que si $T(X)=\overline{T(X)}$, la dimension de l'espace $X/T(X)$ est finie si et seulement si il existe une constante $\nu^2 \in R_+^*$ telle que $q(E^*) \leqslant \nu^2 q[T^*(E^*)]$, quel que soit l'ensemble borné $E^* \subset X^*$.

Si on utilise un théorème de Kato, (voir [1], pp. 37), on arrive à la conclusion que si la dimension de l'espace $X^*/T^*(X^*)$ est finie l'opérateur T satisfait aux conditions : (a) et $\dim X/T(X) < \infty$ si et seulement si il existe $\eta_1^2, \eta_2^2 \in R_+^*$ tels que $q(E) \leqslant \eta_2^2 q[T(E)]$ quel que soit $E \in M(X)$, et $q(E^*) \leqslant \eta_2^2 q[T^*(E^*)]$ quel que soit $E^* \in M(X^*)$.

Nous nous proposons d'utiliser ces résultats dans la théorie de la perturbation des opérateurs et dans la Théorie généralè des algèbres Calkin.

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ON CONVEX FUNCTIONS FROM A FUZZ X TO A FUZZ Y

BY

T. T. BUHĂESCU

1. Introduction. The notion of „fuzz" is suggested by M. A. Erceg [2] : a fuzz is a completely distributive lattice with an involutive reversing order. The fuzzy sets collection in space X according to L. A. Zadeh [6] with the laticeal operations—reunion and intersection—as well as with the involutive reversing order given by complementaries, is a fuzz. Generalizing the properties of the F-type mappings : $F: \mathcal{P}(X) \to \mathcal{P}(Y)$ usually obtained from the function $f: X \to Y$ through $F(A)=\{f(x) \mid x \in A,\ A \in \mathcal{P}(X)\}$ to the mappings between two fuzzy spaces, M. A. Erceg comes to the concept of fuzz function [$\mathcal{P}(X)$ the parts set in a common sense]. In this work it is the convex (concave) fuzz functions that are introduced and their algebrical properties are studied.

2. Preliminaries. A fuzzy set in X according to L. A. Zadeh [6] is determined by its membership function $A: X \to [0, 1]=L$. We shall denote both the fuzzy set A and the membership function with the same letter A. The union and intersection operations as well as the notion of convex fuzzy set are also those of Zadeh [6].

Definition 2.1. *Let X be a real vector space, A and B fuzzy sets in X. The sum $A+B$ and the product by scalars λA, $\lambda \in R$ are given by :*

$$(1) \qquad (A+B)(x)=\sup_{y+z=x} \min\,[A(y), B(z)] \quad \textit{for all} \quad x \in X,$$

$$(2) \qquad (\lambda A)(x)=A\left(\frac{1}{\lambda}x\right) \quad \textit{for any} \quad \lambda \neq 0 \quad \textit{and all} \quad x \in X,$$

$$(3) \qquad (0A)(x)=\begin{cases} 0 & \textit{if} \ x \neq 0, \\ \sup\limits_{z \in X} A(z) & \textit{if} \ x=0. \end{cases}$$

(see A. K. Katsaras and D. B. Liu [3]).

Proposition 2.2. *Let X be a real vector space, and $A, B \in L^X$. For any $p \in (0, 1)$ the convex combination $pA+(1-p)B$ satisfies the relation :*

$$(4) \qquad (pA+(1-p)B)(x)=\sup_{h \in X} \min\left[A(x-h),\ B\left(x+\frac{ph}{1-p}\right)\right] \quad \textit{for all} \quad x \in X.$$

The proof immediately result from 2.1.

Consequence 2.3. *For any* $p\in[0,1]$ *and* $A, B\in L^X$ *the relation:*

(5) $$pA+(1-p)B\supseteq A\cap B \text{ holds.}$$

Proposition 2.4. *The fuzzy set A in the vector space X is convex if and only if the following relation:*

(6) $$pA+(1-p)A=A \text{ holds for any } p\in[0,1].$$

For proof see [1].

Definition 2.5. *Let* L^X *and* L^Y *be two fuzz spaces, with the involutive reversing order denoted „'". A mapping* $F: L^X\to L^Y$ *is called fuzz function if it satisfies the following conditions:*

I. $F(\emptyset)=\emptyset$.

II. $F(\bigcup_i A_i)=\bigcup_i F(A_i)$, *for any index set.*

III. $F^{-1}(A')=[F^{-1}(A)]'$, *where* $F^{-1}: L^Y\to L^X$ *is*

(7) $$F^{-1}(B)=\cap\{A/A\in L^X,\ F(A')\subseteq B'\}, \text{ for all } B\in L^Y.$$

Proposition 2.6. *If* $F: L^X\to L^Y$ *is a fuzz function, then the following relations hold:*

(8) $$(F^{-1}\circ F)(A)\supseteq A \text{ for all } A\in L^X.$$

(9) $$(F\circ F^{-1})(B)\subseteq B \text{ for all } B\in L^Y.$$

The function composition is taken in an usual sense;
For proof see [2].

Definition 2.7. *The fuzzy point in X with the support „u" and the value α is the fuzzy set* $<u,\alpha>$ *given by the following relation:*

(10) $$<u,\alpha>(x)=\begin{cases}\alpha & \text{if } x=u\\ 0 & \text{if } x\neq u\end{cases}$$

(see C. K. Wong [5]).

3. Convex Fuzz Functions. Definition 3.1. *Let X and Y be real vector spaces. The fuzz function* $F: L^X\to L^Y$ *is called convex, concave if for any* $p\in[0,1)$ *and any* $A_1, A_2\in L^X$ *it satisfies the following relations, respectively:*

(11) $$F(pA_1)+(1-p)A_2)\subseteq pF(A_1)+(1-p)F(A_2).$$

(12) $$F(pA_1)+(1-p)A_2)\supseteq pF(A_1)+(1-p)F(A_2).$$

The convex combinations are taken according to definition 2.1.

Lemma 3.2. *Let X be a real vector space. The convex combination of two fuzzy points satisfies the relation:*

(13) $$p<u,\alpha>+(1-p)<v,\beta> = <p\,.\,u+(1-p)\,.\,v,\ \min(\alpha,\beta)>$$

where $p\,.\,u$ *is the product by scalars in X.*

Proof. From definitions 2.7. and 2.1. it results:

(14) $$(p<u,\alpha>)(x)=\begin{cases}\alpha & \text{if } \frac{1}{p}x=u,\\ 0 & \text{if } \frac{1}{p}x\neq u.\end{cases}$$

From 2.7. it also results:

(15) $$<p\,.\,u,\alpha>(x)=\begin{cases}\alpha & \text{if } x=p\,.\,u,\\ 0 & \text{if } x\neq p\,.\,u.\end{cases}$$

From (14) and (15), we obtain:

(16) $$p<u,\alpha> = <p\,.\,u,\alpha>,$$

Consequently it results:

(17) $$\begin{aligned}(p<u,\alpha>+(1-p)<v,\beta>)(x)&=(<p\,.\,u,\alpha>+<(1-p)\,.\,v,\beta>)(x)=\\ &=\sup_{y+z=x}\min[<p\,.\,u,\alpha>(y),<(1-p)\,.\,v,\beta>(z)]=\\ &=\begin{cases}0 & \text{if } y\neq p\,.\,u \text{ or } z\neq(1-p)\,.\,v\\ \min(\alpha,\beta) & \text{if } y=p\,.\,u \text{ and } z=(1-p)\,.\,v.\end{cases}\end{aligned}$$

Likewise (13) is true if $p\in\{0,1\}$ and thus the lemma is proved.

Theorem 3.3. *Let X be a real vector space and* $F: L^X\to L^R$ *a convex (concave) fuzz function. If L. is {0, 1} then the function* $f: X\to R$ *defined through* $f(x)=F(\{x\})$ *for any* $x\in X$ *is convex (concave) in the common sense.*

Proof. By $\{x\}$ we understand the set with the only element x which can be assimilated with the fuzzy point $<x,1>$. According to lemma 3.2, we obtain:

(18) $$\begin{aligned}p\{u\}+(1-p)\{v\}&=p<u,1>+(1-p)<v,1> = <p\,.\,u+(1-p)\,.\,v,1> =\\ &=\{p\,.\,u+(1-p)\,.\,v\}.\end{aligned}$$

On the other hand:

(19) $$\begin{aligned}f(p\,.\,u+(1-p)\,.\,v)&=F(\{p\,.\,u+(1-p)\,.\,v\})=F(p\,.\,\{u\}+(1-p(\{v\})\\ &\subseteq(\supseteq)pF(\{u\})+(1-p)F(\{v\})=p\,.\,f(u)+(1-p)\,.\,f(v).\end{aligned}$$

The last relation leads to:

(20) $$f(p\,.\,u+(1-p)\,.\,v)\leqslant(\geqslant)p\,.\,f(u)+(1-p)\,.\,f(v)$$

and the theorem is proved.

Remark 3.4. The theorem 3.3. shows that the definitions of the fuzz function and of the convexity are in fact natural generalizations of the functions and of the classical convexity.

Proposition 3.5. *If $F : L^X \to L^Y$ is a convex (concave) fuzz function, then the inverse function $F^{-1} : L^Y \to L^X$ is concave (convex), and conversely.*

Proof. a) F convex implies F^{-1} concave. From condition II definition 2.5. it results that F and F^{-1} are monotonous with respect to the order defined by inclusion. Let $p \in [0, 1]$ and $B, C \in L^X$; applying (8), it follows:

(21) $$pF^{-1}(B)+(1-p)F^{-1}(C) \subseteq F^{-1}\{F[pF^{-1}(B)+(1-p)F^{-1}(C)]\}.$$

The right side of the last relation together with definition 3.1. and the monotonicity of the function F^{-1} lead to:

(22) $$F^{-1}\{F[pF^{-1}(B)+(1-p)F^{-1}(C)]\} \subseteq F^{-1}[pF(F^{-1}(B))+(1-p)F(F^{-1}(C))].$$

From (21), (22), considering (9), the monotonicity of the function F^{-1} as well as the monotony of the fuzzy sets sum, we obtain:

(23) $$pF^{-1}(B)+(1-p)F^{-1}(C) \subseteq F^{-1}(pB+(1-p)C),$$

which shows that F^{-1} is concave.

b) F^{-1} concave implies F convex. For any $p \in (0, 1)$. $A, B \in L^X$, applying (9) we obtain:

(24) $$pF(A)+(1-p)F(B) \supseteq F\{F^{-1}[pF(A)+(1-p)F(B)]\}.$$

The monotonicity of function F and the concavity of function F^{-1} applied in the right side of (24) lead to:

(25) $$F\{F^{-1}[pF(A)+(1-p)F(B)]\} \supseteq F[pF^{-1}(F(A)+(1-p)F^{-1}(F(B))].$$

From (24), (25) together with (8) and the monotonicity of F as well as the fuzzy sets sum, it results:

(26) $$pF(A)+(1-p)F(B) \supseteq F(pA+(1-p)B),$$

which shows that F is a convex fuzz function.

Remark 3.6. The previous proposition renders, in a much more general way, the well known theorem for real functions stating that by reversing, we may change convex with concave and conversely.

Theorem 3.7. *If $F : L^X \to L^Y$ is a concave fuzz function, then for any convex fuzzy set A in X, its image $F(A)$ is a convex fuzzy set in Y.*

Proof. Taking in (12) $A_1 = A_2 = A$ we obtain:

(27) $$F(pA+(1-p)A) \supseteq pF(A)+(1-p)F(A).$$

Considering the proposition 2.4 and the consequence 2.3, the last relation leads to:

(28) $$F(A) \supseteq pF(A)+(1-p)F(A) \supseteq F(A).$$

From (28) we obtain

(29) $$pF(A)+(1-p)F(A)=F(A).$$

equality, which according to proposition 2.4., ends the proof.

Consequence 3.8. *If $F : L^X \to L^Y$ is a convex fuzz function, then, through F^{-1}, a convex fuzzy set in X corresponds to a convex fuzzy set in Y. The affirmation results from 3.5. and 3.7.*

Proposition 3.9. *If $F : L^X \to L^Y$ is a convex (concave) fuzz function, then the families of fuzzy sets:*

(30) $$\underline{N}_F^B = \{A \mid A \in L^X,\ F(A) \subseteq B,\ B \in L^Y\}.$$

(31) $$\overline{N}_F^B = \{A \mid A \in L^X,\ F(A) \supseteq B,\ B \in L^Y\}$$

are closed as related to the convex combination in L^X for any B convex fuzzy set in Y.

Proof. Let us show the theorem for example, for $\overline{N}_F^B$. Let F be a concave fuzz function and B a convex fuzzy set in Y. If $A_1, A_2 \in \overline{N}_F^B$, then $F(A_1) \supseteq B$, $F(A_2) \supseteq B$ and consequently

(32) $$pF(A_1)+(1-p)F(A_2) \supseteq pB+(1-p)B = B.$$

The last relation, together with (12) lead to:

(33) $$F(pA_1+(1-p)A_2) \supseteq B,$$

which shows that: $pA_1+(1-p)A_2 \in \overline{N}_F^B$ for any $p \in [0, 1]$.

Remark 3.10. Proposition 3.9. reminds of the well-known theorem of level sets for convex (concave) functions $f : X \to R$.

Proposition 3.11. *Let X and Y be topological vector spaces and L^X, L^Y fuzzy topological spaces endowed with the topology induced according to M. D. Weiss [4]. If $F : L^X \to L^Y$ is an open concave fuzz function, then the degree of separation of the fuzzy sets $F(A_1)$ and $F(A_2)$, if neither of then includes the other one, A_1 and A_2 being convex fuzzy sets, satisfies:*

(34) $$M \geqslant \sup_p \min_y F'(pA_1+(1-p)A_2)(y),$$

where

(35) $$B'(y) = 1 - B(y)$$

for all $y \in Y$

Proof. Taking theorem 3.7. into consideration, the proposition is a consequence of the separation theorem of convex fuzzy sets of L. A. Zadeh [6] and M. D. Weiss [4]. Consequently the maximum degree of separation of the open fuzzy sets $F(A_1)$ and $F(A_2)$ is

(36) $$M = 1 - \sup_y \min [F(A_1)(y), F(A_2)(y)].$$

From the concavity of the fuzz function F, it follows:

(37) $$F(pA_1+(1-p)A_2)(y) \geqslant (pF(A_1)+(1-p)F(A_2))\ (y) \geqslant \geqslant \min [F(A_1)(y), F(A_2)\ (y)].$$

Taking supremum in (37) with respect to $y \in Y$, we obtain:

(38) $$\sup_y F(pA_1+(1-p)A_2)(y) \geqslant \sup_y \min [F(A_1)(y), F(A_2)(y)].$$

From (36) and (38), it follows:

$$(39)\qquad M \geq 1 - \sup_{y} F(pA_1+(1-p)A_2)(y) = \min_{y}[1-F(pA_1+(1-p)A_2)(y)].$$

Taking supremum with respect to $p \in (0, 1)$, from the last relation, (34) is obtained and thus the proposition is proved.

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MIXED TOPOLOGY FOR A FUNCTION SPACE

BY

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§ 1. **Introduction.** In [1] J. C. Kelly introduced the idea of a bitopological space. In this paper, we have considered a bitopological space whose elements are functions. We have investigated a topology called mixed topology which is determined in a natural way by two given topologies determined in a topological space Z. The definition of mixed topology as offered here, shall probably show that it is a natural neighbourhood topology corresponding to a bitopological space; in many results, proved here, it will also show that the idea of mixed topology is a natural extension (might be on a subspace) of the topology on a bitopological space. In section 3, we have discussed the properties of the mixed topology obtained from the point open topology and the compact open topology for a function space Z. In section 4, the properties of the mixed topology obtained from the topology of uniform convergence and the point open topology for a function space Z, have been investigated. The connection between spaces with mixed topology and bitopological spaces makes it possible, more naturally, to apply the theory of topological spaces to the investigation of bitopological spaces.

2. **Preliminaries. Definition 2.1.** *Let E and F be topological spaces and $Z \subset F^E$ be endowed with the compact open topology* $\mathbf{C}$ [2] *and the point open topology* $\mathbf{P}$ [2]. *Consider the bitopological space* $(Z, \mathbf{C}, \mathbf{P})$. *Let* $\mathbf{A}$ *be the family of all* $\mathbf{P}$ *compact sets. Consider the subspaces* $(A, \mathbf{C}_A)$ *for each* $A \in \mathbf{A}$. *Let τ be the finest topology on Z such that each inclusion map* i_A: $A \subset Z$ *is continuous for* $A \in \mathbf{A}$. *Then τ is said to be a mixed topology on Z.*

In [2], J. L. Kelley defined equicontinuous and evenly continuous family of functions as follows:

Definition 2.2. *Let Z be a family of maps of a topological space E into a uniform space* $(F, \mathbf{V})$. *The family Z is equicontinuous at a point x, if and only if for each member V of* $\mathbf{V}$ *there is a neighbourhood U of x such that* $f[U] \subset V[f(x)]$ *for every member f of Z.*

A family Z of functions is equicontinuous iff it is equicontinuous at every point.

Definition 2.3. *Let Z be a family of functions each on a topological space E to a topological space F. The family Z is evenly continuous if and only if for each x in E, each y in F, and each neighbourhood U of y, there is a neighbourhood V of x and a neighbourhood W of y such that* $f[V] \subset U$ *whenever* $f(x) \in W$.