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EXISTENTIAL AND UNIVERSAL QUANTIFIERS IN AN ELEMENTARY TOPOS

BY

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In this article we are concerned with the construction in details of the internal existential and universal quantifiers in an elementary topos and we prove some properties of these quantifiers.

1.01. Let $\mathcal{E}$ be a topos and for any morphism $f: X \to Y$ in $\mathcal{E}$, let

$$\mathcal{E}/X \underset{\pi_f}{\overset{\Sigma_f}{\longrightarrow}} \mathcal{E}/Y, \quad \Sigma_f \dashv f \dashv \pi_f,$$
$$\xrightarrow{f^*}$$

be the adjunction situation described in [1], (ch. IV. §6).

Since f^* is a right adjoint, f induces a functor

$$f^{-1}: \mathrm{Sub}\,(Y) \to \mathrm{Sub}\,(X),$$

called the inverse image functor associated to f; similarly, π_f being a right adjoint, it also induces a functor

$$\forall_f: \mathrm{Sub}\,(X) \to \mathrm{Sub}\,(Y),$$

called the universal quantifier functor associated to f, which is a right adjoint for f^{-1}. If $X' \xrightarrow{h} X$ is a subobject of X, it is easy to verify that the association,

$$(X' \xrightarrow{h} X) \to (\mathrm{Im}(fh) \to Y)$$

can be extended to a functor,

$$\exists_f: \mathrm{Sub}(X) \to \mathrm{Sub}\,(Y),$$

called the existential quantifier functor associated to f, which is a left adjoint for f^{-1}. In Ens, for any subset Y' of Y,

$$f^{-1}(Y') = \{x \in X / f(x) \in Y'\},$$

$$\exists_f(X') = f(X') = \{y \in Y \mid \textit{there exists } x \in X \textit{ with } f(x) = y\},$$

$$\forall_f(X') = \{y \in Y \mid f^{-1}(y) \subseteq X'\} = \{y \in Y / \forall x \textit{ with } f(x) = y,$$

$$\textit{then } x \in X'\} = Cf(X'C)$$

where $C(.)$ denotes the complement operator ; if f is the first projection $p_1 : X \times Y \to X$ and $R \subseteq X \times Y$ then :

$$\exists_{p_1}(R) = \{x \in X \mid \textit{there exists } y \in Y \textit{ such that } (x, y) \in R\}.$$

and

$$\forall p_1(R) = \{x \in X \mid \text{for any } y \in Y, (x, y) \in R\} \quad \text{i.e.,}$$

$\exists p_1$ and $\forall p_1$ coincide with the existential and universal quantifiers. For any $X \in |\mathcal{E}|$, we denote by i_x : $\mathrm{Sub}(X) \to \mathcal{E}/X$ the inclusion functor and by $\sigma_X : \mathcal{E}/X \to \mathrm{Sub}(X)$ the functor which associates to each object $X' \to X$ from $\mathcal{E}/X$, the object $\mathrm{Im}(h) \xrightarrow{h} X$ from $\mathrm{Sub}(X)$. Thus, we obtain the following diagram of categories and functors

$$\begin{tikzcd}
\mathcal{E}/X \arrow[rr, shift left=2, "\Sigma_f"] \arrow[d, shift right, "\sigma_X"'] \arrow[rr, shift right=2, "\Pi_f"'] & & \mathcal{E}/Y \arrow[ll, "f^*"'] \arrow[d, shift right, "\sigma_Y"'] \\
\mathrm{Sub}(X) \arrow[u, shift right, "1_X"'] \arrow[rr, shift left=2, "\exists_f"] \arrow[rr, shift right=2, "\forall_f"'] & & \mathrm{Sub}(Y) \arrow[ll, "f^{-1}"'] \arrow[u, shift right, "1_Y"']
\end{tikzcd}$$

$$\Sigma_f \dashv f^* \dashv \Pi_f \qquad \sigma_X \dashv 1_X \qquad \exists_f \dashv f^{-1} \dashv \forall_f$$

where the equalities below are easily verifiable:

$$f^* \circ i_Y = i_X \circ f^{-1} \qquad f^{-1} \circ \sigma_Y = \sigma_X f^*,$$

$$i_Y \circ \forall_f = \pi_f \circ i_X, \qquad \sigma_Y \circ \Sigma_f = \exists_f \circ \sigma_X,$$

$$\exists_f = \sigma_y \Sigma_f i_X.$$

1.02. Proposition. *Any pullback from an elementary topos*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\
C \arrow[r, "k"'] & D
\end{tikzcd}$$

induces the commutative diagrams

$$\begin{tikzcd}
\mathrm{Sub}(B) \arrow[r, "f^{-1}"] \arrow[d, shift right, "\exists_g"'] \arrow[d, shift left, "\forall_g"] & \mathrm{Sub}(A) \arrow[d, shift right, "\forall_h"'] \arrow[d, shift left, "\exists_h"] \\
\mathrm{Sub}(D) \arrow[r, "k^{-1}"'] & \mathrm{Sub}(C)
\end{tikzcd}$$

Proof. Results immediately from [1] ch. II, 9.12, 9.28 noticing that an elementary topos is obviously a regular category with existential and universal quantifiers. From [1] ch. II, 9.26 it results that any elementary topos $\mathcal{E}$ is a regular category with the implication „$\to$" and from [1] ch II, 9.24, it results that for any morphisms f from $\mathcal{E}$, the functor f^{-1} commutes with the implication.

1.03. Proposition. *Let $f : X \to Y$ be a morphism from an elementary topos $\mathcal{E}$. For any subobjects X', X'' of X and for any subobjects Y', Y'' of Y, the following relations hold.*

1°. $\exists_f(X') \subseteq Y' \Leftrightarrow X' \subset f^{-1}(Y')$;
2°. $Y' \subseteq \forall_f(X') \Leftrightarrow f^{-1}(Y') \subseteq X'$;
3°. $f^{-1} \circ \forall_f(X') \subseteq X'$;
4°. $X' \subseteq f^{-1}\exists_f(X')$;
5°. $\forall_f(X' \cap X'') = \forall_f(X') \cap \forall_f(X'')$;
6°. $\exists_f(X' \cup X'') = \exists_f(X') \cup \exists_f(X'')$;
7°. $\exists_f(X' \cap X'') \subseteq \exists_f(X') \cap \exists_f(X'')$;
8°. $\exists_f(f^{-1}(Y') \cap Y'') = Y' \cap \exists_f(X'')$;
9°. $\forall_f(X' \cup X'') \supseteq \forall_f(X') \cup \forall_f(X'')$;
10°. $\exists_f(f^{-1}(Y') \to X'') \subseteq Y' \to \exists_f(X'')$;
11°. $\exists_f(X' \to f^{-1}(Y'')) \subseteq \forall_f(X') \to Y''$;
12°. $\exists_f(X' \to X'') \subseteq \forall_f(X') \to \exists_f(X'')$;
13°. $\forall_f(f^{-1}(Y') \to X'') = Y \to \forall_f(X'')$;
14°. $\forall_f(X' \to f^{-1}(Y'')) = \exists_f(X') \to Y''$;
15°. $\neg\exists_f(X') = \forall_f(\neg X')$;
16°. $\neg\forall_f X' \supseteq \exists_f(\neg X')$;
17°. $\forall_f(X' \to X'') \subseteq \forall_f(X') \to \forall_f(X'')$;
18°. $f^{-1}(Y'') \supseteq f^{-1}(Y') \cap X'' \Leftrightarrow Y'' \supseteq Y' \cap \exists_f(X'')$.

Proof. The first six relations result from the adjunction situation $\exists_f \dashv f^{-1} \dashv \forall_f$.

7°. $\exists_f(X' \cap X'') \subseteq \exists_f(X') \cap \exists_f(X'') \Leftrightarrow X' \cap X'' \subseteq f^{-1}(\exists_f(X') \cap \exists_f(X'')) = f^{-1}\exists_f(X') \cap f^{-1}\exists_f(X'')$ and it is sufficient to consider relation 4°.

8°. Is obtained from [1] ch. II, 9.13.

9°. $\forall_f(X') \cup \forall_f(X'') \subseteq \forall_f(X' \cup X'') \Leftrightarrow X' \cup X'' \supseteq f^{-1}(\forall_f(X') \cup \forall_f(X'')) = f^{-1}\forall_f(X') \cup f^{-1}\forall_f(X'')$ and it is sufficient to consider relation 3°.

10°. $\exists_f(f^{-1}(Y') \to X'') \subseteq Y' \to \exists_f(X'') \Leftrightarrow f^{-1}(Y') \to X'' \subseteq f^{-1}(Y') \to \exists_f(X'')) = f^{-1}(Y') \to f^{-1}\exists_f(X'')$ and it is sufficient to consider that $f^{-1}(Y') \to (\quad)$ is a covariant functor.

11°. $\exists_f(X' \to f^{-1}(Y'')) \subseteq \forall_f(X') \to Y'' \Leftrightarrow X' \to f^{-1}(Y'') \subseteq f^{-1}(\forall_f(X') \to Y'') = f^{-1}\forall_f(X') \to f^{-1}(Y'')$ and it is sufficient to consider relation 3° and the fact that $(\quad) \to f^{-1}(Y'')$ is obviously a contravariant functor.

12°. $\exists_f(X' \to X'') \subseteq \forall_f(X') \to \exists_f(X'') \Leftrightarrow X' \to X'' \subseteq f^{-1}(\forall_f(X') \to \exists_f(\mathbf{X}'')) = f^{-1}\forall_f(X') \to f^{-1}\exists_f(X'')$. Then we take into account that $f^{-1}\forall_f(X') \subseteq X'$ and $X'' \subseteq f^{-1}\exists_f(X'')$.

13°. It must be shown that if Y_1 is a subobject of Y, then $Y_1 \subseteq Y' \to \forall_f(X'') \Leftrightarrow f^{-1}(Y_1) \subseteq f^{-1}(Y') \to X''$. Indeed, $Y_1 \subseteq Y' \to \forall_f(X'') \Leftrightarrow Y_1 \cap Y' \subseteq \forall_f(X'') \Leftrightarrow X'' \supseteq f^{-1}(Y_1 \cap Y') = f^{-1}(Y_1) \cap f^{-1}(Y') \Leftrightarrow f^{-1}(Y_1) \subseteq f^{-1}(Y') \to X''$.

14°. It must be proved that if Y_1 is a subobject of Y, then $Y_1 \subseteq \exists_f(X') \to Y'' \Leftrightarrow f^{-1}(Y_1) \subseteq X' \to f^{-1}(Y'')(\Leftrightarrow Y_1 \subseteq \forall_f(X' \to f^{-1}(Y''))$. Really, $Y_1 \subseteq \exists_f(X') \to Y'' \Leftrightarrow Y_1 \cap \exists_f(X') \subseteq Y'' \Leftrightarrow \exists_f(X') \subseteq Y_1 \to Y'' \Leftrightarrow X' \subseteq f^{-1}(Y_1 \to Y'') = f^{-1}(Y_1) \to f^{-1}(Y'') \Leftrightarrow X' \cap f^{-1}(Y_1) \subseteq f''(Y'') \Leftrightarrow f^{-1}(Y_1) \subseteq X' \to f^{-1}(Y'')$.

15°. From the definition $\neg\exists_f(X') = \exists_f(X') \to 0$ and therefore if Y_1 is a subobject of Y, $Y_1 \subseteq \neg\exists_f(X) \Leftrightarrow Y_1 \cap \exists_f(X') \subseteq 0 \Leftrightarrow \exists_f(X') \subseteq Y_1 \to 0 \Leftrightarrow X' \subseteq$

$\subseteq f^{-1}(Y'_1\to 0)=f^{-1}(Y'_1)\to f^{-1}(0)=f^{-1}(Y'_1)\to 0\Leftrightarrow X'\cap f^{-1}(Y'_1)\subseteq 0\Leftrightarrow f^{-1}(Y'_1)\subseteq X'\to$ $\to 0=\neg X'\Leftrightarrow Y_1\subseteq \forall_f(\neg X')$.

16°. $\exists_f(\neg X')\subseteq\neg\forall_f(X')\Leftrightarrow\neg X'\subseteq f^{-1}(\neg\forall_f(X'))=\neg f^{-1}\forall_f(X')\neg\Leftrightarrow X'\cap f^{-1}\forall_f(X')\subseteq$ $\subseteq 0\Leftrightarrow f^{-1}\forall_f(X')\subseteq\neg X'\to 0=\neg\neg X'$ and it is enough to consider that $f^{-1}\circ\forall_f(X')X'\subseteq\neg\neg X'$.

17°. $\forall_f(X_f\to X'')\subseteq\forall_f(X')\to\forall_f(X'')\Leftrightarrow\forall_f(X'\to X'')\cap\forall_f(X')\subseteq\forall_f(X'')\Leftrightarrow$ $\Leftrightarrow\forall_f((X'\to X'')\cap X')\subseteq\forall_f(X'')$ and it is enough to take into account the facts that $(X'\to X'')\cap X'\subseteq X''$ and that $\forall_f$ preserves the order ($\forall_f$ is a functor).

18°. $f^{-1}(Y')\cap X''\subseteq f^{-1}(Y'')\Leftrightarrow X''f^{-1}(Y')\to f^{-1}(Y'')=f^{-1}(Y'\to Y'')\Leftrightarrow$ $\Leftrightarrow\exists_f(X'')\subseteq Y'\to Y''\Leftrightarrow\exists_f(X'')\cap Y'\subseteq Y''$.

1.04. Proposition. *If $X\xrightarrow{g}Y\xrightarrow{f}Z$ are two composite morphisms from $\mathcal{E}$, then $\forall_g(\exists_f(X')\to Y')\forall_h(X'\to f^{-1}(Y'))$ where $h=gf$, X' is a subobject of X and Y' a subobject of Y.*

Proof. From $g^{-1}\circ\forall_g(\exists_f(X')\to Y')\subseteq\exists_f(X')\to Y'$ it results that $g^{-1}\circ$ $\circ\forall_g(\exists_f(X')\to Y')\cap\exists_f(X')\subseteq Y'$ and therefore $f^{-1}\circ g^{-1}\circ\forall_g(\exists'(X')\to Y')\cap f^{-1}$ $\exists_f(X')\subseteq f^{-1}(Y')$, i.e. $h^{-1}\circ\forall_g(\exists_f(X')\to Y')\cap X'\subseteq f^{-1}(Y')$ because $X'\subseteq f^{-1}\circ$ $\circ\exists_f(X')$, But $h^{-1}\forall_g(\exists_f(X')\to Y')\subseteq X'\to f^{-1}(Y')$ and consequently $\forall_g(\exists_f(X')\to$ $\to Y')\subseteq\forall_h(X'\to f^{-1}(Y'))$.

1.05. Proposition. *Let $\mathcal{E}$ be an elementary topos. The association $A\to PA$, $f\to\exists f$ given by the following commutative diagram with the pullback right square,*

$$\begin{tikzcd}
\in_A \arrow[r] \arrow[d, "j_A"'] & \exists_{PA\times f}(\in_A) \arrow[rr] \arrow[d] & & 1 \arrow[d, "v"] \\
PA\times A \arrow[r, "PA\times f"] & PA\times B \arrow[r, "\exists f\times B"] \arrow[rr, bend right] & PB\times B \arrow[r, "e_B"] & \Omega
\end{tikzcd}$$

defines a functor $\exists:\mathcal{E}\to\mathcal{E}$, called the internal existential quantifier.

Proof. It is obvious that $\exists 1_A=1_{PA}$. If $A\xrightarrow{f}B\xrightarrow{g}C$ are two composite morphisms in $\mathcal{E}$, then $\exists gf=\exists g\,\exists f$. It is enough to show that $\overline{\exists g}(\exists f\times C)=\overline{\exists(gf)}$ and for that, to prove they are characteristic morphisms of a same subobject of $PA\times C$. Applying proposition 1.02 to the following pullback

$$\begin{tikzcd}
PA\times B \arrow[r, "\exists f\times B"] \arrow[d, "PA\times g"'] & PB\times B \arrow[d, "PB\times g"] \\
PA\times C \arrow[r, "\exists f\times C"'] & PB\times C
\end{tikzcd}$$

we have $(\exists f\times C)^{-1}(\exists_{PB\times g}(\in B))=\exists_{PA\times g}((\exists f\times B)^{-1}(\in_B))=\exists_{PA\times g}\exists_{PA\times f}$ $(\in_A)=\exists_{PA\times gf}(\in_A)$. If f is a monomorphism, $\overline{\exists_f}$ is the characteristic morphism of the subobject, $\in_A\xrightarrow{j_A}PA\times A\xrightarrow{PA\times f}PA\times B$ and f built in the preceding proposition coincides with $\exists f$ built in (1.10) [1], Ch. IV.

In Ens, $\exists_{PA\times f}(\in_A)=\{(A',b)/A'\subseteq A,\ b\in B$ so that $\exists a\in A$ with $a\in A'$ and $f(a)=b\}$ and it can beasily noticed that $\exists f(A')=f(A')$, $\forall A'\in P(A)$.

1.06. Proposition. *Let $\mathcal{E}$ be an elementary topos. The association $A\to PA$ $f\to\forall f$ given by the following commutative diagram with the pullback square*

$$\begin{tikzcd}
\in_A \arrow[d, "j_A"'] & \forall_{PA\times f}(\in_A) \arrow[rr] \arrow[d] & & 1 \arrow[d, "v"] \\
PA\times A \arrow[r, "PA\times f"'] & PA\times B \arrow[r, "\forall f\times B"] \arrow[rr, bend right, "f"'] & PB\times B \arrow[r, "e_B"] & \Omega
\end{tikzcd}$$

defines a functor $\forall:\mathcal{E}\to\mathcal{E}$ called the internal universal quantifier.

Proof. It is obvious that $\forall_{1_A}=1_{PA}$; if $A\xrightarrow{f}B\xrightarrow{g}C$ are two composite morphisms in $\mathcal{E}$, then $\forall gf=\forall g\circ\forall f$. It is sufficient to show that $\overline{\forall g}(\forall f\times C)=$ $=\overline{\forall gf}$ and for that, to show they are the characteristic morphisms of the same subobject of $PA\times C$. Applying proposition 1.02 to the following pullback

$$\begin{tikzcd}
PA\times B \arrow[r, "f\times B"] \arrow[d, "PA\times g"'] & PB\times B \arrow[d, "PB\times g"] \\
PB\times C \arrow[r, "f\times C"] & PB\times C
\end{tikzcd}$$

we have $(\forall f\times C)^{-1}(\forall_{PB\times g}(\in_B))=\forall_{PA\times g}((\forall f\times B)^{-1}(\in_g))=\forall_{PA\times g}\forall_{PA\times f}(\in_A)=$ $=\forall_{PA\times gf}(\in_A)$.

In Ens, $\forall_{PA\times f}(\in_A)=\{(\varphi_b,A',b)/A'\subseteq A,\ b\in B,\ \varphi_b:f^{-1}(b)\to\in_A,\ \varphi_b(a)=$ $=(A',a),\ \forall a\in f^{-1}(b)\}=\{(A',b)/A'\subseteq A,\ b\in B$ such that $f^{-1}(b)\subseteq A'\}=\{(A',b)/$ $/A'\subseteq A,\ b\in B$ so that $a\in f^{-1}(b)$ we have $a\in A'\}$ and it can be easily noticed that $\forall_f(A')=C\,\exists f(CA')$, $\forall A'\in P(A)$, where C is the operator of taking the complement.

1.07. Proposition. *If*

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

is a pullback in an elementary topos $\mathcal{E}$, then the following diagrams

$$\begin{tikzcd}
PB \arrow[r, "Pk"] \arrow[d, shift right, "\forall g"'] \arrow[d, shift left, "\exists g"] & PA \arrow[d, shift right, "\exists h"'] \arrow[d, shift left, "\forall h"] \\
PD \arrow[r, "Pf"] & PC
\end{tikzcd}$$

are commutative.

Proof. To show that $\exists h \circ Pk = Pf \circ \exists g$, it is sufficient to prove that $\overline{\exists h}((Pk \times C) = e_C(Pf \times C)(\exists g \times C) = e_D(PD \times f)(\exists g \times C) = e_D(\exists g \times D)(PB \times f) = \overline{\exists g}(PB \times f)$ and for that, to show that they are characteristic morphisms of the same subobject of $PB \times C$.

By applying Proposition 1.02 to the following pullback

$$\begin{tikzcd} PB \times A \arrow[r, "Pk \times A"] \arrow[d, "PB \times h"'] & PA \times A \arrow[d, "PA \times h"] \\ PB \times C \arrow[r, "Pk \times C"] & PA \times C \end{tikzcd}$$

we have $(Pk \times C)^{-1}(\exists_{PA \times h}(\in_A)) = \exists_{PB \times h}(Pk \times A)^{-1}(\in_A)) = \exists_{PB \times h}(Pb \times k)^{-1}(\in_B) = (PB \times f)^{-1}{}_{PB \times g}(\in_B)$. Here the second equality obtained by considering the definition of Pk and the last equal ty can be deduced by applying Proposition 1.02 to the following pullback (the functor $PB \times (\ \ )$ commutes with pulbacks)

$$\begin{tikzcd} PB \times A \arrow[r, "PB \times k"] \arrow[d, "PB \times h"'] & PB \times B \arrow[d, "PB \times g"] \\ PB \times C \arrow[r, "PB \times f"] & PB \times D \end{tikzcd}$$

To show that $\forall h \times Pk = Pf \forall g$, it is sufficient to show that $\overline{\forall h}(Pk \times C) = e_C(Pf \times C)(\forall g \times C) = e_D(PD \times f)(\forall g \times C) = e_D(\forall g \times D)(PB \times f) = \overline{\forall g}(PB \times f)$. We mimic the proof of the first case substituting everywhere „$\exists$" by „$\forall$"; to be more exact, we apply Proposition 1.02 to the pullback *) and obtain

$$(Pk \times C)^{-1}(\underset{PA \times h}{\forall}(\in_A)) = \underset{PB \times h}{\forall}((Pk \times A)^{-1}(\in_A)) =$$

$$= \underset{PB \times h}{\forall}((PB \times k)^{-1})(\in_B)) = (PB \times f)^{-1}(\underset{PB \times g}{\forall}(\in_B)).$$

1.08. Corollary. *If $f : A \to B$ is a monomorphism, then $\forall f$ and $\exists f$ are also monomorphisms and Pf is an epimorphism.*

Proof. Applying the previous proposition to the pullback

$$\begin{tikzcd} A \arrow[r, equal] & A \arrow[d, "f"] \\ A \arrow[r] & B \end{tikzcd}$$

we obtain the commutative diagram

$$\begin{tikzcd} PA \arrow[r, equal] \arrow[d, shift right, "\exists f"'] \arrow[d, shift left, "\forall f"] & PA \arrow[d, equal] \\ PB \arrow[r, "Pf"] & PA \end{tikzcd}$$

i.e. $Pf \circ \exists f = 1_{PA} = Pf \circ \forall f$.

1.09. Proposition. *If $f : A \to B$ is a monomorphism, then $e_B \circ (\exists f \times f) = e_A$.*

Proof. To show that $e_B \circ (\exists f \times f) = e_B \circ (\exists f \times B)(PA \times f) = \overline{\exists f}(PA \times f) = e_A$, it is sufficient to show that they are the characteristic morphisms of the same subobject of $PA \times A$, namely of the subobject $\in_A \xrightarrow{j_A} PA \times A$.

If Proposition 1.02 is applied to the following pullback

$$\begin{tikzcd} PA \times A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "g"] & PA \times A \arrow[d, "PA \times f"] \\ PA \times A \arrow[r, "PA \times f"'] & PA \times B \end{tikzcd}$$

we have $(PA \times f)^{-1}(\exists_{PA \times f}(\in_A)) = \in_A$. The proof is complete.

1.10. Proposition. *If*

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "f"] \\ A \arrow[r, "g"] & C \end{tikzcd}$$

is a pullback in an elementary topos $\mathcal{E}$, with g (hence also with α) a monomorphisms, then $\exists$ preserves this pullback.

Proof. It must be shown that the following square

$$\begin{tikzcd} PD \arrow[r, "\exists\alpha"] \arrow[d, "\beta"'] & PB \arrow[d, "\exists f"] \\ PA \arrow[r, "\mathrm{E}g"] & PC \end{tikzcd}$$

is a pullback. The commutativity derives from the fact that $\exists$ is a functor.

If $\overline{x} : X \to PA$ and $\overline{y} : X \to PB$ are two morphisms so that $\exists g . \overline{x} = \exists f \overline{y}$, then $\overline{\exists g}(\overline{x} \times C) = \overline{\exists f}(\overline{y} \times C)$ and therefore they are the characteristic morphism of a same subobject of $X \times C$, i.e. $(\overline{x} \times C)^{-1}(\underset{PA \times g}{\exists}(\in_A)) = (\overline{y} \times C)^{-1}(\underset{PB \times f}{\exists}(\in_B))$ or, applying Proposition 1.02 to the following pullbacks:

$$\begin{tikzcd} X\times A \arrow[r, "\bar{x}\times A"] \arrow[d, "X\times g"'] & PA\times A \arrow[d, "PA\times g"] \\ X\times C \arrow[r, "\bar{x}\times C"] & PA\times C \end{tikzcd} \qquad \begin{tikzcd} X\times A \arrow[r, "\bar{y}\times B"] \arrow[d, "X\times f"'] & PB\times B \arrow[d, "PB\times f"] \\ X\times C \arrow[r, "\bar{y}\times C"] & PB\times C \end{tikzcd}$$

$\exists_{X\times a}((\bar{x}\times A)^{-1}(\in_A)) = \exists_{X\times f}((\bar{y}\times B)^{-1}(\in_B))$.

Since $X\times g$ is a monomorphism, there exists an epimorphism p: $(\bar{y}\times B)^{-1}(\in_B)\to(\bar{x}\times A)^{-1}(\in A)$ such that $(X\times g)(\bar{x}\times A)^{-1}(j_A)p=(X\times f)(\bar{y}\times B)^{-2}(j_B)$ and since the diagram

$$\begin{tikzcd} X\times D \arrow[r, "X\times\alpha"] \arrow[d, "X\times\beta"'] & X\times B \arrow[d, "X\times f"] \\ X\times A \arrow[r, "X\times g"] & X\times C \end{tikzcd}$$

is a pullback, there exists a unique morphism $q:(\bar{y}\times B)^{-1}(\in_B)\to X\times D$ such that $(X\times\beta)q=(\bar{x}\times A)^{-1}(j_A)p$ and $(X\times\alpha)q=(\bar{y}\times B)^{-1}(j_B)$.

From the last equality it results that q is a monomorphism; if γ is its characteristic morphism and $\bar{\gamma}: X\to PD$ is the morphism corresponding to γ by Cartesian adjunction, then $\exists\alpha\,.\,\gamma=\bar{y}$ and $\exists\beta\,.\,\alpha=\bar{x}$ and is uniquely determined by this property, because $\exists\alpha$ is a monomorphism.

To show that $\exists\alpha\bar{\gamma}=\bar{y}$ it is sufficient to show that $\overline{\exists\alpha}\,.\,(\bar{\gamma}\times B)=e_B(\bar{y}\times B)$ and for that, to show that they are the characteristic morphisms of a same subobject of $X\times B$. Applying Proposition 1.02 to the following pullback

$$\begin{tikzcd} X\times D \arrow[r, "\bar{\gamma}\times D"] \arrow[d, "X\times\alpha"'] & PD\times D \arrow[d, "PD\times\alpha"] \\ X\times B \arrow[r, "\bar{\gamma}\times B"] & PD\times B \end{tikzcd}$$

we obtain $(\bar{\gamma}\times B)^{-1}(\exists_{PD\times\alpha}(\in_D))=\exists_{X\times\alpha}(\bar{\gamma}\times D)^{-1}(\in_D))=\exists_{X\times\alpha}((\bar{y}\times B)^{-1}(\in_B))=$ $=(\bar{y}\times B)^{-1}(\in_B)$ (for $X\times\alpha$ is a monomorphism). To show that $\exists\beta\,.\,\gamma=\bar{x}$, it is sufficient to show that $\overline{\exists\beta}(\bar{\gamma}\times A)=e_A(\bar{x}\times A)$ and for that, to show that theory are the characteristic morphisms of the same subobject of $X\times A$.

Applying Proposition 1.02 to the following pullback

$$\begin{tikzcd} X\times D \arrow[r, "\bar{\gamma}\times D"] \arrow[d, "X\times\beta"'] & PD\times D \arrow[d, "PD\times\beta"] \\ X\times A \arrow[r, "\bar{\gamma}\times A"] & PD\times A \end{tikzcd}$$

we obtain

$(\bar{\gamma}\times A)^{-1}(\exists_{PD\times\beta}(\in_D))=\exists_{X\times\beta}((\bar{\gamma}\times D)^{-1}(\in_D))=\exists_{X\times\beta}((\bar{y}\times B)^{-1}(\in_B))=\exists_{(\bar{\gamma}\times\beta)q}(\in_B)=$
$=\exists_{(\bar{x}\times A)^{-1}j_A}(j_A)p(\in_B)=\exists_{(\bar{x}\times A)^{-1}}(j_A)((\bar{x}\times A)^{-1}(\in_A))=(\bar{x}\times A)^{-1}(\in_A)$.

(Th last but one equality derives from the fact that p is an epimorphism and the last equality from the fact that $(\bar{x}\times A)^{-1}(j_A)$ is a monomorphism).

1.11. **Proposition.** *The morphism* $\{\cdot\}_A: A\to PA$, $A\in|\mathcal{E}|$, *are the components of a functorial morphism* $\{\cdot\}: 1\mathcal{E}\to\exists$, *i.e.* for any $f: A\to B$ we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\{\cdot\}_A"] \arrow[d, "f"'] & PA \arrow[d, "\exists f"] \\ B \arrow[r, "\{\cdot\}_B"] & PB \end{tikzcd}$$

Proof. It is sufficient to show that $\overline{\exists f}(\{\cdot\}_A\times B)=e_B(\{\cdot\}_B\times B)(f\times B)$ and for that, to prove that they are the characteristic morphism sof a same subobject of $A\times B$. Applying Proposition 1.02 to the following pullback

$$\begin{tikzcd} A\times A \arrow[r, "\{\cdot\}_A\times A"] \arrow[d, "A\times f"'] & PA\times A \arrow[d, "PA\times f"] \\ A\times B \arrow[r, "A^B"] & PA\times B \end{tikzcd}$$

we have $(\{\cdot\}_A\times B)^{-1}(\exists_{PA\times f}(\in_A))=\exists_{A\times f}((\{\cdot\}_A\times A)^{-1}(\in_A))=\exists_{A\times f}(A\xrightarrow{\Delta_A}A\times A)=$ $=<1_A, f>$; or, $<1_A, f>=(f\times B)^{-1}(\Delta_B)$ as it results from the following pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\langle 1_A, f\rangle"'] & B \arrow[d, "\Delta_B"] \\ A\times B \arrow[r, "f\times B"] & B\times B \end{tikzcd}$$

1.12. *Remark.* If f is a monomorphism, the diagram *) is a pullback.

Let $u: C\to PA$ and $v: C\to B$ be such that $\{.\}_B\,.\,v=\exists fu$.

Then

$$C\times B\xrightarrow{u\times B}PA\times B\xrightarrow{\exists_f\times B}PB\times B\xrightarrow{e_B}\Omega=$$

$$=C\times B\xrightarrow{v\times B}B\times B\xrightarrow{\{\cdot\}_B\times B}PB\times B\xrightarrow{e_B}\Omega;$$

i.e. $|\overline{\exists f}(u\times B)$, $s_B(v\times B)$ are the characteristic morphisms of the same subobject of $C\times B$.

From the diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "\langle 1_C, v\rangle"'] \arrow[dr, phantom, "p.b"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta_B p.b"] & 1 \arrow[d, "v"] \\
C\times B \arrow[r, "v\times B"'] & B\times B \arrow[r, "s_B"'] & \Omega
\end{tikzcd}$$

one derives that $s_B(v\times B)=\chi(<1_C, v>)$ (the characteristic morphism of the subobject $<1_C, v>$).

However, $\overline{\exists f}(u\times B)=\chi((u\times B)^{-1}(\underset{PA\times f}{\exists}(\in_A))=\chi(<\gamma_1, f\gamma_2>)$ with γ_1, γ_2 given by the diagram below.

$$\begin{tikzcd}
D \arrow[r] \arrow[d, "\langle \gamma_1, \gamma_2\rangle"'] \arrow[dr, phantom, "p.b."] & \in_A \arrow[d, "j_A"] \\
C\times A \arrow[r, "u\times A"] \arrow[d, "C\times f"'] \arrow[dr, phantom, "p.b."] & PA\times A \arrow[d, "PA\times f"] \\
C\times B \arrow[r, "u\times B"'] & PA\times B
\end{tikzcd}$$

Hence, there exists an isomorphism $\rho: C\to D$, such that $<\gamma_1, f\circ\gamma_2>\rho= =<1_C, v>$ where from $\gamma_1\rho=1_C$ and $f\gamma_2=v$. If $w=\gamma_2\rho$ then $fw=v$ and to show that $\{\cdot\}_A w=u$, it is sufficient to prove that

$C\times A\xrightarrow{w\times A}A\times A\xrightarrow{\{\cdot\}_A\times A}PA\times A\xrightarrow{e_A}\Omega=C\times A\xrightarrow{u\times A}PA\times A\xrightarrow{e_A}\Omega$, so, $s_A(w\times A), e_A(u\times A)$ are the characteristic monomorphism of the same subobject of $C\times A$.

From the diagram below

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "\langle 1_C, w\rangle"'] \arrow[dr, phantom, "p.b"] & A \arrow[r] \arrow[d, "\Delta_A"] \arrow[dr, phantom, "p.b"] & 1 \arrow[d, "v"] \\
C\times A \arrow[r, "w\times A"'] & A\times A \arrow[r, "s_A"'] & \Omega
\end{tikzcd}$$

it results that $s_A(w\times A)=\chi(<1_C, w>)=\chi(<\gamma_1\rho, \gamma_2\rho>)=\chi(<\gamma_1, \gamma_2>\rho)= =\chi(<\gamma_1, \gamma_2>)=e_A(u\times A)$.

The uniqueness of w derives from the fact that f is a monomorphism

1.13. Proposition. *Let $\mathcal{E}$ be an elementary topos. For any $f: A\to B$ from $\mathcal{E}$, the diagrams*

$E(B,\Omega)$ $\forall f$ $E(A,\Omega)$ $E(1\times B,\Omega)$ $E(1\times A,\Omega)$

$$\begin{tikzcd}
\mathrm{Sub}(B) \arrow[r, shift left, "\exists f"] \arrow[r, shift right, "f^{-1}"'] \arrow[d, phantom, "\wr"] & \mathrm{Sub}(A) \arrow[d, phantom, "\wr"] \\
\mathcal{E}(B,\Omega) \arrow[d, phantom, "\wr"] & \mathcal{E}(A,\Omega) \arrow[l, "\forall f"'] \arrow[d, phantom, "\wr"] \\
\mathcal{E}(1\times B,\Omega) \arrow[d, phantom, "\wr"] & \mathcal{E}(1\times A,\Omega) \arrow[d, phantom, "\wr"] \arrow[l, phantom, "\mathcal{E}(1,\exists f)"] \\
\mathcal{E}(1,PB) \arrow[r, shift left, "\mathcal{E}(1,Pf)"] & \mathcal{E}(1,PA) \arrow[l, shift left, "\mathcal{E}(1,\forall f)"]
\end{tikzcd}$$

are commutative, where the vertical monomorphisms are the canonical isomorphisms.

Proof. a) Let $B'\xrightarrow{m}B$ be a subobject of B, let $\varphi_m: B\to\Omega$ be the characteristic morphism corresponding to m, $\lambda_m= 1\times B\xrightarrow{p_B}B\xrightarrow{\varphi_m}\Omega$ be the element from $\mathcal{E}(1\times B,\Omega)$ corresponding to φ_m and let $[m]: 1\to PB$ be the unique morphism such that $e_B([m]\times B)=\lambda_m$.

If $A'\xrightarrow{m}A=f^{-1}(B'\xrightarrow{m}B)$ then $\varphi_{m'}=\varphi_m f$, $\lambda_{m'}p_A=\varphi_m f p_A=\varphi_m p_B).1\times f)= =\lambda_m(1\times f)$ and $[m']: 1\to PA$ is the unique morphism such that $e_A([m]\times \times A)=\lambda_m$. To show that $Pf[m]=[m']$, it is sufficient to prove that

$$1\times A\xrightarrow{[m]\times A}PB\times A\xrightarrow{Pf\times A}PA\times A\xrightarrow{e_A}\Omega=1\times A\xrightarrow{[m']\times A}PA\times A\xrightarrow{e_A}\Omega.$$

However $e_A\circ(Pf\times A(([m]\times A)=e_B\circ(PB\times f)([m\times A)=e_B([m]\times B)(1\times f)= =\lambda_m(1\times f)=\lambda_{m'}=e_A([m'[\times A)$.

b). Let $A'\xrightarrow{m}A$ be a subobject of A, let $\varphi_m: A\to\Omega$ be its characteristics morphism, let $\lambda_m=\varphi_m p_A$ and $[m]: 1\to PA$ be the unique morphism such that $e_A([m]\times A)=\lambda_m$.

Let $B'\xrightarrow{m'}B=\exists_f(A'\xrightarrow{m}A)$, $\varphi_{m'}: B\to\Omega$ be the characteristic morphism of m', $\lambda_{m'}=\varphi_{m'}p_B: 1\times B\to\Omega$ and $[m']: 1\to PB$ the unique morphism such that $e_B.([m']\times B)=\lambda_{m'}$. To show that $\exists f[m]=[m']$ it is sufficient to prove that

$$1\times B\xrightarrow{[m]\times B}PA\times B\xrightarrow{\exists f\times B}PB\times B\xrightarrow{e_B}\Omega=1\times B\xrightarrow{[m']}PB\ B\xrightarrow{e_B}\Omega,$$

and for that, to show that they are the characteristic morphism of the same subobject of $1\times B$.

From $e_B([m']\times B)=\lambda_{m'}=\varphi_{m'}p_B$ it results that the subobject corresponding to $e_B([m]\times B)$ is $B\xrightarrow{p_B^{-1}\cdot m'}1\times B$. Applying Proposition 1.02 to the pullbacks

$$(*)\quad\begin{tikzcd}
1\times A \arrow[r, "{[m]\times A}"] \arrow[d, "1\times f"'] & PA\times A \arrow[d, "PA\times f"] \\
1\times B \arrow[r, "{[m]\times B}"'] & PA\times B
\end{tikzcd}
\qquad
\begin{tikzcd}
1\times A \arrow[r, "p_A"] \arrow[d, "1\times f"'] & A \arrow[d, "f"] \\
1\times B \arrow[r, "p_B"'] & B
\end{tikzcd}$$

we obtain

$$([m]\times B)^{-1}\circ \underset{PA\times f}{\exists}(\in_A)=\underset{1\times f}{\exists}(([m]\times A)^{-1}(\in_A)=$$

$$=\underset{1\times f}{\exists}(A'\xrightarrow{p_m^{-1}\cdot m}1\times A)=p_A^{-1}\circ\exists_f(A'\xrightarrow{m}A)=p_B^{-1}(B'\xrightarrow{m'}B)=$$

$$=B'\xrightarrow{p_B^{-1}\cdot m}1\times B.$$

c) Let $A'\xrightarrow{m}A$ be a subobject of A, let $\varphi_m\colon A\to\Omega$ be its characteristic morphism, let $\lambda_m=\varphi_m p_A\colon 1\times A\to\Omega$ and $[m]\colon 1\to PA$ be the unique morphism such that $e_A\circ([m]\times A)=\lambda_m$. Let $B'\xrightarrow{m'}B=\forall_f(A'\xrightarrow{m}A)$ be a morphism and $\varphi_{m'}\colon B\to\Omega$ be its characteristic morphism, let $\lambda_{m'}=\varphi_{m'}p_B\colon 1\times B\to\Omega$ be a morphism and let $[m']\colon 1\to PB$ be the unique morphism such that $e_B([m']\times B)=\lambda_{m'}$. To show that $\forall_f[m]=[m']$, it is enough to prove that $\forall f([m]\times B)=\lambda_m$, and for that, to show they are the characteristic morphisms of a same subobject $B'\xrightarrow{p_B^{-1}\cdot m'}1\times B$ of $1\times B$. Applying the Proposition 1.02 to the pullbacks *1 we have

$$([m]\times B)^{-1}\circ\forall(\in_A)=\underset{PA\times f}{\forall}\ \underset{1\times f}{\forall}(([m]\times A)^{-1}(\in_A))=\underset{1\times f}{\forall}(A'\xrightarrow{p_m^{-1}\circ m}1\times A)=$$

$$=p_B^{-1}\circ\forall_f(A'\xrightarrow{m}A)=p_B^{-1}(B'\xrightarrow{m'}B)=B'\xrightarrow{p_B^{-1}m'}1\times B.$$

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THE PRECISE ANNULI CONTAINING THE ZEROS OF POLYNOMIALS

BY

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Let $p(z)=a_0+\ldots+a_nz^n$ with $a_0\neq0$ be a polynomial over complex numbers. The mathematical literature is replete [1] with results concerning the bounds for the moduli of zeros of $p(z)$. Below we give formulas for evaluating the precise minimum r and the precise maximum R of the moduli of zeros of $p(z)$. Thus, $r\leqslant|z|\leqslant R$ is the precise annulus which contains all the zeros of $p(z)$. Perhaps, in general, the actual evaluation of our formulas is impractical. However, it seems that many of the known bounds are actually various approximations of our formulas. Let

$$A(z)=\frac{1}{a_0+\ldots+a_nz^n}\qquad\text{and}\qquad B(w)=\frac{w^n}{a_0w^n+\ldots+a_n}.$$

Based on the notations introduced above, we prove:

Theorem. *The minimum r and the maximum R of the moduli of zeros of the polynomial $p(z)$ are given by:*

$$r=\frac{1}{\varlimsup\limits_{n\to\infty}\sqrt[n]{\left|\dfrac{A^{(n)}(0)}{n!}\right|}}\qquad\text{and}\qquad R=\varlimsup_{n\to\infty}\sqrt[n]{\left|\frac{B^{(n)}(0)}{n!}\right|}$$

Proof. Clearly, the radius of convergence of the Taylor series of $A(z)$ which is valid in the disk about 0 is equal to r. But then, the formula for r mentioned in the Theorem is derived via the Cauchy-Hadamard formula for the radius of convergence of that Taylor series.

Again, clearly, the inner radius of convergence of the Laurent series of $A(z)$ which is valid in the infinite annulus about 0 is equal to R. For $k=0, 1, 2,\ldots$ if we replace in that Laurent series z^{-k} by w^k, we obtain the Taylor series (about 0) of $B(w)$. But then the formula for R mentioned in the Theorem is the inverse of the Cauchy-Hadamard formula for the radius of convergence of that Taylor series.

Corollary. *Let $p(z)=a_0+\ldots+a_nz^n$ with $a_0\neq0$ be a polynomial over complex numbers and let $A(z)$ and $B(w)$ be defined as in the above. Then the precise annulus which contains all the zeros of $p(z)$ is given by:*