From (36) and (38), it follows:

$$M \geq 1 - \sup_{y} F(pA_1 + (1-p)A_2)(y) = \min_{y} \{1 - F(pA_1 + (1-p)A_2)(y)\}. \tag{39}$$

Taking supremum with respect to $p \in (0, 1)$, from the last relation, (34) is obtained and thus the proposition is proved.

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MIXED TOPOLOGY FOR A FUNCTION SPACE

BY

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§ 1. **Introduction.** In [1] J. C. Kelly introduced the idea of a bitopological space. In this paper, we have considered a bitopological space whose elements are functions. We have investigated a topology called mixed topology which is determined in a natural way by two given topologies determined in a topological space Z. The definition of mixed topology as offered here, shall probably show that it is a natural neighbourhood topology corresponding to a bitopological space; in many results, proved here, it will also show that the idea of mixed topology is a natural extension (might be on a subspace) of the topology on a bitopological space. In section 3, we have discussed the properties of the mixed topology obtained from the point open topology and the compact open topology for a function space Z. In section 4, the properties of the mixed topology obtained from the topology of uniform convergence and the point open topology for a function space Z, have been investigated. The connection between spaces with mixed topology and bitopological spaces makes it possible, more naturally, to apply the theory of topological spaces to the investigation of bitopological spaces.

2. **Preliminaries. Definition 2.1.** *Let E and F be topological spaces and $Z \subset F^E$ be endowed with the compact open topology* $\mathbf{C}$ [2] *and the point open topology* $\mathbf{P}$ [2]. *Consider the bitopological space* $(Z, \mathbf{C}, \mathbf{P})$. *Let* $\mathbf{A}$ *be the family of all* $\mathbf{P}$ *compact sets. Consider the subspaces* $(A, \mathbf{C}_A)$ *for each* $A \in \mathbf{A}$. *Let τ be the finest topology on Z such that each inclusion map* $i_A : A \subset Z$ *is continuous for* $A \in \mathbf{A}$. *Then τ is said to be a mixed topology on Z.*

In [2], J. L. Kelley defined equicontinuous and evenly continuous family of functions as follows:

Definition 2.2. *Let Z be a family of maps of a topological space E into a uniform space $(F, \mathbf{V})$. The family Z is equicontinuous at a point x, if and only if for each member V of $\mathbf{V}$ there is a neighbourhood U of x such that $f[\bar{U}] \subset V[f(x)]$ for every member f of Z.*

A family Z of functions is equicontinuous iff it is equicontinuous at every point.

Definition 2.3. *Let Z be a family of functions each on a topological space E to a topological space F. The family Z is evenly continuous if and only if for each x in E, each y in F, and each neighbourhood U of y, there is a neighbourhood V of x and a neighbourhood W of y such that $f[V] \subset U$ whenever $f(x) \in W$.*

§3. **Mixed topology for a bitopological space** $(Z, \mathbf{C}, \mathbf{P})$. **Theorem 3.1.** *$(Z, \tau)$ is a Hausdorff space if F is a Hausdorff space.*

Proof. Let $f \neq g$ be any two distinct points of Z. We know that if F is a Hausdorff space then $(Z, \mathbf{P})$ is a Hausdorff space. So there exist two disjoint $\mathbf{P}$ open sets U and V such that $f \in U$, $g \in V$ and $U \cap V = O$. Since $\mathbf{P} \subset \mathbf{I}$, U and V are also $\mathbf{I}$ open sets. Now $i_A^{-1}(U) = A \cap U$ is $\mathbf{C}_A$ open and $i_A^{-1}(V) = A \cap V$ is $\mathbf{C}_A$ open, where $A \in \mathbf{A}$. Consequently U and V are τ open sets. Hence (Z, τ) is a Hausdorff space.

Theorem 3.2. *Let F be a regular space and the members of Z are continuous. Then (A, τ_A) is regular for every $A \in \mathbf{A}$.*

Proof: We know that if F is a regular space and the members of Z are continuous then $(Z, \mathbf{C})$ is regular. Then $(A, \mathbf{C}_A)$ is regular. Now $i_A : (A, \mathbf{I}_A) \to (A, \tau_A)$ is a homeomorphism. So (A, τ_A) is regular for every $A \in \mathbf{A}$.

Theorem 3.3. *Let F be a Hausdorff space and $(Z, \mathbf{C})$ be a T_4 space. Then (A, τ_A) is a T_4 space for every $A \in \mathbf{A}$.*

Proof: First we observe that F^E is a Hausdorff space in the point open topology. So $(Z, \mathbf{P})$ is a Hausdorff space. Now every $A \in \mathbf{A}$ is $\mathbf{P}$ compact. So A is $\mathbf{P}$ closed in Z. Again $\mathbf{P} \subset \mathbf{C}$. So A is $\mathbf{C}$ closed in Z. By hypothesis $(Z, \mathbf{C})$ is a T_4 space. Consequently $(A, \mathbf{C}_A)$ is T_4 space for every $A \in \mathbf{A}$. Now $i_A : (A, \mathbf{C}_A) \to (A, \tau_A)$ is a homeomorphism. Hence (A, τ_A) is a T_4 space for every $A \in \mathbf{A}$.

Theorem 3.4. *If F is compact and Z is pointwise closed then τ is identical with $\mathbf{I}$.*

Proof: If F is compact, F^E is compact in the point open topology. Since Z is pointwise closed, $(Z, \mathbf{P})$ is compact. Then $Z \in \mathbf{A}$. By the construction of τ, $i_z : (Z, \mathbf{C}) \to (Z, \tau)$ is a homeomorphism. Hence τ is identical with $\mathbf{I}$.

Theorem 3.5. *If E is a locally compact Hausdorff space and each member of Z is continuous on every compact subset of E, then the mixed topology τ is a topology of joint continuity* [2].

Proof: First we shall show that every member of Z is a continuous mapping of E into F. Suppose $f \in Z$. Then the restriction f/K on each compact subset K of E to F is continuous. Let $\{V_x$ for each $x \in E$ where V_x is a relatively compact open neighbourhood of $x\}$ be a covering of E. Let U be an open set in F. Then $f^{-1}(U) = \cup \{f_{V_x}^{-1}(U)$ for each $x \in E\}$ where f_{V_x} is the restriction f/V_x. Since $f_{V_x} : V_x \to F$ is continuous, $f_{V_x}^{-1}(U)$ is open in V_x and hence open in E. Then $f^{-1}(U)$ is open in E and so f is a continuous mapping E into F.

Let $\Phi : Z \times E \to F$ be a mapping defined as $\Phi(f, x) = f(x)$ for $f \in Z$ and $x \in E$. We shall prove that Φ is continuous. Let M be an open set in F and let $(f, x) \in \Phi^{-1}(M)$. Since E is locally compact for every $x \in E$, there is a relatively compact open neighbourhood V_x of x. Again f is continuous. there is an open neighbourhood V of x such that $f(V) \subset M$. Since E is T_3 space, V contains a closed neighbourhood W of x. Then $W \cap V_x$ is a neighbourhood of x and $\overline{W \cap V_x} \subset \overline{V_x}$ and so $\overline{W \cap V_x}$ is compact. Let $\overline{W \cap V_x} = P$. Then $P \subset \overline{W} = W \subset V$. So $f(P) \subset f(V) \subset M$. Let $T(P, M) = \{g \in Z : g(P) \subset M\}$. Then $T(P, M)$ is $\mathbf{C}$ open. Now $i_A^{-1}[T(P, M)] = A \cap T(P, M)$ is $\mathbf{C}_A$ open. So $T(P, M)$ is τ open. Then $T(P, M) \times P$ is a neighbour-

hood of (f, x) in $Z \times E$, contained in $\Phi^{-1}(M)$. Hence $\Phi^{-1}(M)$ is open in $Z \times E$ when Z is endowed with the topology τ. Hence τ is a topology of joint continuity.

Theorem 3.6. *If E is regular or Hausdorff and each member of Z is continuous on every compact subset of E, then τ is jointly continuous on compacts* [2].

Proof: Let K be a compact subset of E. Let P be a map of $Z \times K$ into F, where $P(f, x) = f(x)$. Suppose $x \in K$, U be open in F and $(f, x) \in P^{-1}(U)$. We shall show that $P^{-1}(U)$ is open in $Z \times K$ where Z is endowed with the mixed topology τ.

If E is regular, then K is regular. If E is Hausdorff, then K is compact Hausdorff and hence regular. Since f is continuous on K, there exists a neighbourhood V of x such that $f(v) \subset U$. Since K is regular and V is a neighbourhood of the compact set $\{x\}$, there exists a closed neighbourhood M of x such that $M \subset V$. Since M is a closed subset of the compact set K, M is compact. We note that $f(M) \subset U$. Now $T(M, U) = \{g : g(M) \subset U\}$ is a subbasic open set in $\mathbf{C}$ and $f \in T(M, U)$. Again $i_A^{-1}[T(M, U)] = A \cap T(M, U)$ is $\mathbf{C}_A$ open. So $T(M, U)$ is τ open. Then $T(M, U) \times M$ is a neighbourhood of (f, x) in $Z \times K$ where Z is endowed with the topology τ.

Now we shall show that $T(M, U) \times M \subset P^{-1}(U)$. For any $y \in M$ and $g \in T(MU,)$, we have $g(y) \in U$. So for any $(g, y) \in T(M, U) \times M$, $P(g, y) = g(y) \in U$. Hence $T(M, U) \times M \subset P^{-1}(U)$. So $P^{-1}(U)$ is open in $Z \times K$ when Z is endowed with τ. Hence τ is jointly continuous on compacta.

Theorem 3.7. *Let Z be evenly continuous family on function on a topological space E to a space F. Let $(Z, \mathbf{I}, \mathbf{P})$ be a bitopological space. Then τ is jointly continuous on Z.*

Proof: Let $\Phi : Z \times E \to F$ be defined as $\Phi(f, x) = f(x)$, $f \in Z$ and $x \in E$. Let U be a neighbourhood of $f(x)$. Since Z is evenly continuous there is a neighbourhood V of x and a neighbourhood W of $f(x)$ such that $f[V] \subset U$, whenever $f(x) \in W$. Then $M = \{f : f \in Z$ and $f(x) \in W\}$ is a $\mathbf{P}$ neighbourhood. Since $\mathbf{P} \subset \mathbf{C}$, M is a $\mathbf{C}$-neighbourhood. Again $i_A^{-1}(M) = A \cap M$ is a $\mathbf{C}_A$ neighbourhood. So M is a τ neighbourhood of f. Then $M \times V$ is a neighbourhood of (f, x) and $\Phi(M \times V) \subset U$. Hence Φ is continuous.

Theorem 3.8. *Let E be a topological space and $(F, \mathbf{V})$ be a uniform space. Suppose $(Z, \mathbf{C}, \mathbf{P})$ be a bitopological space and Z be an equicontinuous family. Then the mixed topology τ is jointly continuous.*

Proof: Let $\Phi : Z \times E \to F$ be defined as $\Phi(f, x) = f(x)$ where $f \in Z$ and $x \in E$. Let $(f, x) \in Z \times E$. Since Z is equicontinuous for a member V of $\mathbf{V}$ there exists a neighbourhood U of x such that $g(U) \subset V[g(x)]$ for all g in Z. Let $M = \{t : t(x) \in V[f(x)]\}$ be a $\mathbf{P}$ neighbourhood of f. Then M is a $\mathbf{C}$ neighbourhood. Again $i_A^{-1}(M) = A \cap M$ is a $\mathbf{C}_A$ neighbourhood. So M is a τ neighbourhood of f. If $g \in M$ and $y \in U$ then $g(y) \in V[g(x)]$ and $g(x) \in V[f(x)]$; consequently $g(y) \in V \circ V[f(x)]$. Therefore $\Phi(M \times U) \subset V \circ V[f(x)]$ where $(M \times U)$ is a neighbourhood of (f, x) in $Z \times E$ when Z is endowed with τ. Hence τ is jointly continuous.

§4. **Topology of uniform convergence.** Let Z be a family of functions on a topological space E, to a uniform space $(F, \mathbf{V})$. Let $\mathbf{U}$ be the uniformity of uniform convergence [2]. Let $\tau(\mathbf{U})$ be the topology of uniform convergence. Consider the bitopological space $(Z, \tau(\mathbf{U}), \mathbf{P})$. Let τ be the mixed topology on Z, constructed from $\tau(\mathbf{U})$ and $\mathbf{P}$ exactly in the same way as in section 2.

Theorem 4.1. *Let $(Z, \tau(\mathbf{U}), \mathbf{P})$ be a bitopological space and each member of Z is continuous. Then the mixed topology τ is jointly continuous.*

Proof: Let $\Phi : Z \times E \to F$ be a mapping defined as $\Phi(f, x) = f(x)$ for $f \in Z$ and $x \in E$. Suppose $(f, x) \in Z \times E$. Then $\Phi(f, x) = f(x)$. Let $V_1[f(x)]$ be a neighbourhood of $f(x)$, where V_1 is a member of $\mathbf{V}$. For V_1 there exists a member V of V such that $V \circ V \subset V_1$. Then $V[f(x)]$ is also a neighbourhood of $f(x)$. Since f is a continuous mapping of E into F, there exists a neighbourhood U of X such that $f(U) \subset V[f(x)]$. Let $W(V)[f]$ be a neighbourhood of f in the $\tau(\mathbf{U})$ topology. Since $i_A^{-1}[W(V)[f]] = A \cap [W(V)[f]]$ is a $(\tau(\mathbf{U}))_A$ open set, $W(V)[f]$ is τ open. Then $W(V)[f] \times U$ is a neighbourhood of (f, x). We shall show that $\Phi(W(V)[f] \times U) \subset V_1[f(x)]$. Let $(g, y) \in W(V)[f] \times U$. Then $\Phi(g, y) = g(y)$ i.e. we have to show that $(f(x), g(y)) \in V_1$. Now $(f(x), f(y)) \in V$ and $(f(y), g(y)) \in V$. So $(f(x), g(y)) \in V \circ V \subset V_1$. Hence τ is jointly continuous.

In [2] J. L. Kelley defined the topology of uniform convergence on compacta as follows:

Definition 4.1. *Let Z be a family of functions on a set E to a uniform space $(F, \mathbf{V})$ and $\mathbf{R}$ be the family of all compact subsets of E. Then the uniformity $\mathbf{U}|\mathbf{R}$ of uniform convergence on compacta, has for a subbase the family of all sets of the form $\{(f, g) : (f(x), g(x)) \in V$ for all x in $K\}$, for V in $\mathbf{V}$ and K in R. The topology $\tau(\mathbf{U}|\mathbf{R})$ of $\mathbf{U}|\mathbf{R}$ is called the topology of uniform convergence on compacta.*

Theorem 4.2. *Let Z be the family of continuous functions on a topological space to a uniform space $(F, \mathbf{V})$. Let $(Z, \tau(\mathbf{U}|\mathbf{R}), \mathbf{P})$ be a bitopological space and τ be the mixed topology of $\tau(\mathbf{U}|\mathbf{R})$ and $\mathbf{P}$. Then $\tau_A = \mathbf{C}_A$ for every $\mathbf{P}$ compact subset A of Z and $\mathbf{C}$ is the compact open topology on Z.*

Proof: First we shall show that $\mathbf{C}_A \subset \tau_A$. Let K be a compact subset of E and U be an open subset of F. Then $T(K, U) = \{f : f \in Z \text{ and } f(K) \subset U\}$ is a $\mathbf{C}$ open set. Let $f \in T(K, U)$. Then $f(K) \subset U$. Since $f(K)$ is compact, there exists a member V of V such that $V[f(K)] \subset U$. Now $M = \{(f, g) | (f(x), g(x)) \in V$ for all x in $K\}$ is a member of $\mathbf{U}|\mathbf{R}$. If $g \in M$, then $g(x) \in V[f(x)]$ for each x in K and finally $g(K) \subset U$. Then $g \in T(K, U)$. So $T(K, U) \in \tau(\mathbf{U}|\mathbf{R})$. From the construction of τ, $i_A^{-1}[T(K, U)] = A \cap T(K, U)$ is $(\tau(\mathbf{U}|\mathbf{R}))_A$ open. Hence $T(K, U) \in \tau$. Consequently $\mathbf{C} \subset \tau$. Therefore $\mathbf{C}_A \subset \tau_A$.

Now we shall show that $\tau_A \subset \mathbf{C}_A$. Let M be a τ_A open set. Then $i_A^{-1}(M) = M$ is a $(\tau(\mathbf{U}|\mathbf{R}))_A$ open set. For $f \in M$ there exists a $\tau(\mathbf{U}|\mathbf{R})$ open set $V(f) = \{g : (f(x), g(x)) \in V$ for all x in a compact subset K of E and $V \in \mathbf{V}\}$ such that $f \in V(f) \subset A \cap M$. For $V \in \mathbf{V}$, there exists a closed symmetric member W of $\mathbf{V}$ such that $W \circ W \circ W \subset V$. We choose $x_1, x_2, \dots, x_n$ in K such that the sets $W[f(x_i)]$ cover $f[K]$. Let $K_i = K \cap f^{-1}[W[f(x_i)]]$. Then K_i is compact for each i. Suppose U_i be the interior of $W \circ W[f(x_i)]$. Then $T(K_i, U_i)$ for each i is compact open and $f \in T(K_i, U_i)$. Let $g \in T(K_i, U_i)$. Then $g[K_i] \subset U_i$ and for each x in K there is i such that $x \in K_i$. Hence $g(x) \in W \circ W[f(x_i)]$. Again $f(x) \in W[f(x_i)]$. So $(g(x), f(x)) \in V$

for each x in K i.e. for each x in k_1. Therefore $T(K_i, U_i) \subset V(f)$. Consequently $f \in A \cap T(K_i, U_i) \subset A \cap V(f) \subset M$. So $M \in \mathbf{C}_A$ and hence $\tau_A \subset \mathbf{C}_A$.

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