According to [10], they fall under type (I') of Strubecker's classification [4]. From our viewpoint, the surfaces (3.3) *are transformed by collineation of class* (I) for $U=au^3$, $V=bv^3$, and admit the group generated by $X=u\partial/\partial u+v\partial/\partial v$.

Jonas's parabolic tetrahedral surface. It may be shown that β, γ satisfy the condition $\beta_u=\gamma_v$, hence this is a particular Jonas surface (see [3]), and *collinear with the surface of class* (II) *with* $U=uv^3$, $V=-v^3$.

Strubecker's $K_{3,3}$ surface given by:

$$27x_0^2(x_1^2+x_2^2)^2-2x_0x_1x_2+4x_3^3)^2=0, \tag{3.4}$$

is also a particular Jonas surface, having $\beta_u=\gamma_v$, $\beta_u+\beta_v\neq 0$, $\gamma_u+\gamma_v\neq 0$. It admits the group generated by $X=u\partial/\partial u+v\partial/\partial v$.

We denote by $K_{m,n}$ non-ruled surfaces whose real asymptotes in one family are of order $m\geqslant 3$ and those of other — of order $n\geqslant 3$. Grimm studied the $K_{4,4}$ surface given by:

$$x_1=1\;;\; x_2=u+v\;;\; x_3=u^3+v^3\;;\; x_4=\frac{1}{2}(u^4v^4)+uv(v^2-u^2), \tag{3.5}$$

whose order is 9 and which has a *singular asymptote*. This surface belongs to the third class, hence is *minimal projective*. It is homographic with the surface of class (III) given by $U=u^4/4$, $V=-v^4/4$ *and admits the group generated by* $X=u\partial/\partial u+v\partial/\partial v$.

Finally, we *note that Grimm's notion of singular asymptotes indicates that the invariants* β *and* γ *tend to* ∞ *along this curve.*

BIBLIOGRAPHY

1. Terracini A. — *Sulle superficie le cui asintotiche dei due sistemi sono cubiche sghembe.* Selecta, Edizione Cremonese, Roma, Vol. I, pp. 130—155.
2. Wilczynski E. J. — *Über Flächen mit umbestimten Direktrixkurven.* Mathematische Annalen, vol. 76, 1914, pp. 122—166.
3. Fubini G. - E. Cech — *Geometria proiettiva differenziale*, Bologna, N. Zanicheli 1926, T. I.
4. Strubecker K. — *Le superficie le cui asintotiche dei due sistemi appartengono a complessi lineari*, Atti del IV congresso dell'Unione Matematica Italiana, Edizione Cremonese, Roma, 1953.
5. Fubini G. — *On a class of surfaces.* Princeton, J. P. Proc. N.A.S. 1942, pp. 178—185.
6. Marcus F. — *Again on the surfaces which allow* ∞^2 *projective transformation into themselves.* An. şt. Univ. Iaşi T. XXII, 1976, pp. 35—48.
7. Marcus F. — *Sur les surfaces de troisième espèce de Terracini*, Czeckoslowak Math. Journal, 6 (81) Praga 1956, p. 4559—562.
8. Terracini A. — *Su una superfice del sesto ordine e della sesta classe le cui asintotiche sono cubiche sghembe.* Rend. R. Acc. Lincei (5) Vol. XXIX Fas. 11, 1920, pp. 356—361.
9. Lane E. P. Mac Queen, M. L. — *Surfaces whose curves are twisted cubics*, Amer. Journal of Math., Vol. 60, p. 337—344, 1938.
10. Grimm W. — *Über Flachen mit zwei Scharen von Kubischen Asymptotenlinien*, Disertation Fakultät für Mathematik der Universität Karlsruhe (T H) 1972.

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s. I a, 1981, f. 1

CRITICAL SECTIONS OF A RIEMANNIAN FIBRE BUNDLE

BY

LILIANA MAXIM RĂILEANU

Introduction. In his paper „Critical mappings of Riemannian manifolds" [1], D. D. Bleecker studies the maps from one Riemannian compact connected C^∞ manifold M to another C^∞ Riemannian manifold N, which are critical for all invariantly defined functionals on the space of maps between the given Riemannian manifolds. Some possible directions for further research appear in § 6 of Bleecker's work. The ninth open problem is: „A map $f: M\to N$ can be regarded as a section of the product bundle $M\times N\to M$. This suggests that one should try to formulate a notion of a critical section of a (Riemannian) fibre bundle, and generalize the results here." In this note I give the solution of this problem.

§1. Notations and definitions. Let $\mathcal{F}$ be a smooth fibering of the Riemannian C^∞ manifold (P, g_P^0) with the base space the compact, connected C^∞ manifold M and the projection π. Let $\mathcal{M}$ (resp. $\mathcal{P}$) be the space of all C^∞ Riemannian metrics on M (resp. P). Denote the space of diffeomorphisms of M by $\mathcal{D}(M)$. This is a strong ILB- and a strong ILH- Lie group modeled on $\{\Gamma(TM), \gamma^k(TM), k\geqslant \dim M+5\}$, resp. on $\{\Gamma(TM), \Gamma^k(TM), k\geqslant 1\}$, where by $\Gamma(TM)$, $\gamma^k(TM)$ and $\Gamma^k(TM)$ we denote the space of all C^∞ vector fields with C^∞- topology, all C^∞- vector filds with C^k- topology and the completion of $\Gamma(TM)$ with respect to the H^k- norms $\|\cdot\|_k$, respectively (see [4]). The tangent space of the fibres of π defines a smooth involutive subbundle $F=\operatorname{Ker} T\pi$ of TP. Let E be the orthogonal complementary subbundle of F related to g_p^0, i.e. $TP=F\oplus E$ (Whitney sum). Let $\mathcal{D}_{\mathcal{F}}$ be the space of fibre preserving diffeomorphisms which is a strong ILB- and a strong ILH- Lie subgroup of $\mathcal{D}(P)$ modeled on $\{\Gamma(F)\oplus\Gamma_{\mathcal{F}}(E), \gamma^k(F)\oplus\gamma^k_{\mathcal{F}}(E), k\geqslant \dim M+5\}$ and resp. on $\{\Gamma(F)\oplus\Gamma_{\mathcal{F}}(E), \Gamma^k(F)\oplus\Gamma^k_{\mathcal{F}}(E), k\geqslant 1\}$ where $\Gamma_{\mathcal{F}}(E)$ is the set of $Y\in\Gamma(E)$ such that $T\pi(Y)$ is constant along the fibres, that is $(T\pi(Y))(p')=(T\pi(Y))(p'')$ if p', p'' are contained in the same fibre of π. Consider the projection $\tilde{\pi}:\mathcal{D}_{\mathcal{F}}\to\mathcal{D}(M)$ defined by $\tilde{\pi}(\eta)(\pi(p))=\pi(\eta(p))$ for $\eta\in\mathcal{D}_{\mathcal{F}}$, $p\in P$, the local expression of which is given by $\omega(u, v)=T\pi(v)$, $u\in\Gamma(F)$, $v\in\Gamma_{\mathcal{F}}(E)$ (see [4]). Let $\tilde{\mathcal{D}}=\{(\sigma,\eta)\in \in\mathcal{D}(M)\times\mathcal{D}_{\mathcal{F}}, \tilde{\pi}(\eta)=\sigma\}=\text{graph }\tilde{\pi}$. Using [2], 5.8.9 it is easy to see that $\tilde{D}$ is a strong ILB- and a strong ILH- Lie subgroup of $\mathcal{D}(M)\times\mathcal{D}_{\mathcal{F}}$ such that we have the following identification:

$$T(\mathrm{id}_M, \mathrm{id}_P)\,\tilde{\mathcal{D}}=\{(\tilde{Y}, (X, Y)),\ X\in\Gamma(F),\ Y\in\Gamma_{\mathcal{F}}(E)\}.$$

Here $\tilde{Y}$ denotes the vector field induced by Y on the base space M: $\tilde{Y}(m)= T\pi(Y)(p)$, $p\in\pi^{-1}(m)$.

Now, let $\Gamma(\mathcal{F})$ be the Fréchet manifold (resp. Banach manifold) of C^∞ (resp. C^k) cross-sections of $\mathcal{F}$ (see [6] Th. 3.6.) with:

$$T_s\Gamma(\mathcal{F}) \simeq \Gamma(s^*F)). \text{ (resp. } T_s\Gamma^k(\mathcal{F}) \simeq \Gamma^k(s^*F)).$$

Particularly, if $P = M \times N$, $\pi = \mathrm{pr}_1$, then $\Gamma(\mathcal{F}) = C^\infty(M, N)$ and, in a canonical way, F is identified with TN, $\Gamma(s^*F)$ with $\Gamma_s(TN)$ or $\mathcal{V}_s$ in the notations from [1] and $\tilde{\mathcal{D}} = \mathcal{D}(N) \times \mathcal{D}(M)$. There is a left action ρ of $\tilde{\mathcal{D}}$ on $\mathcal{M} \times \Gamma(\mathcal{F}) \times \mathcal{P}$ given by $\rho((\sigma, \eta), (g_M, s, g_P)) = (\sigma^{-1*} g_M, \eta \circ s \circ \sigma^{-1}, \eta^{-1*} g_P)$. In the next section we shall study this action.

§2. **The action of the groop $\tilde{\mathfrak{D}}$.** For $(g_M, s, g_P) \in \mathcal{M} \times \Gamma(\mathcal{F}) \times \mathcal{P}$, let $I_{(g_M, s, g_P)}$ be the isotropy group of (g_M, s, g_P): thus $I_{(g_M, s, g_P)} = \{(\sigma, \eta) \in \bar{D}, \rho((\sigma, \eta), (g_M, s, g_P)) = (g_M, s, g_P)\} = \{(\sigma, \eta) \in I_{g_M} \times I_{g_P} \cap \tilde{\mathfrak{D}}$ such that $\eta \circ s \circ \sigma^{-1} = s\}$.

This is a closed subgroup of the Lie group $I_{g_M} \times I_{g_P}$, hence a Lie subgroup. Let $\Psi_{(g_M, s, g_P)} : \tilde{\mathcal{D}} \to \mathcal{M} \times \Gamma(\mathcal{F}) \times \mathcal{P}$ be the orbit map and $O_{(g_M, s, g_P)} = \Psi_{(g_M, s, g_P)}(\tilde{\mathcal{D}})$ be the orbit of (g_M, s, g_P) under the action ρ. Because of whell know technical difficulties related to the implicit functions theorem for Fréchet manifolds, we enlarge the previous sets and we work in the corresponding Sobolev spaces [6] $\mathcal{M}^k$, $H^k(\mathcal{F})$, $\mathcal{P}^k$, $\mathcal{D}^k(M)$, $\mathcal{D}^k_{\mathcal{F}}$, $\bar{\mathcal{D}}^k$. The set $\bar{\mathcal{D}}^k$ is also a topological group and, for $k \geqslant \dim P + 5$, the action $\rho : \bar{\mathcal{D}} \times \mathcal{M} \times \Gamma(\mathcal{F}) \times \mathcal{P} \to \mathcal{M} \times \Gamma(\mathcal{F}) \times \mathcal{P}$ can be extended to an action $\rho^k : \bar{\mathcal{D}}^{k+1} \times (\mathcal{M}^k \times H^k(\mathcal{F}) \times \mathcal{P}^k) \to \mathcal{M}^k \times \Gamma^k(\mathcal{F}) \times \mathcal{P}^k$. Moreover, if $(g_M, s, g_P) \in \mathcal{M}^{k+r} \times H^k(\mathcal{F}) \times \mathcal{P}^{k+r}$, $\Psi(g_M, s, g_P)$ can be extended to a C^r, $r \geqslant 0$, map from $\bar{\mathcal{D}}^{k+r+1}$ which has their values in $\mathcal{M}^k \times H^k(\mathcal{F}) \times \mathcal{P}$ (see [5]). For $r \geqslant 1$, $\Psi(g_M, s, g_P)$ has tangent in identity given by: $\Psi' = T_{(\mathrm{id}_M, \mathrm{id}_P)} \Psi(g_M, s, g_P) : T_{(\mathrm{id}_M, \mathrm{id}_P)} \bar{\mathcal{D}}^{k+r+1} \to S^{2,k}(M) \times H^k(s^*F) \times S_0^{2,k}(P)$.

$$\Psi'(\tilde{Y}, (X, Y)) = (-L_{\tilde{Y}} g_M, X \circ s - \pi_V(s_* \tilde{Y}) - L_{(X,Y)} g_P),$$

where $s_* \in \mathrm{Hom}(TM, s^*(TP))$, $S_0^2(P)$ is the space of compactly supported sections in $S^2(P)$, L is the Lie differentiation and π_V is the vertical projection. Now, on $H^k(s^*F)$ we consider the restriction of the inner product $<,>$ from $\mathcal{V}_s = H^k_s(M, TP)$ (see [1]). So, we say that $(u, \Phi, v) \in S^{2,k}(M) \times H^k(s^*F) \times S_0^{2,k}(P)$ is normal to the orbit of (g_M, s, g_P) at (g_M, s, g_P) if, for all vector fields $X \in \Gamma(F)$, $Y \in \Gamma_{\mathcal{F}}(E)$, $(u, \Phi, v)(-L_{\tilde{Y}} g_M, X \circ s - \pi_V(s_* \tilde{Y}), -L_{(X,Y)} g_P) = <u, -L_{\tilde{Y}} g_M> + <\Phi, X \circ s - \pi_V(s^* \tilde{Y})> + <v, -L_{(X,Y)} g_P> = 0$.

To avoid the difficulty in the explicitation of this condition of normality (see also and [1]) we consider in $\mathcal{P}$ a subset $\mathcal{P}_{g_P^0}$ of the Riemannian structures on P which induces the fixed Riemannian structure g_P^0 on the vertical vector bundle F. That is:

$$\mathcal{P}_{g_P^0} = \{g_P \in \mathcal{P} \text{ such that } g_P(\bar{X}, \bar{Y}) = g_P^0(\pi_V \bar{X}, \pi_V \bar{Y}) + g_M(\pi_* \pi_H \bar{X}, \pi_{**} \pi_H \bar{Y}), \ \forall \bar{X}, \bar{Y} \in \Gamma(TP), \ \forall g_M \in \mathcal{M}\},$$

where π_V and π_H are the projections of TP onto the subbundles F and $E = F\perp$ (related to g_P^0).

If $g_P \in P_{g_P^0}$ we denote by $\theta(g_P)$ the Riemannian metric $g_M \in \mathcal{M}$ which defines this cannonically constructed metric g_P on P. We consider now the following subset in $\mathcal{M} \times \Gamma(\mathcal{F}) \times P_{g_P^0}$:

$$\Theta_{g_P^0} = \{(g_M, s, g_P) \in \mathcal{M} \times \Gamma(\mathcal{F}) \times \mathcal{P}_{g_P^0}, \ \theta(g_P) = g_M\},$$

which is a closed submanifold in $\mathcal{M} \times \Gamma(\mathcal{F}) \times \mathcal{P}_{g_P^0}$ whose tangent space at a generic element (g_M, s, g_P) may be identified with:

$$\{(u, \Phi, u^*), u \in S^2(M), \Phi \in \Gamma(s^*F), u^*(\bar{X}, \bar{Y}) = u(\pi_* \pi_H \bar{X}, \pi_* \pi_H \bar{Y}), \ \forall \bar{X}, \bar{Y} \in \Gamma(TP)\}.$$

The group $\tilde{\mathcal{D}}_{g_P^0} = \{(\sigma, \eta) \in \tilde{\mathcal{D}}, \eta^* g_F^0 = g_F^0$, g_F^0 being the restriction of g_P^0 to F, $\eta_* \pi_H = \pi_H \circ \eta_*\}$ is a string ILB-and ILH Lie subgroup of $\bar{\mathcal{D}}$ such that we have the folowing identification: $T_{\mathrm{id}_M, \mathrm{id}_P} \bar{\mathcal{D}}_{g_P^0} = \{(\tilde{Y}, (X, Y)), X \in \Gamma(F), Y \in \Gamma_{\mathcal{F}}(E)$, X being a Killing field g_F^0 i.e. $(L_X g_P^0)(X', Y') = 0$, $\forall X', Y' \in \Gamma(F)\}$.

This group acts on the left on $\Theta_{g_P^0}$ by the restriction of ρ to $\bar{\mathcal{D}}_{g_P^0} \times \Theta_{g_P^0}$:

$$\rho_{g_P^0} : \bar{\mathcal{D}}_{g_P^0} \times \Theta_{g_P^0} \to \Theta_{g_P^0}, \ \rho_{g_P^0}((\sigma, \eta), (g_M, s, g_P)) = (\sigma^{-1*} g_M, \eta \circ s \circ \sigma^{-1}, \eta^{-1*} g_P),$$

$$\eta^{-1*} g_P(\bar{X}, \bar{Y}) = g_P^0(\pi_V \bar{X}, \pi_V \bar{Y}) + \sigma^{-1*} g_M(\pi_* \pi_H \bar{X}, \pi^* \pi_H \bar{Y}).$$

So, we have the following:

Theorem 1. *If* $(g_M, s, g_P) \in \Theta^{k+r}_{g_P^0} = \{(g_M, s, g_P) \in \mathcal{M}^{k+r} \times H^k(\mathcal{F}) \times P^{k+r}_{g_P^0}, \theta(g_P) = g_M\}$ *then its orbit* $O^k_{(g_M, s, g_P)}$ *under the action of* $\bar{\mathcal{D}}^{k+r+1}_{g_P^0}$ *is a* C^r *closed submanifold in* $\Theta^k_{g_P^0}$ *with tangent space at* (g_M, s, g_P) *given by:* $T_{(g_M, s, g_P)} O^k_{(g_M, s, g_P)} = \{(-L_{\tilde{Y}} g_M, X \circ s - \pi_V(s_* \tilde{Y}), (-L_{\tilde{Y}} g_M)^*), \forall X \in H^{k+r}(F)$, *X Killing for* g_F^0, $\forall Y \in H^{k+r}(E)$

$$(L_{\tilde{Y}} g_M)^*(\bar{X}, \bar{Y}) = L_{\tilde{Y}} g_M(\pi_* \pi_H \bar{X}, \pi_* \pi_H \bar{Y}), \quad \forall \bar{X}, \bar{Y} \in H^{k+r}(TP)\}.$$

Proof is symilar with the proof of Theorem 1 from [5].

Now, $(u, \Phi, u^*) \in S^{2,k}(M) \times H^k(s^*F) \times S_0^{2,k}(P)$ is normal to the orbit $O^k_{(g_M, s, g_P)}$ iff:

$$<u, -L_{\tilde{Y}} g_M> + <\Phi, X \circ s - \pi_V(s_* \tilde{Y})> + <u^*, (-L_{\tilde{Y}} g_M)^*> =$$
$$= 2 <u, -L_{\tilde{Y}} g_M> + <\Psi, X \circ s - \pi_V(s^* \tilde{Y})> = 0$$

because $\pi : (P, g_P) \to (M, g_M)$ is a Riemannian mapping (see [1], [3]). This condition is equivalent with:

(1) $<\Phi, X \circ s> = 0$ for all Killing fields X for g_F^0;

(2) $\qquad 4 <\mathrm{Div}(u)\# - (\pi_V \circ s_*)t, \tilde{Y}> = 0,$

where $(\pi_V \circ s_*)^t : s^*F \to TM$ is the adjoint of $\pi_V \circ s_*$. Since this is a vector bundle map and Φ is a section of the bundle s^*F, we have that $(\pi_V \circ s_*)^t(\Phi)$ is a vector field on M and $<\Phi, \pi_V(s_* \tilde{Y})> = <(\pi_V \circ s_*)^t(\Phi), \tilde{Y}>$.

The condition (2) is equivalent with:

(2)' $\qquad 4 \,\mathrm{Div}(u)\# - (\pi_V \circ s_*)^t = 0.$

§3. **Definitions and principal results.** In analogy with Definition 2.2 from [1] we have:

Definition 1. *Let* (P, g_P^0) *be a fixed Riemannian manifold and* $\mathcal{F}$ *be a smooth fibering of this manifold with the base space the* C^∞ *connected compact manifold* M *and the projection* π. *For* $g_M \in \mathcal{M}$ *let* $g_P \in \mathcal{P}_{g_{g_P^0}}$ *with* $\theta(g_P) = g_M$. *We say that the section* $s : (M, g_M) \to (P, g_P)$ *of the Riemannian fibration* $\pi : P \to M$ *is a critical section if for all* $(u, \Phi, u^*) \in S^2(M) \times \Gamma(s^*F) \times S_0^2(P)$, *which* *are normal to the orbit of* (g_M, s, g_P) *(under the action of* $\mathcal{D}_{g_P^0}$ *on* $\Theta_{g_P^0}$*) at* (g_M, s, g_P) *and are left fixed by the isotropy subgroup* $I_s = I_{(g_M, s, g_P)}$ *we have* $\Phi = 0$.

Using the previous observations, we have the following equivalent

Definition 2. *We say that s is a critical section of the Riemannian fibration* $\pi : (P, g_P) \to (M, g_M)$ *if, for all* $(u, \Phi, u^*) \in S^2(M) \times \Gamma(s^*F) \times S^2(P)$ *satisfiyng the conditions* (α), (β), (γ) *below, we have* $\Phi = 0$:

(α) $4\,\mathrm{Div}(u)^{\#} - (\pi_V o s^*)^t = 0$;

(β) $<\Phi, X o s> = 0$ for all Killing fields X for g_P^0;

(γ) $\eta_* o \Phi o \sigma^{-1} = \Phi$, $\sigma_*^{-1}(u) = u$, $\forall (\sigma, \eta) \in I_s$.

Now, we consider a functional $F : \Theta g_P^0 \to R$ which is constant on the orbit of the action of $\mathcal{D}_{g_P^0}$ and is smooth in the sense that for each $(g_M, s, g_P) \in \Theta_{g_P^0}$, we have a triple $(D_1F, D_2F, D_3F) \in S^2(M) \times \Gamma(s^*F) \times S_0^2(P)$, $D_3F_2 = (D_1F)^*$, such that for all $(u, \Phi, u^*) \in S^2(M) \times \Gamma(s^*F) \times S_0^2(P)$, we have:

$$\frac{d}{dt}(F(g_M + tu, \mathrm{Exp}_s(t\Phi), g_P + tu))\Big|_{t=0} = <D_1F, u> + <D_2F, \Phi> +$$
$$+ <D_3F, u^*> = 2 <D_1F, u> + <D_2F, \Phi>,$$

where Exp is the exponential map associated to a bundle spray over P (see [6] T. 12.10).

Theorem 2. *Under the above assumptions,* $(D_1F, D_2F, (D_1F)^*$ *meets conditions* (α), (β), (γ) *of Definition 2.*

Proof. Let X be a Killing vector field for g_P^0 and $Y \in \Gamma_{\mathcal{J}}(E)$. Let σ_t be the one-parameter group associated to $\tilde{Y}$ on M and η_t be the one-parameter group associated to (X, Y) on P. Then, at $t = 0$, we have:

$$0 = \frac{d}{dt} F(\sigma_t^{-1} g_M, \eta_t o s o \sigma_t^{-1}, \eta_t^{-1} g_P) = \frac{d}{dt}(g_M^{-1} - tL_{\tilde{Y}} g_M, \mathrm{Exp}_s(t(X o s -$$
$$-(\pi_V o s)(\tilde{Y}))), g_P - t(L_{\tilde{Y}} g_M)^*) = 2 <D_1F, -L_{\tilde{Y}} g_M> + <D_2F, X o s -$$
$$-(\pi_V o s_*)(\tilde{Y})> = <4\,\mathrm{Div}(D_1F)^{\#} - (\pi_V o s_*)^t (D_2F), \tilde{Y}> + <D_2F, X o s>.$$

Thus we proved the conditions (α), (β).

To prove (γ), suppose $(\sigma, \eta) \in I_s$ and $\psi^0_{(g_M, s, g_P)} : D_{g_P^0} \to \Theta_{g_P^0}$ the orbit map of the action $\rho_{g_P^0}$. Because $F = F o \psi^0_{(g_M, s, g_P)}(\sigma, \eta)$ we have, for all $(u, \Phi, u^*) \in S^2(M) \times \Gamma(s^*F) \times S_0^2(P)$:

$$2 <D_1F, u> + <D_2F, \Phi> = <DF o \psi^0_{(g_M, s, g_P)}(\sigma, \eta), (u, \Phi\, u^*)> = \frac{d}{dt}(F o \psi^0_{(g_M, s, g_P)}$$
$$(\sigma, \eta)(g_M + tu, \mathrm{Exp}_s t\Phi, g_P + tu^*))|_{t=0} = \frac{d}{dt}(F(g_M + t\sigma^{-1*}u, \mathrm{Exp}_{\eta o s o \sigma} - 1(t\eta^* o \Phi o \sigma^{-1}),$$
$$g_P + t\eta^{-1*}(u)^*)|_{t=0} = \frac{d}{dt} F(g_M + t\sigma^{-1}u, \ \mathrm{Exp}_s(t\eta_* \, o \Phi \, o \, \sigma^{-1}), \ g_P + tu^*))|_{t=0} =$$
$$= <D_1F, \sigma^{-1*}u> + <D_2F, \eta_* o \Phi o \sigma^{-1}> + <(D_1F)^*, \eta^{-1*}(u^*)> = <\sigma^* D_1F, u>$$
$$<\eta_*^{-1} o D_2F o \sigma, \Phi> + <(D_1F)^*, (\sigma^{-1*}u)^*> = 2 <\sigma^* D_1F, u> + <\eta_*^{-1} o D_2F o \sigma, \Phi>.$$

Thus $\sigma^* D_1F = D_1F$, $\eta_*^{-1} o\ D_2F\ o\ \sigma = D_2F$ and (γ) is proved.

Theorem 3. *If s is a critical section of the Riemannian fibration* $\pi : (P, g_P) \to (M, g_M)$ *then s is harmonic if* π *has totally geodesic fibres.*

Proof. Consider the energy functional $F : \Theta_{g_P^0} \to R$ defined by:

$$F(g_M, s, g_P) = 1/2 <g_M, s^* g_P>,$$

which is constant on the orbit of the action $\rho_{g_P^0}$. From [3] it is known that $D_2F = -\Delta s$, where Δs is the trace of the covariant derivative of s_* in the bundle Hom (TM, s^*TP), $\nabla s_* : M \to L(TM, \mathrm{Hom})(TM, s^*TP)) = L^2(TM, s^*TP)$. But, from [3], Corollary of Prop. (C), p. 133, $\tau(s) = -\Delta s$ is always vertical, hence $D_2F \in \Gamma(s^*F)$. Moreover $2\ D_1F = 1/2\ \mathrm{trace}(s^* g_P) g_M - s^* g_P$. The previous theorem applies and we have $-\Delta s = D_2F = 0$. As in [1], we say:

Definition 3. *The section s is a hypercritical section of the Riemannian fibration* π *only if* (γ) *from Definition 2 implies* $\Phi = 0$.

We denote by $I_s(M) \subset I_{g_M}$ and $I_s(P) \subset I_{g_P}$ the projections of I_s onto the first and the second factors of $I_{g_M} \times I_{g_P} \cap \tilde{\mathcal{D}}$.

We observe that, if $\eta_* o \Phi o \sigma^{-1}$ for all $(\sigma, \eta) \in Is$, the vector field $(\pi_V o s^*)^t(\Phi)$ is invariant under $I_s(M)$. Indeed, or every $V_m \in T_m M$, we have:

$$g_M((\sigma_*(\pi_V o s_*)^t(\Phi))_m, V_m) = g_M(\sigma_*(\pi_V o s^*)^t(\Phi)(\sigma^{-1}(m)), V_m) =$$
$$= g_M((\pi_V o s_*)^t(\Phi)(\sigma^{-1}(m), \sigma_*^{-1} V_m) = g_P(\Phi(\sigma^{-1}(m)), \pi_V o s^* o \sigma^{-1}(V_m)) =$$
$$= g_P(\eta_* \Phi(\sigma^{-1}(m), \eta_*(\pi_V o s_*) o \sigma_*^{-1} V_m) = g_P(\Phi(m), \pi_V o \eta_* o s o \sigma_*^{-1}(V_m)) =$$
$$= g_P(\Phi(m), \pi_V o s_*(V_m)) = g_M((\pi_V o s_*)^t(\Phi)(m), V_m).$$

So
$$\sigma_*(\pi_V o s_*)^t(\Phi) = (\pi_V o s_*)^t(\Phi), \ \forall \sigma \in I_s(M).$$

This fact permits to give the following result which generalizes the Corollary 3.2 from [1].

Proposition 4. *The section s of the Riemannian fibration* $\pi : (P, g_P) \to (M, g_M)$ *is critical iff for all* $\Phi \in \Gamma(s^*F)$ *satisfying* (α') (β'), (γ') *below, we have* $\Phi = 0$.

(α') $<(\pi_V o s_*)^t(\Phi), \tilde{Y}> = 0$ for all Killing fields $\tilde{Y}$ on (M, g_M);

(β') $<\Phi, X o s> = 0$ for all Killing fields X for g_P^0;

(γ') $\eta_* o \Phi o \sigma^{-1} = \Phi\ \forall (\sigma, \eta) \in I_s$.

This characterization of the critical sections of a Riemannian fibration is very useful to prove the following result which generalizes Theorem 3.4 from [1].

Theorem 5. *If s is a critical section of the Riemannian fibration* $\pi : (P, g_P) \to (M, g_M)$, *then* $I_s(P)$ *acts transitively on the set* $s(M) \subset P$. *Thus* $s(M)$ *is an embedded submanifold of P.*

Proof is similar with that of T. 3.4 from [1], if we put $\Phi \in \Gamma(s^*F)$, $\Phi = \pi_V o \nabla h o s$. This Φ satisfies (γ') because $\eta_* : TP \to TP$ preserves the subbundle $F = \mathrm{Ker}\ T\pi$. Also, since the normal bundle of $s(M)$ is contained in F, Th. 3.7 from [1] is true, without modifications. Hence:

Theorem 6. *If s is a critical section of the Riemannian fibration* $\pi : (P, g_P) \to (M, g_M)$, *then* $s(M)$ *is a minimal submanifold of* (P, g_P).

Definition 4. *If* $\Phi \in \Gamma(s^*F)$ *have the form* $\Phi = X o s + (\pi_V o s_*)(\tilde{Y})$, *where X is a Killing field for* g_P^0 *and* $\tilde{Y}$ is *Killing on* (M, g_M), Φ *is called infinitesimal isodeformation of s* (see [1], *Definition* 3.8).

Let $\mathcal{K}$ be the vector space of infinitesimal isodeformations of s. It is an invariant subspace of the vector space of I_s- invariant fields in $\Gamma(s^*F)$. Indeed, for $(\sigma, \eta) \in I_s$, we have:

$$\eta_* o(X o s + (\pi_V o s_*))(\tilde{Y})) o \sigma^{-1} = (\eta_* o X o \eta^{-1}) o s + (\pi_V o s_*)(\sigma_* o \tilde{Y} o \sigma^{-1}).$$

So, the following characterization of critical maps (see Th. 3.10) from [1] holds:

Theorem 7. *The section s is critical iff every I_s-invariant field from $\Gamma(s^*F)$ is an infinitesimal isodeformation.*

Examples. (1) Every critical mapping $f : (M, g_M) \to (N, g_N)$ in the sense of [1] is also a critical section regarded as a section of the product bundle $(M \times N, g_M \times g_N) \to (N, g_N)$. So, all examples from § 4 [1] can be regarded as examples of critical sections.

(2) Let $\pi : E \to M$ be a vector bundle, g^0_E a fixed Riemannian structure on E, g_M a Riemannian structure on the base space M and s^0 the zero section of π. Then $s^0 : (M, g_M) \to (E, g_E)$, $(g_M, s^0, g_E) \in \Theta_{g^0_E}$, is a critical section of π for all $g_M \in \mathcal{M}$. Indeed, $I_{s^0} = \{(\sigma, \eta) \in I_{g_M} \times I_{g_E}$ where η is a diffeomorphism of vector bundle π over $\sigma\}$; let $\Phi \in \Gamma(s^*F)$, Φ be I_{s^0}-invariant (i.e. $\eta_* \circ \Phi \circ \sigma^{-1} = \psi$ for all $(\sigma, \eta) \in I_{s^0}$) and let $(id_M, \eta) \in I_{s^0}$, where η is defined by $\eta(v) = -v$, for all $v \in \pi^{-1}(m)$, $\forall\ m \in M$. Then: $-\Phi(m) = (\eta_* \circ \Phi \circ id_M)(m) = \Phi(m)$. Thus $\Phi(m) = 0$ for any $m \in M$ and s^0 is hypercritical, hence a critical section of π.

(3) More generally, if $\pi : P \to M$ is a fibre bundle such that, for all $m \in M$, $\pi^{-1}(m)$ is a Lie group, then the section $s^0 : (M, g_M) \to (P, g_P)$, $s^0(m) = e_M$, where e_m is the unity element of $\pi^{-1}(m)$, is a hypercritical hence a critical section of π.

REFERENCES

1. Bleecker David D. — *Critical mappings of Riemannian manifolds*, Trans. of Amer. Math. Soc., V. 254, 1979, 319—3.
2. Bourbaki N. — *Variétés différentielles et analytiques, Fascicule de résultats, Paragraphes* 1 à 7, Hermann, Paris 1967.
3. Eells J. and Sampson J. — *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., 86, 1964, 109—160.
4. Omori H. — *Infinite Dimensional Lie Transformation Groups*, Lecture Notes in Math, vol. 427, Springer-Verlag, Berlin, 1974.
5. Maxim-Răileanu L. — *Critical mappings of Riemannian manifolds and the slice theorem for an action of the diffeomorphism group*, (to appear).
6. Palais R. — *Foundations of global Non-linear analysis*, W. A. Benjamin, Inc. New York, Amsterdam, 1968.

Faculty of Mathematics
Universsity of Iaşi
Iaşi, R. S. România

Received 21. V. 1980

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVII, s. I a, 1981, f. 1

SEMI-INVARIANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

BY

AUREL BEJANCU and NECULAI PAPAGHIUC

0. Introduction. The differential geometry of several classes of submanifolds in a Sasakian manifold has been investigated by many people. Interesting results have been obtained by G. Ludden, M. Okumura, K. Yano [3], K. Yano and M. Kon [6] in studying anti-invariant submanifolds of Sasakian manifolds and by M. Kon [4] in studying invariant submanifolds. Moreover, K. Yano and M. Kon have studied generic submanifolds of odd dimensional spheres which are generalizations of anti-invariant submanifolds [7].

Recently in [1] one of the present authors introduced the notion of CR-submanifold in a Kaehler manifold. CR-submanifolds appear as a natural generalization of both complex submanifolds and totally real submanifolds.

The purpose of the present paper is to introduce and study a similar class of submanifolds in a Sasakian manifold. In §1 we introduce the notion of semi-invariant submanifold of a Sasakian manifold and give fundamental formulas for later use. The integrability of all distributions involved in the definition of a semi-invariant submanifold is studied in §2. Finally, we give a complete characterization for totally umbilical semi-invariant submanifolds of a Sasakian manifold.

1. Semi-invariant submanifolds of a Sasakian manifold. First, we recall the definition and some properties of a Sasakian manifold.

Let $\tilde{M}$ be a $(2n+1)$-dimensional differentiable manifold of class C^∞ and F, η, ξ be a tensor field of type (1.1), a 1-form and a vector filed on $\tilde{M}$ respectively, which satisfy

$$F^2 = -I + \eta \otimes \xi, \quad F\xi = 0, \quad \eta(FX) = 0, \quad \eta(\xi) = 1 \tag{1.1}$$

for any vector field X on $\tilde{M}$, where I is the identity tensor on $\tilde{M}$.

Then $\tilde{M}$ is called an almost contact manifold and (F, η, ξ) an almost contact structure on $\tilde{M}$.

Now, suppose on $\tilde{M}$ is given a Riemannian metric tensor field g which satisfies the equations

$$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \tag{1.2}$$

$$\eta(X) = g(X, \xi), \tag{1.3}$$

for any vector fields X, Y tangent to $\tilde{M}$. Then $\tilde{M}$ is called an almost contact metric manifold. When we have

$$d\eta(X, Y) = g(X, FY) \tag{1.4}$$