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UNIFORMITIES WITH A DILATATION FUNCTION

BY

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1. The notion of a probabilistic uniform structure, introduced in [3], can be considered as axiomatization of some situations in which we know the probability of the events „$(x, y) \in U$ for every $U \in \mathfrak{U}$", where $\mathfrak{U}$ is an uniformity on a set X. This model is the probabilistic variant of the uniform structures; the relations between the probabilistic topological structures (see [2]), the probabilistic uniform structures and the Menger's probabilistic metric structures (see [1]) are analogous with the relations between the topological structures, the uniform structures and the metric structures on a set.

The dilatation function is a convenient tool for an investigation of probabilistic uniformities; so, a left-continuous probabilistic uniformity prove to be an uniform structure equipped with an estimate operator of „the entourages dilatation". In a metric space, it is possible to estimate the dilatation of entourages of uniformity generated by metric; we infer that the left-continuous probabilistic uniformities are placed between the uniform structures and the metric structures.

The purpose of this paper is to study the notion of *dilatation function* and to give a characterization of left-continuous probabilistic uniformities. We establish also, three necessary and sufficient conditions in order that a dilatation function be induced by a metric.

Let I be the closed unit interval; a *t*-function is a mapping $T : I \times I \to I$ for which $T(\lambda', \mu') \leqslant T(\lambda'', \mu'')$ if $\lambda' \leqslant \lambda''$, $\mu' \leqslant \mu''$.

Let X be a set and let $\Delta = \{(x, x) : x \in X\}$ be the diagonal of the cartesian product $X \times X$. A semiuniformity $\mathfrak{V}$ on X is a filter in $X \times X$ which satisfies the following conditions:

a) $\Delta \subseteq U$, for every $U \in \mathfrak{V}$,

b) $U^{-1} \in \mathfrak{V}$, for every $U \in \mathfrak{V}$.

Let $\mathfrak{U} = \{\mathfrak{U}_\lambda : \lambda \in I\}$ be a family of semiuniformities on X and let T be a *t*-function. The pair $(\mathfrak{U}, T)$ is called a *probabilistic uniform structure* on X, if the following conditions hold:

(PU0) $\mathfrak{U}_0 = \{X \times X\}$,

(PU1) $\mathfrak{U}_\mu \subseteq \mathfrak{U}_\lambda$, for every $\lambda, \mu \in I$, with $\mu \leqslant \lambda$,

(PU2) for every $U \in \mathfrak{U}_{T(\lambda, \mu)}$ there exists $V \in \mathfrak{U}_\lambda$, $W \in \mathfrak{U}_\mu$ such that $V \circ W \subseteq U$.

A *left-continuous probabilistic uniformity* on X is a probabilistic uniformity $(\mathfrak{U}, T)$ for which:

$$\mathfrak{U}_\lambda = \cup \{\mathfrak{U}_\mu : \mu < \lambda\}, \text{ for every } \lambda \in I - \{0\}.$$

2. The dilatation function. Definition 1. *Let $\mathcal{V}$ be an uniformity on X and let T be a t-function. A dilatation function for $(\mathcal{V}, T)$ is a function $d : \mathcal{V} \to [0, 1)$ which satisfies the following conditions :*

(D1) $d(X \times X) = 0$,

(D2) $d(U \cap V) = \sup(d(U), d(V))$, for every $U, V \in \mathcal{V}$,

(D3) $d(U^{-1}) = d(U)$, for every $U \in \mathcal{V}$,

(D4) *for every* $\lambda, \mu \in I$ $U \in \mathcal{V}$ *with* $d(U) < T(\lambda, \mu)$ *there exists* $V, W \in \mathcal{V}$ *with* $d(V) < \lambda$, $d(W) < \mu$ *such that* $V \circ W \subseteq U$.

An uniformity with a dilatation function is a triple $(\mathcal{V}, T, d)$ consisting of an uniformity $\mathcal{V}$, a t-function T and a dilatation function d for $(\mathcal{V}, T)$.

Let $\mathfrak{D}_T$ be the family of all uniformities with a dilatation function and let $\mathfrak{U}_T$ be the family of all left-continuous probabilistic uniformities on X with the same t-function T, which satisfies the condition $T^{-1}(1) \neq \emptyset$.

Theorem 1. *The mapping $\alpha : \mathfrak{D}_T \to \mathfrak{U}_T$, defined by $\alpha((\mathcal{V}, T, d)) = (\mathfrak{U}, T)$, where $\mathfrak{U} = \{\mathfrak{U}_\lambda\}_{\lambda \in I}$, $\mathfrak{U}_\lambda = \{U \in \mathcal{V} : d(U) < \lambda\}$ for every $\lambda \in (0, 1]$ and $\mathfrak{U}_0 = \{X \times X\}$, is a bijection.*

Proof. First, we note that, for every uniformity with a dilatation function $(\mathcal{V}, T, d)$, $\alpha((\mathcal{V}, T, d))$ is a left-continuous probabilistic uniformity, hence the definition of α is right. Now, for every $(\mathfrak{U}, T) \in \mathfrak{U}_T$, $\mathfrak{U} = \{\mathfrak{U}_\lambda : \lambda \in I\}$, $\mathfrak{U}_1$ is an uniformity on X. Define a mapping $d : \mathfrak{U}_1 \to [0, 1)$ letting $d(U) = \inf\{\lambda : U \in \mathfrak{U}_\lambda\}$, for every $U \in \mathfrak{U}_1$. Since $(\mathfrak{U}, T)$ is left-continuous, $d(U) < 1$ for every $U \in \mathfrak{U}_1$. (D1) $d(X \times X) = \inf\{\lambda : X \times X \in \mathfrak{U}_\lambda\} = 0$; (D2) let $U, V \in \mathfrak{U}_1$; it follows from $U \cap V \subseteq U, V$ that $d(U \cap V) \geqslant \sup(d(U), d(V))$. Now, for every $\varepsilon > 0$, from the definition of d, there exists λ', λ'' such that $U \in \mathfrak{U}_{\lambda'}$, $V \in \mathfrak{U}_{\lambda''}$ and $d(U) > \lambda' - \varepsilon$, $d(V) > \lambda'' - \varepsilon$. Then $\sup(d(U), d(V)) > \sup(\lambda', \lambda'') - \varepsilon$. From $U \cap V \in \mathfrak{U}_{\sup(\lambda', \lambda'')}$, it follows $d(U \cap V) < \sup(d(U), d(V)) + \varepsilon$. This give the desired proof of (D2). (D3) is a consequence of the equality $\{\lambda : U \in \mathfrak{U}_\lambda\} = \{\lambda : U^{-1} \in \mathfrak{U}_\lambda\}$. (D4) For every $\lambda, \mu \in I$, $U \in \mathfrak{U}_1$ with $d(U) < T(\lambda, \mu)$, it follows from the definition of d that $U \in \mathfrak{U}_{T(\lambda, \mu)}$, hence, there exists $V \in \mathfrak{U}_\lambda$, $W \in \mathfrak{U}_\mu$ such that $V \circ W \subseteq U$. Obviously, $d(V) \leqslant \lambda$ and $d(W) \leqslant \mu$. From the left-continuity of $(\mathfrak{U}, T)$, $d(V) \leqslant \lambda$ and $d(W) \leqslant \mu$. Hence $(\mathfrak{U}_1, T, d) \in \mathfrak{D}_T$. For any $\lambda \in (0, 1]$ $U \in \mathfrak{U}_\lambda$ iff $d(U) < \lambda$; we shall thus have $\alpha((\mathfrak{U}_1, T, d)) = (\mathfrak{U}, T)$.

Now, let $(\mathfrak{U}', T, d')$ be another uniformity with a dilatation function such that $\alpha((\mathfrak{U}', T, d)) = (\mathfrak{U}, T)$. It follows from $\mathfrak{U}_1 = \{U \in \mathfrak{U}' : d'(U) < 1\}$ that $\mathfrak{U}_1 \subseteq \mathfrak{U}'$; on the other hand, for every $U \in \mathfrak{U}'$ $d'(U) < 1$, hence $U \in \mathfrak{U}_1$. Thus $\mathfrak{U}_1 = \mathfrak{U}'$. The fact that $d = d'$ is quite obvious, as we remark that for every $U \in \mathfrak{U}_1$, $\lambda \in I$, $d'(U) < \lambda$ iff $d(U) < \lambda$.

Remark 1. Each concept which is based on the notion of a left-continuous probabilistic uniformity can be described in terms of a uniformity with a dilatation function, because each left-continuous probabilistic uniformity $(\mathfrak{U}, T)$ is completely determined by $\alpha^{-1}((\mathfrak{U}, T))$.

3. Metrization theorems. The purpose of this section is to compare uniformities with dilatation function and pseudometrics on a set. The comparison is an example of the standard procedure for testing the effectiveness of a generalization. In this case it will be shown that each pseudometric generates an uniformity with a dilatation function. It will also be given three necessary and sufficient conditions in order that an uniformity with a dilatation function can be derived from a pseudometric.

Let T_m be the t-function defined by $T_m(\lambda, \mu) = \max(\lambda + \mu - 1, 0)$, for every $\lambda, \mu \in I$ and let X be a set.

Definition 2. *Let $(\mathfrak{U}, T_m, d)$ be an uniformity with a dilatation function on X. We shall say that d is complete iff :*

$d(\cap\{U_\gamma : \gamma \in \Gamma\}) = \sup\{d(U_\gamma) : \gamma \in \Gamma\}$, *for every* $\{U_\gamma : \gamma \in \Gamma\} \subseteq \mathfrak{U}$ *with* $\cap\{U_\gamma : \gamma \in \Gamma\} \in \mathfrak{U}$.

For every $x, y \in X$ we note $C(x, y) = X \times X - \{(x, y)\}$.

Remark 2. A dilatation function d for $(\mathfrak{U}, T_m)$ is complete iff : $d(U) = \sup\{d(C(x, y)) : (x, y) \overline{\in} U\}$, for every $U \in \mathfrak{U}$.

Let us denote by $\mathfrak{D}_m$ the family of all uniformities with a complete dilatation function with the t-function T_m, and by $\mathfrak{M}$ the family of all pseudometrics $\rho : X \times X \to I$.

Theorem 2. *The mapping $\beta : \mathfrak{M} \to \mathfrak{D}_m$ defined by $\beta(\rho) = (\mathfrak{U}, T_m, d)$, where $\mathfrak{U}$ is the uniformity generated by ρ and $d : \mathfrak{U} \to [0, 1)$, $d(U) = 1 - \inf\{\rho(x, y) : (x, y) \overline{\in} U\}$ for every $U \in \mathfrak{U} - \{X \times X\}$ and $d(X \times X) = 0$, is a one-to-one mapping.*

Proof. For every $\rho \in \mathfrak{M}$, let $\mathfrak{U}$ be the uniformity generated by ρ : for any $U \in \mathfrak{U}$ there is $\alpha > 0$ such that $B_\alpha = \{(x, y) : \rho(x, y) < \alpha\} \subseteq U$. Consequently, if $(x, y) \overline{\in} U$ then $\rho(x, y) \geqslant \alpha$; hence $d(U) = 1 - \inf\{\rho(x, y) : (x, y) \overline{\in} U\} \leqslant 1 - \alpha < 1$. (D1) and (D3) follow from the definition of d. (D4) If $\lambda, \mu \in I$, $U \in \mathfrak{U}$ with $d(U) < T_m(\lambda, \mu) = \lambda + \mu - 1$ then $\alpha = \inf\{\rho(x, y) : (x, y) \overline{\in} U\} > (1 - \lambda) + (1 - \mu)$. There are $\alpha_1 > 1 - \lambda$ and $\alpha_2 > 1 - \mu$ such that $\alpha = \alpha_1 + \alpha_2$. Then $d(B_{\alpha_1}) \leqslant 1 - \alpha_1 < \lambda$, $d(B_{\alpha_2}) \leqslant 1 - \alpha_2 < \mu$ and for each $(x, y) \in B_{\alpha_1} \circ B_{\alpha_2}$ there exists z such that $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y) < \alpha_1 + \alpha_2 = \alpha$; hence $(x, y) \in U$. For every $U \in \mathfrak{U}$ if $(x, y) \overline{\in} U$ then $C(x, y) \in \mathfrak{U}$ and $d(C(x, y)) = 1 - \rho(x, y)$. Then $d(U) = 1 - \inf\{\rho(x, y) : (x, y) \overline{\in} U\} = \sup\{1 - \rho(x, y) : (x, y) \overline{\in} U\} = \sup\{d(C(x, y)) : (x, y) \overline{\in} U\}$, hence d is a complete dilatation function for $(\mathfrak{U}, T_m)$. If $\rho_1, \rho_2 \in \mathfrak{M}$ and $\beta(\rho_1) = \beta(\rho_2)$, $\beta(\rho_1) = (\mathfrak{U}^1, T_m, d_1)$, $\beta(\rho_2) = (\mathfrak{U}^2, T_m, d_2)$, then $\mathfrak{U}^1 = \mathfrak{U}^2$ and $d_1 = d_2$. For any $x, y \in X$ if $\rho_1(x, y) = 0$ then $\rho_2(x, y) = 0$ (if $\rho_2(x, y) > 0$ it follows $C(x, y) \in \mathfrak{U}^2 = \mathfrak{U}^1$; then there is $\alpha > 0$ such that $(x, y) \in B^1_\alpha = \{(u, v) : \rho_1(u, v) < \alpha\}$, hence $\rho_1(x, y) \geqslant \alpha > 0$). If $\rho_1(x, y) > 0$ there is $\alpha > 0$ such that $C(x, y) \supseteq B^1_\alpha$, hence $C(x, y) \in \mathfrak{U}^1 = \mathfrak{U}^2$. Then it follows that $d_1(C(x, y)) = d_2(C(x, y))$. But $d_1(C(x, y)) = 1 - \rho_1(x, y) = d_2(C(x, y)) = 1 - \rho_2(x, y)$. Hence $\rho_1(x, y) = \rho_2(x, y)$ for every $x, y \in X$.

Remark 3. It may be noticed that, if $\rho \in \mathfrak{M}$ and $\beta(\rho) = (\mathfrak{U}, T_m, d)$, then d estimates the dilatation of entourages of uniformity $\mathfrak{U}$. If $\rho_1, \rho_2 \in \mathfrak{M}$ are two pseudometrics which induce the same uniformity, then $\rho_1 = \rho_2$ iff their dilatation functions coincide.

We shall now state three necessary and sufficient conditions in order that an uniformity with a dilatation function $(\mathfrak{U}, T_m, d) \in \beta(\mathfrak{M})$.

Let us observe that if $(\mathfrak{U}, T_m, d) \in \mathfrak{U}_m$ and $\Delta_{\mathfrak{U}} = \cap \mathfrak{U}$ then $\mathfrak{U}^* = \{A \subseteq X \times X : A \supseteq \bigcap_{\gamma \in \Gamma} U_\gamma$ where $\{U_\gamma\}_{\gamma \in \Gamma} \subseteq \mathfrak{U}\}$ is the filter generated by $\Delta_{\mathfrak{U}}$

Define a mapping $d^*:\mathfrak{U}^*\to I$ letting $d^*(A)=\inf\{\sup_{\gamma\in\Gamma} d(U_\gamma) : \{U_\gamma\}_{\gamma\in\Gamma}\subseteq\mathfrak{U},\ A\supseteq\bigcap_{\gamma\in\Gamma}\mathfrak{U}_\gamma\}$, for every $A\in\mathfrak{U}^*$. The triple $(\mathfrak{U}^*, T_m, d^*)$ is called the canonical extension of the uniformity with a dilatation function $(\mathfrak{U}, T_m, d)$.

Remark 4. For every $A\in\mathfrak{U}^*$ $d^*(A)=\sup\{d^*(C(x, y)) : (x, y)\overline{\in} A\}$.

Proof. For every $(x, y)\overline{\in} A$ $C(x, y)\supseteq A$ and $d^*(A)\geq d^*(C(x, y))$. Conversely, for every $\varepsilon>0$ and $(x, y)\overline{\in} A$ there exists $\{U_\gamma^{x,y}: \gamma\in\Gamma_{x,y}\}\subseteq\mathfrak{U}$ such that $C(x, y)\supseteq\cap\{U_\gamma^{x,y}:\gamma\in\Gamma_{x,y}\}$ and $\sup\{d(U_\gamma^{x,y}):\gamma\in\Gamma_{x,y}\}<d^*(C(x, y))+\varepsilon$. Then $\sup\{\sup\{d(U_\gamma^{x,y}):\gamma\in\Gamma_{x,y}\}:(x, y)\overline{\in} A\}\leq\sup\{d^*(C(x, y)):(x, y)\overline{\in} A\}+\varepsilon$. But $A=\cap\{C(x, y):(x, y)\overline{\in} A\}\supseteq\cap\{\cap\{U_\gamma^{x,y}:\gamma\in\Gamma_{x,y}\}:(x, y)\overline{\in} A\}$, hence $d^*(A)\leq\sup\{\sup\{d(U_\gamma^{x,y}):\gamma\in\Gamma_{x,y}\}:(x, y)\overline{\in} A\}$. Therefore $d^*(A)\leq d^*(C(x, y))$.

Remark 5. d^* is an extension of d.

Proof. If $C(x, y)\in\mathfrak{U}$ then $d^*(C(x, y))\leq d(C(x, y))$ and for every $\varepsilon>0$ there exists $\{U_\gamma:\gamma\in\Gamma\}\subseteq\mathfrak{U}$ such that $C(x, y)\supseteq\cap\{U_\gamma:\gamma\in\Gamma\}$ and $d^*(C(x, y))>\sup\{d(U_\gamma):\gamma\in\Gamma\}-\varepsilon$; then there is $\gamma_0\in\Gamma$ such that $(x, y)\overline{\in} U_{\gamma_0}$, hence $d(U_{\gamma_0})\geq d(C(x, y))$. Then $d^*(C(x, y))>d(C(x, y))-\varepsilon$ and it follows that $d^*(C(x, y))=d(C(x, y))$. For every $U\in\mathfrak{U}$ $d(U)=\sup\{d(C(x, y)):(x, y)\overline{\in} U\}=\sup\{d^*(C(x, y)):(x, y)\overline{\in} U\}=d^*(U)$.

Now, for every $(\mathfrak{U}, T_m, d)\in\mathfrak{U}_m$ denote by:

$$u(U)=\{(x, y) : C(x, y)\in\mathfrak{U},\ d(C(x, y))>d(U)\}\cup\Delta_{\mathfrak{U}},$$

$$U_\alpha=\{(x, y) : C(x, y)\in\mathfrak{U},\ d(C(x, y))>\alpha\}\cup\Delta_{\mathfrak{U}},$$

for every $U\in\mathfrak{U}$ and $\alpha\in[0, 1)$.

Theorem 3. *Let $(\mathfrak{U}, T_m, d)$ be an uniformity with a complete dilatation function and let $(\mathfrak{U}^*, T_m, d^*)$ be the canonical extension of $(\mathfrak{U}\ T_m, d)$; suppose $\mathfrak{U}^*\neq\mathfrak{U}$. Then the following statements are equivalent:*

(i) $(\mathfrak{U}, T_m, d)\in\beta(\mathfrak{M})$;

(ii) *the restriction of d^* to $\mathfrak{U}^*-\mathfrak{U}$ is* 1;

(iii) *for every $U\in\mathfrak{U}$ $u(U)\in\mathfrak{U}$ and* $1\in\overline{d(\mathfrak{U})}$;

(iv) *for every $\alpha\in[0, 1)$ $U_\alpha\in\mathfrak{U}$,*

where $\overline{d(\mathfrak{U})}$ is the closure in the usual topology for I of image of $\mathfrak{U}$ under d.

Proof. (i)$\Rightarrow$(ii): If $(\mathfrak{U}, T_m, d)\in\beta(\mathfrak{M})$ there is $\rho\in\mathfrak{M}$ such that $(\mathfrak{U}, T_m, d)=\beta(\rho)$. For every $A\in\mathfrak{U}^*-\mathfrak{U}$ $d^*(A)=\sup\{d^*(C(x, y)):(x, y)\overline{\in} A\}$. But $\mathfrak{U}^*=\{A\subseteq X\times X : A\supseteq\Delta_{\mathfrak{U}}\}$ and if $(x, y)\overline{\in} A$ then $\rho(x, y)>0$; hence $C(x, y)\in\mathfrak{U}$. Therefore $d^*(A)=1-\inf\{\rho(x, y):(x, y)\overline{\in} A\}$. If $\alpha=\inf\{\rho(x, y):(x, y)\overline{\in} A\}>0$ then $B_\alpha\subseteq A$ and $A\in\mathfrak{U}$. Hence $d^*|_{\mathfrak{U}^*-\mathfrak{U}}=1$.

(ii)$\Rightarrow$(iii): For every $U\in\mathfrak{U}$ $\Delta_{\mathfrak{U}}\subseteq u(U)$; hence $u(U)\in\mathfrak{U}^*$. $d^*(u(U))=\sup\{d^*(C(x, y)):(x, y)\overline{\in} u(U)\}=\sup\{d(C(x, y)):C(x, y)\in\mathfrak{U},\ d(C(x, y))\leq d(U)\}\leq d(U)<1$. Therefore $u(U)\in\mathfrak{U}$. If $1\overline{\in}\overline{d(\mathfrak{U})}$ there is $\varepsilon>0$ such that $d(U)<1-\varepsilon$ for every $U\in\mathfrak{U}$. Now, for every $A\in\mathfrak{U}^*-\mathfrak{U}$ and $\{U_\gamma:\gamma\in\Gamma\}\subseteq\mathfrak{U}$ such that $A\supseteq\bigcap_{\gamma\in\Gamma}U_\gamma$ we have $d^*(A)\leq\sup\{d(U_\gamma):\gamma\in\Gamma\}\leq 1-\varepsilon$; hence $d^*(A)\neq 1$.

(iii)$\Rightarrow$(iv): For every $\alpha\in[0, 1)$ there is $U\in\mathfrak{U}$ such that $\alpha<d(U)<1$. Then $u(U)\in\mathfrak{U}$ and by $u(U)\subseteq U_\alpha$, $U_\alpha\in\mathfrak{U}$.

(iv)$\Rightarrow$(i): Define a function $\rho:X\times X\to I$ by letting $\rho(x, y)=1-d(C(x, y))$ if $(x, y)\overline{\in}\Delta_{\mathfrak{U}}$ and $\rho(x, y)=0$ iff $(x, y)\in\Delta_{\mathfrak{U}}$. Since $(x, x)\in\Delta_{\mathfrak{U}}$ $\rho(x, x)=0$. It is evident that $\rho(x, y)=\rho(y, x)$. Now, let x, y, z be three points; if $\rho(x, y)>\rho(x, z)+\rho(y, z)$ and $\rho(x, z)>0=\rho(y, z)$ we have $d(C(x, y))<d(C(x, z))=T_m(d(C(x, z)), 1)$. It follows by (D4) that there are $U, V\in\mathfrak{U}$ such that $d(U)<d(C(x, z))$, $d(V)<1$ and $U\circ V\subseteq C(x, y)$. But $d(U)<d(C(x, z)$ implies that $(x, z)\in U$; hence $(x, y)\in U\circ V\subseteq C(x, y)$ $((y, z)\in\Delta_{\mathfrak{U}}\subseteq V)$. If $\rho(x, z)>0$, $\rho(z,y)>0$ and $\rho(x, y)>\rho(x, z)+\rho(y, z)$ then $d(C(x, y))<T_m(d(C(z, y), d(C(x, z)))$. Hence there are $U, V\in\mathfrak{U}$ such that $d(U)<d(C(z, y))$, $d(V)<d(C(x, z))$ and $U\circ V\subseteq C(x, y)$. It follows that $(z, y)\in U$ and $(x, z)\in V$; hence $(x, y)\in U\circ V\subseteq C(x, y)$. If $\rho(x, z)=0=\rho(z, y)$ then $(x, y)\in\Delta_{\mathfrak{U}}\circ\Delta_{\mathfrak{U}}=\Delta_{\mathfrak{U}}$ hence $\rho(x, y)=0$. It follows that $\rho\in\mathfrak{M}$. Now, let $\beta(\rho)=(\mathfrak{U}', T_m, d')$. For every $U\in\mathfrak{U}$ $d(U)<1$ hence there is $r>0$ such that $d(U)<1-r$. For any $(x, y)\in B_r=\{(u, v):\rho(u, v)<r\}$ $d(C(x, y))>1-r>d(U)$ hence $(x, y)\in U$. Therefore $\mathfrak{U}'\supseteq\mathfrak{U}$. For every $\alpha\in(0, 1]$ $B_\alpha=\{(x, y):\rho(x, y)<\alpha\}=\{(x, y):C(x, y)\in\mathfrak{U},\ d(C(x, y))>1-\alpha\}\cup\Delta_{\mathfrak{U}}=U_{1-\alpha}\in\mathfrak{U}$. Hence $\mathfrak{U}\supseteq\mathfrak{U}'$. For every $U\in\mathfrak{U}$ $d'(U)=\sup\{d'(C(x, y)):(x, y)\overline{\in} U\}=\sup\{1-\rho(x, y):(x, y)\overline{\in} U\}=\sup\{d(C(x, y)):(x, y)\overline{\in} U\}=d(U)$. Therefore $(\mathfrak{U}, T_m, d)=\beta(\rho)$.

Remark 6. If $(\mathfrak{U}, T_m, d)\in\mathfrak{D}_m$ and $\mathfrak{U}^*=\mathfrak{U}$ *then* $(\mathfrak{U}, T_m, d)\in\beta(\mathfrak{M})$.

Proof. Define a function $\rho:X\times X\to I$ by letting $\rho(x, y)=1-d(C(x, y))$ iff $(x, y)\overline{\in}\Delta_{\mathfrak{U}}$ and $\rho(x, y)=0$ iff $(x, y)\in\Delta_{\mathfrak{U}}$. Notice first that we already know that $\rho\in\mathfrak{M}$, $\beta(\rho)=(\mathfrak{U}', T_m, d)$ and $\mathfrak{U}'\supseteq\mathfrak{U}$ (see the part (iv)$\Rightarrow$(i) in the proof of the previous theorem). It means that in order to show that $(\mathfrak{U}, T_m, d)\in\beta(\mathfrak{M})$ we have only to proof $\mathfrak{U}'\subseteq\mathfrak{U}$. It will follow if we remark that $\Delta_{\mathfrak{U}}\in\mathfrak{U}$.

REFERENCES

1. Menger K. — *Statistical metrics*, Proc. Nat. Acad. Sci. USA, 28 (1942), 535–537.
2. Frank M. J. — *Probabilistic topological spaces*, J. Math. Anal. Appl. 34 (1971), 67–81.
3. Florescu L. — *Structures syntopogènes probabilistes*, Com. Math. Debrecen (to appear) (see Zbl. 381. 54001. 1979).

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