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SECOND ORDER ELASTIC EFFECTS IN A TRIANGULAR HOLE WITH ROUNDED CORNERS UNDER UNIFORM NORMAL PRESSURE

BY

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1. **Introduction and preliminaries.** The well known complex variable technique in infinitesimal elasticity was extended and made available for use in the second order elasticity by G r e e n, A d k i n s and their associates [1, 2]. This technique was further developed and modified by B h a r g a v a, P a n d e and L a l [3, 5] so that a larger and more complex set of problems became tractable. This technique mainly consists of expressing the elastic field in terms of function φ, called Airy's stress function, and the displacement vector D which is also a complex function. The function are expressible in terms of two sets of complex functions $\Omega(z)$, $\omega(z)$ and $\Delta(z)$, $\delta(z)$. The first set covers the infinitesimal case while the first and second jointly determine the second order case. Thus linear elasticity is a particular case of second order elasticity. A brief resumé of the mathematical preliminaries, essential for the problem is given below. Let

$$\zeta = x_1 + i x_2 \text{ and } z = y_1 + i y_2, \tag{1}$$

be the complex coordinates of a point of an elastic body in its undeformed state B_0 and the deformed state B respectively. The Airy's stress function φ and the displacement vector

$$D = u_1 + i u_2, \tag{2}$$

being analytic and expandable as below

$$\begin{aligned} \varphi &= \mu\varepsilon\{\varphi_0(z, \bar{z}) + \varepsilon\varphi_1(z, \bar{z}) + \ldots\}, \\ D &= \varepsilon\{D_0(z, \bar{z}) + \varepsilon D_1(z, \bar{z}) + \ldots\}, \end{aligned} \tag{3}$$

ε being a parameter depending upon the physical condition of the problem. Equilibrium equations are satisfied if

$$T^{11} = \overline{T}^{22} = -4\frac{\partial^2\varphi}{\partial\bar{z}^2}; \quad T^{12} = 4\frac{\partial^2\varphi}{\partial z\partial\bar{z}}, \tag{4}$$

where

$$\begin{aligned} T^{11} = \overline{T}^{22} &= p_{y_1y_1} - p_{y_2y_2} + 2i p_{y_1y_2}, \\ T^{12} &= p_{y_1y_1} + p_{y_2y_2}. \end{aligned} \tag{5}$$

If the resultant stress along the boundary is zero

$$\frac{\partial\varphi}{\partial z}=\frac{\partial\varphi}{\partial\bar z}=0. \tag{6}$$

The entire elastic field is determined by a set of potential functions $\{\Omega(z), \omega(z)\}$ in the infinitesimal case and $\{\Omega(z), \omega(z)\}$ and $\{\Delta(z), \delta(z)\}$ in the second order case.
We have

$$\begin{aligned}\varphi_0(z, \bar z)&=\bar z\Omega(z)+z\bar\Omega(\bar z)+\omega(z)+\bar\omega(\bar z)\\ D_0(z, \bar z)&=\varkappa\Omega(z)-z\bar\Omega'(\bar z)-\bar\omega'(\bar z),\end{aligned} \tag{7}$$

and

$$\frac{\partial\varphi_1}{\partial\bar z}=\Delta(z)+z\bar\Delta'(\bar z)+\bar\delta'(\bar z)+\nu\Gamma(z, \bar z)+B_3\Omega(z)\bar\Omega'(\bar z)-B_1 z\{\bar\Omega'(\bar z)\}^2, \tag{8}$$

$$\begin{aligned}D_1(z, \bar z)=\varkappa\bar\Delta(z)&-z\bar\Delta'(\bar z)-\bar\delta(\bar z)-\nu\Lambda(z, \bar z)-B_3\Omega(z)\bar\Omega'(\bar z)+\\ &+B_1''z\{\Omega'(\bar z)\}^2+B_1'\int^{\bar z}\bar\Omega'(\bar z)\bar\omega''(\bar z)d\bar z-B_4\int^{z}\{\Omega'(z)\}^2dz,\end{aligned} \tag{9}$$

where $\varkappa=(5\lambda+6\mu)/(3\lambda+2\mu)$ for the case of plane stress, λ and μ are Lame's constants, ν, B_i, B_1, B_1'' are elastic constants given in the [5].

For the uniqueness and single valuedness of φ_0 and D_0 the following condition should be satisfied:

$$\begin{gathered}[\Omega'(z)]_c=0\ ;\ [\omega''(\bar z)]_c=0\ ;\ [\varkappa\Omega(z)-\bar\omega'(\bar z)]_c=0,\\ [\Omega'(z)]=0\ ;\ [\delta''(z)+B_3\bar\Omega(z)\Omega''(z)]_c=0,\end{gathered} \tag{10}$$

$$[\varkappa\Delta(z)-\bar\delta'(\bar z)]_c=[B_3\Omega(z)\bar\Omega'(\bar z)+B_4\int^{z}\{\Omega'(z)\}^2dz-B_1'\int^{z}\bar\Omega'(\bar z)''\bar\omega(\bar z)d\bar z]_c. \tag{11}$$

where $[\quad]_c$ denotes the change in value of the function inside the brackets during a complete circuit of a contour c lying entirely within the deformed body.

3. The problem. Let a homogeneous, isotropic infinite medium of compressible material having a hole in it. Let a uniform normal pressure P acts at each point of the hole, so that in the final state the boundary of the hole becomes an equilateral triangle. It is known [4] that the mapping function

$$z=\omega(z)=A(1/\zeta+\zeta^2/3), \tag{12}$$

maps the region exterior to an equilateral triangle on the region $|\zeta|\leqslant 1$ and the triangular boundary is changed in the boundary of a unit circle $\gamma(\sigma=1)$. Here A is a real constant which determines the dimensions of the equilateral triangle. It is relevant to point out that as the point Z traces the triangle in the Z plane in the anticlockwise sense, the corresponding point traces the circle in the clockwise sense in the ζ-plane. We distinguish a point on the boundary of the circle γ in the ζ-plane by σ and note that $\sigma=e^{-i\theta}$ where θ is measured from the real ζ-axis in the anticlockwise sense.

The potential functions $\Omega(z)$, $\omega'(z)$ and $\Delta(z)$, $\delta(z)$ will be transformed as follows

$$\begin{aligned}\Omega(z)&=\Omega\left\{A\left(\frac{1}{\zeta}+\frac{\zeta^2}{3}\right)\right\}=\Omega_1(\zeta),\\ \omega'(z)&=\omega'\left\{A\left(\frac{1}{\zeta}+\frac{\zeta^2}{3}\right)\right\}=\omega_1'(\zeta),\\ \Delta(z)&=\Delta\left\{A\left(\frac{1}{\zeta}+\frac{\zeta^2}{3}\right)\right\}=\Delta_1(\zeta),\\ \delta'(z)&=\delta'\left\{A\left(\frac{1}{\zeta}+\frac{\zeta^2}{3}\right)\right\}=\delta_1'(\zeta).\end{aligned} \tag{13}$$

The potential functions $\{\Omega(z), \omega'(z)\}$ which determine the infinitesimal case may be evaluated from the integro-differential equation

$$F^0(\sigma)=\Omega(\sigma)+\frac{\omega(\sigma)}{\bar\omega'(\sigma)}\bar\Omega'(\bar\sigma)+\bar\omega'(\bar\sigma), \tag{14}$$

after computing $F^0(\sigma)$ appropriately for the specific problem.

In our case

$$F^0(\sigma)=-AP\left(\frac{1}{\sigma}+\frac{\sigma^2}{3}\right). \tag{15}$$

Thus substituting for $F^0(\sigma)$ in (14) and integrating the equation round the unit circle γ after multiplying both sides by $1/\sigma(\sigma-\zeta)$ we have

$$\Omega_1(\zeta)=-\frac{AP\zeta^2}{6}. \tag{16}$$

Similarly taking the conjugate complex of (16) and integrating round γ after multiplying both sides by $1/(\sigma-\zeta)$, we get

$$\omega_1'(\zeta)=\frac{11AP\zeta}{6(2\zeta^3-3)}. \tag{17}$$

Thus the first set of potential functions which determine the linear elasticity case of our problem is determined. Since the resultant force along the boundary is zero, equation (8) gives

$$\frac{\partial\varphi_1}{\partial\bar z}=\Delta(z)+z\bar\Delta'(\bar z)+\bar\delta'(\bar z)+\nu\Gamma(z, \bar z)+B_3\Omega(z)\bar\Omega'(\bar z)-B_1 z\{\bar\Omega'(\bar z)\}^2=0. \tag{18}$$

Substituting for $\Gamma(z, \bar z)$, $\Omega(z)$, $\bar\Omega'(\bar z)$ in equation (18) and multiplying by $\zeta/\sigma(\sigma-\zeta)$, integrating round γ

$$\Delta_1(\zeta)=-\frac{P^2A\nu\zeta^2(\varkappa\zeta^3-3)}{6(3-2\zeta^3)}, \tag{19}$$

taking conjugate complex of (18), multiplying by $1/(\sigma-\zeta)$ and integrating round γ, we have

$$(20)\quad \delta_1'(\zeta)=\frac{P^2A\nu\zeta}{12(3-2\zeta^3)^2}\Big[12\varkappa\zeta^6+(22+3\varkappa)\zeta^3+(15-6\varkappa)+ \\ +\frac{2(3\zeta^3+1)}{(3-2\zeta^3)}\{4\varkappa\zeta^6+(6-15\varkappa)\zeta^3+18\}\Big]+\frac{B_3P^2A\zeta}{6(3-2\zeta^3)}.$$

Since $\{\Omega_1(\zeta),\ \omega_1'(\zeta)$ and $\{\Delta_1(\zeta),\ \delta_1'(\zeta)\}$ are known the hoop-stress and displacements may be written with the help of equations (4, 5, 7).

Taking $\varepsilon=1/\mu$ the displacements on the internal boundary are given by

$$(21)\quad D(z,\bar{z})=\frac{AP}{6\mu}\left[\frac{3}{\sigma}-\varkappa\sigma^2\right]+\frac{AP^2B_1'}{12\mu^2}\Big[\frac{11\sigma^2-3}{\sigma(3\sigma^3-2)}+ \\ +\frac{11}{18(2)^{1/3}(3)^{2/3}}\Big\{\log_e\frac{(3)^{2/3}\sigma^2-2(6)^{1/3}\sigma+(2)^{2/3}}{(3)^{2/3}\sigma^2+2(6)^{1/3}\sigma+(2)^{2/3}}+ \\ +253\tan^{-1}\Big\{\frac{2(3)^{1/3}\sigma+(2)^{1/3}}{\sqrt{3}(2)^{1/3}}\Big\}\Big\}\Big]-\frac{AB_4}{12\mu^2}\Big[\sigma^2+ \\ +\frac{1}{(2)^{2/3}(3)^{1/3}}\Big\{\log_e\frac{(2)^{2/3}\sigma^2-2(6)^{1/3}\sigma+(3)^{2/3}}{(2)^{2/3}\sigma^2+2(6)^{1/3}\sigma+(3)^{2/3}}+ \\ +253\tan^{-1}\Big\{\frac{(2)^{4/3}\sigma+(3)^{1/3}}{(3)^{5/6}}\Big\}\Big\}\Big].$$

To evaluate the hoop-stress at the inner boundary, then the stress components $T^{\alpha\beta}$ may be expressed in terms of $T'^{\alpha\beta}$. Thus, if ψ is the inclination of the normal on y_1-axis at any point of the equilateral triangle, $T^{\alpha\beta}$ is expressed as follows:

$$(22)\quad \begin{aligned} T^{11}e^{-2i\psi}&=T'^{11}=p_{\xi\xi}-p_{\eta\eta}-2ip_{\xi\eta}. \\ T^{22}e^{2i\psi}&=T'^{22}=p_{\xi\xi}-p_{\eta\eta}-2ip_{\xi\eta} \end{aligned}$$

and $T^{12}=T'^{12}=p_{\xi\xi}+p_{\eta\eta};$ where

$$e^{-2i\psi}=\frac{(3\sigma^3-2)}{\sigma(3-2\sigma^3)}.$$

The stress components in the equilateral triangular system so obtained give $p_{\xi\eta}=0$ and

$$(23)\quad \overline{T'}^{11}=\overline{T}^{22}=p_{\xi\xi}-p_{\eta\eta}\quad T'^{12}=T^{12}=p_{\xi\xi}+p_{\eta\eta}.$$

The hoop-stress at any point of the triangular boundary is obtained by putting $p_{\xi\xi}=-p$ in the first or second equations of (23). Thus

$$(24)\quad p_{\eta\eta}=\frac{24P}{(3\sigma^3-2)(3-2\sigma^3)}(2\sigma^6-\sigma^3+2)-\frac{4P^2\nu}{\mu(3\sigma^3-2)^2(3-2\sigma^3)^2}[36\varkappa\sigma^{12}- \\ -6(5+16\varkappa)\sigma^9+2(65+48\varkappa)\sigma^6-6(5+16\varkappa)\sigma^3+36\varkappa]- \\ -\frac{8P^2\nu}{\mu(3\sigma^3-2)^3(3-2\sigma^3)^3}[108\varkappa\sigma^{18}+18(13+3\varkappa)\sigma^{15}-3(529+ \\ +484\varkappa)\sigma^{12}+4(1107+416\varkappa)\sigma^9-3(529+484\varkappa)\sigma^6+18(13+3\varkappa)\sigma^3+ \\ +108\varkappa]+\frac{32B_3P^2\sigma^3}{\mu(3-2\sigma^3)(3\sigma^3-2)}.$$

4. Conclusions. Equations (21) and (2) giving displacements on the inner boundary show that in the undeformed state the hole is very much different from its triangular hole with bounded corners in the deformed state. Also, the first expression of the equation (24) shows that hoop-stress remains unaffected by the elastic constants in the infinitesimal case, while the hoop-stress in second order case is influenced by the elastic properties of the material. The numerical values for hoop-stress have been given in appendix.

APPENDIX

Table shows the values of hoop stress $p_{\eta\eta}/P$ at the boundary of the equilateral triangle for infinitesimal case and for compressible and incompressible materials with second order effects, P/μ being taken to be equal to 1/200.

$p_{\eta\eta}/P$ / θ	Infinitesimal case	Second order effects : Compressible	Incompressible
0°	7.20000	7.92399	6.26813
10°	−0.82439	−1.06400	−1.00086
20°	−4.72519	−4.92310	−4.79420
30°	−3.64672	−3.78153	−3.74263
40°	16.76733	15.86676	15.72138
50°	9.35815	8.70435	8.76104
60°	−4.03595	−4.15721	−4.12191
70°	−4.61115	−4.15721	−4.68290
80°	0.79954	0.49557	0.56988
90°	5.93773	6.22927	5.28410

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