4\alpha_2-2\alpha_3) \bmod 2n}$ $A(BC\circ B) = y^{(\alpha_1-2\alpha_2-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3-4\alpha_2-2\alpha_3) \bmod 2n}$
(6) when β_1 = odd, β_2 = even, β^3 = odd	$(A\,B\circ C)B = y^{(\alpha_1-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3-2\alpha_2+2\alpha_3) \bmod 2n}$ $A(BC\circ B) = y^{(\alpha_1-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3-2\alpha_2+2\alpha_3) \bmod 2n}$
(7) when β_1 = odd, β_2 = odd, β_3 = even	$A(BC\circ B) = y^{(\alpha_1-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3-2\alpha_2+2\alpha_3) \bmod 2n}$ $A(BC\circ B) = y^{(\alpha_1-\alpha_3) \bmod n}\, x^{(\beta_1+2\beta_2-2\alpha_2+2\alpha_3) \bmod 2n}$
(8) wehen β_1 = odd, β_2 = odd, β_3 = odd	$(A\,B\circ C)B = y^{(\alpha_1+\alpha_3-2\alpha_2) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3+4\alpha_3) \bmod 2n}$ $A(BC\circ B) = y^{(\alpha_1+\alpha_3-2\alpha_2) \bmod n}\, x^{(\beta_1+2\beta_2+\beta_3+4\alpha_3) \bmod 2n}$

5. Theorem 3. *Let L be defined as all formal symbols*

$$z^i y^j x^k \quad i=0,1;\quad j=0,1;\quad k=0,1,2,\dots,n-1,$$

where $z^2=y^2=x^n=e$ *and*

$$z^{\alpha_1} y^{\beta_1} x^{\gamma_1} \circ z^{\alpha_2} y^{\beta_2} x^{\gamma_2} = \begin{cases} z^{(\alpha_1+\alpha_2) \bmod 2}\, y^{(\beta_1+\beta_2) \bmod 2}\, x^{(\gamma_1+\gamma_2) \bmod n} & \text{if } \alpha_2 \equiv 1 \pmod 2 \\ z^{(\alpha_1+\alpha_2) \bmod 2}\, y^{(\beta_1+\beta_2) \bmod 2}\, x^{\nu} & \text{if } \alpha_2 \equiv (1 \bmod 2) \end{cases}$$

where

$$\nu = \begin{cases} (\gamma_1+\gamma_2) \bmod n \text{ if } \beta_2 \equiv 0 \pmod 2 \\ (\gamma_1+(n-1)\gamma_2) \bmod n \text{ if } \beta_1 \equiv y \pmod 2 \end{cases};$$

then $L(\circ)$ *is a Bol-loop of order* $4n$ $(n>2)$ *with identity* e.

Proof: Similar to the proof of Theorem 2.

In tables III and IV we display Bol-loops of order 12 and 16 respectively, which we believe have not been given previously.

TABLE III

(Bol-loop of order 12)

1	2	3	4	5	6	7	8	9	10	11	12
2	3	1	5	6	4	8	9	7	11	12	10
3	1	2	6	4	5	9	7	8	12	10	11
4	5	6	1	2	3	10	12	11	7	9	8
5	6	4	2	3	1	11	10	12	8	7	9
6	4	5	3	1	2	12	11	10	9	8	7
7	8	9	10	11	12	1	2	3	4	5	6
8	9	7	11	12	10	2	3	1	5	6	4
9	7	8	12	10	11	3	1	2	6	4	5
10	11	12	7	8	9	4	6	5	1	3	2
11	12	10	8	9	7	5	4	6	2	1	3
12	10	11	9	7	8	6	5	4	3	2	1

TABLE IV

(Bol-loop of order 16)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	4	1	6	7	8	5	10	11	12	9	14	15	16	13
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	1	2	3	8	5	6	7	12	9	10	11	16	13	14	15
5	6	7	8	1	2	3	4	13	16	15	14	9	12	11	10
6	7	8	5	2	3	4	1	14	13	16	15	10	9	12	11
7	8	5	6	3	4	1	2	15	14	13	16	11	10	9	12
8	5	6	7	4	1	2	3	16	15	14	13	12	11	10	9
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
10	11	12	9	14	15	16	13	2	3	4	1	6	7	8	5
11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
12	9	10	11	16	13	14	15	4	1	2	3	8	5	6	7
13	14	15	16	9	10	11	12	5	8	7	6	1	4	3	2
14	15	16	13	10	11	12	9	6	5	8	7	2	1	4	3
15	16	13	14	11	12	9	10	7	6	5	8	3	2	1	4
16	13	14	15	12	9	10	11	8	7	6	5	4	3	2	1

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