En considérant l'application $\tau_2 : \mathcal{D}^1_2(M) \to V\mathcal{D}^1_1(E)$ donnée par

$$\tau_2\left(f^i_{hj}\frac{\partial}{\partial x^i}\otimes dx^h \otimes dx^j\right)(t_p) = \left(f^i_{hj}y^j_l\frac{\partial}{\partial y^i_l}\otimes dx^h\right), \tag{41}$$

on a

Proposition 4.3. *Un champ de tenseurs $F \in \mathcal{D}^1_1(E)$ est invariant à droite si et seulement s'il existe $\alpha \in \mathcal{F}(M)$ et $f \in \mathcal{D}^1_2(M)$ tels que*

$$F = \alpha^V I + \tau_2(f), \tag{42}$$

où τ_2 est donnée par (41).

On obtient aussi

Proposition 4.4. *L'ensemble $R\mathcal{D}^1_1(E)$ des champs de tenseurs de type (1,1) sur E, invariants à droite, est un sous-module du module $V\mathcal{D}^1_1(E)$ sur $\mathcal{F}(M)^V$, isomorphe avec le module $\mathcal{F}(M) \times \mathcal{D}^1_2(M)$ sur $\mathcal{F}(M)$.*

Un raisonnement analogue à celui fait pour les champs de vecteurs invariants nous donne :

Proposition 4.5. *Un champ de tenseurs de type (1,1) sur E est invariant à gauche (resp. à droite) si et seulement si sa dérivée de Lie par rapport à tout champ de vecteurs invariant à droite (resp. à gauche) est égale à zero.*

Enfin, comme un champ de tenseurs (1,1) sur E est uniquement déterminé par ses valeurs sur les champ de vecteurs de la forme S^V avec $S \in \mathcal{D}^1_1(M)$ et X^H avec $X \in \mathcal{D}^1(M)$ et X^H le relèvement horizontal, calculé par rapport à une connexion quelconque sur M [2], on peut donner pour les champs de tenseurs de type (1,1) invariants sur E, la caractérisation suivante :

Proposition 4.6. *Un champ de tenseurs de type (1,1) sur E est invariant à gauche (resp. à droite) si et seulement s'il existe une fonction $\alpha \in \mathcal{F}(M)$, un champ de tenseurs $f \in \mathcal{D}^1_2(M)$ et une connexion linéaire ∇ sur M tels que pour tous $S \in \mathcal{D}^1_1(M)$ et $X \in \mathcal{D}^1(M)$ on a*

$$F(S^V) = \alpha^V S^V, \quad F(X^H) = \alpha^V X^H + \tau_1(i_X f), \tag{43}$$

$$(\text{resp. } F(S^V) = \alpha^V S^V, \quad F(X^H) = \alpha^V X^H + \tau_2(i_X f)),$$

où τ_1, τ_2 sont les applications définies respectivement par (36) et (41) et $i_X f$ le champ de tenseurs de type (1,1) sur M, donné par

$$(i_X f)(Y) = f(X, Y).$$

BIBLIOGRAPHIE

1. Bourbaki, N. — *Eléments de mathématique. Algèbre.* Chap. 2, Hermann, Paris, 1962, p. 51—52.
2. Cruceanu, V. — *Le fibré des tenseurs de type (1,1) sur une variété différentiable.* An. şt. Univ. Iaşi, T. XXVIII (1982), s. I a, f. 1, p. 45—54.
3. Cruceanu, V. — *Structures remarquables sur le fibré des tenseurs de type (1,1).* Mathematica (Cluj), 23 (1981).
4. Kobayashi, S. and Nomizu, K. — *Foundations of differential geometry.* Vol. 1, Interscience, 1963.

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Faculté de Mathématiques
Université de Iaşi
6600 — Iaşi
R. S. România

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TANGENT AND FRAME BUNDLES OF ORDER TWO

BY

C. T. J. DODSON and M. S. RADIVOIOVICI

I. The Tangent Bundle of Order two. Given a smooth n-manifold M with atlas $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$, we construct a bundle $T^{(2)}M$ from equivalence classes of curves in M which agree up to their acceleration. We adopt a coordinate-free approach to achieve essentially the same space that was constructed locally by Yano and Ishihara [3] (cf. also [2]).

Definition. $(\forall x \in M)$ *$C_x = \{$smooth $f : (-\varepsilon, \varepsilon) \to M \mid f(0) = x$, some $\varepsilon > 0\}$ and $(\forall f, h \in C_x)$ $f \sim_x h \Leftrightarrow \dot f(0) = \dot h(0)$,*

$$f \approx_x h \Leftrightarrow \dot f(0) = \dot h(0) \text{ and } (\ddot f(0) = \ddot h(0)$$

where $\dot f$ and $\ddot f$ are the tangent velocity and acceleration vector fields along the curve f :

$$\dot f : (-\varepsilon, \varepsilon) \to TM : t \to D_t f(1)$$

$$\ddot f : (-\varepsilon, \varepsilon) \to TTM : t \to D_t \dot f(1).$$

Remark. For each $x \in M$, $\sim_x$ and $\approx_x$ are equivalence relations. Moreover, $C_x/\sim_x$ is set-theoretically equivalent to T_xM. It can readily be given the same vector space structure and so we have $\bigcup_{x\in M} C_x/\sim_x$ a natural vector bundle with fibre $\mathbf{R}^n$ over M. Precisely, T is a functor from Man' to Vbun (cf. [1]).

We can begin in a similar way to use $\approx_x$:

$$T^{(2)}_x M = C_x/\approx_x, \quad T^{(2)}M = \bigcup_{x\in M} T^{(2)}_x M.$$

Then we have a fibre bundle $(T^{(2)}M, \Pi^{(2)}_T, M)$ with

$$\Pi^{(2)}_T : T^{(2)}M \to M : [f]_{\approx_x} \to x$$

continuous in the coinduced topology. In fact a $3n$-manifold structure is readily coinduced through the atlases :

$$(TU, D\varphi_\alpha)_{\alpha\in A} \text{ for } TM,$$

$$(TTU_{\alpha'}, DD\varphi_\alpha)_{\alpha\in A} \text{ for } TTM.$$

It is clearly possible to coinduce a vector space structure on $T^{(2)}_x M$ by requiring that $DD\varphi_\alpha|_x$ yields an isomorphism. But in general that structure will not linearly agree with a structure similarly coinduced via another chart about x. Historically, it was the failure of acceleration to transform like a vector that prompted the introduction of covariant differentiation of vector fields. The appropriate covariant derivative here arises from a linear connection ; that provides a consistent assignment of vector space structures to the fibres $T^{(2)}_xM$.

Theorem 1.1. *$T^{(2)}M$ is a vector bundle with fibre $\mathbf{R}^{2n}$ over M if and only if M has a linear connection ∇.*

Proof. (i) Given ∇ an M and a chart $(U_\alpha, \varphi_\alpha)$ about $x \in M$, then there is a diffeomorphism on $\Pi^{(2)}_T{}^{\leftarrow}(U_\alpha) = T^{(2)}U_\alpha$ given by

$$\begin{tikzcd}[column sep=small]
\Phi_\alpha : T^{(2)}U_\alpha \arrow[rr] \arrow[dr] & & U_\alpha \times \mathbf{R}^{2n} : & {[f]_{\approx_x}} \arrow[rr] \arrow[dr] & & (x, (\dot f^i_\alpha), (\ddot f^i_\alpha + \dot f^k_\alpha \dot f^m_\alpha \Gamma^i_{km})). \\
& U_\alpha \times TU_\alpha \times TU_\alpha \arrow[ur] & & & (x, \dot f(0), \nabla_{\dot f(0)}(\dot f(0))) \arrow[ur] &
\end{tikzcd}$$

This commutes with the projections onto U_α and yields fibre maps that are isomorphisms compatible under changes of chart about x. Each such diffeomorphism is well-defined because

$$f \approx_x h \Leftrightarrow \dot f(0) = \dot h(0) \text{ and } \nabla_{\dot f(0)} \dot f(0) = \nabla_{\dot h(0)} \dot h(0).$$

(ii) Suppose that $T^{(2)}M$ is a vector bundle with fibre $\mathbf{R}^{2n}$ over M. Then about every $x \in M$, there is a chart or subchart (U, φ) and a diffeomorphism

$$\Phi : T^{(2)}U \to U \times \mathbf{R}^{2n} : [f]_{\approx_x} \to (x, (f^i_1), (f^i_2)),$$

with each fibre map an isomorphism compatible with similar maps from other intersecting charts. We construct a chart $(U, \bar\varphi)$ about x using Φ by putting

$$D_x\bar\varphi \dot f(0) = (f^i_1) = (\bar\varphi \circ f)^{\cdot}|_0.$$

Now we shall construct a connection ∇ in LM locally by requiring that

$$D_x\bar\varphi \nabla_{\dot f(0)} \dot f(0) = (f^i_2).$$

To do this we must solve for Γ^i_{km} the equations

$$f^i_2 = \dot f^i_1 + f^k_1 f^m_1 \Gamma^i_{km}$$

in such a way that they yield the components of a connection satisfying

$$\nabla_{\partial_k} \partial_m = \Gamma^i_{km} \partial_i$$

where (∂_i) is the frame field determined by $(U, \bar\varphi)$. Suitable Γ^i_{km} can be found from two types of curves :

(a) For some $\varepsilon > 0$ we can find $h \in C_x$ satisfying for some k

$$\bar\varphi \circ h(t) = (0, \dots, 0, t, 0, \dots, 0), \text{ so } \dot h(t) = \delta^i_k \partial_i.$$

Then we find Γ^i_{kk} from $\varphi[h]_{\approx_x} = (x, (h^i_1), (h^i_2))$ by

$$h^i_2 = \Gamma^i_{kk}.$$

So n such curves yield all Γ^i_{kk} for $k = 1, \dots, n$.

(b) Next we find

$$\Gamma^i_{km} + \Gamma^i_{mk} + \Gamma^i_{kk} + \Gamma^i_{mm} = g^i_2$$

for a curve $g \in C_x$ satisfying for some k, m

$$\bar\varphi \circ g(t) = (0, \dots, 0, t, 0, \dots, 0, t, 0, \dots, 0) \text{ with } \dot g(t) = (\delta^i_k + \delta^i_m)\partial_i.$$

Finally we assume symmetry of the connection by putting

$$\Gamma^i_{km} = \Gamma^i_{mk} \text{ for all } i, k, m.$$

This process uses $1/2n^2(n+1)$ independent members of C_x to fix the Γ^i_{km}, as we expect it should. It remains to check that we get the same connection via another chart map $\bar\psi$ which determines frame fields $(\tilde\partial_i)$ about x and $1/2n^2(n+1)$ functions $\tilde\Gamma^i_{km}$. We know that there is a smooth map $\bar\varphi \circ \bar\psi^{-1}$ about $\bar\psi(x)$ and a corresponding change of frame

$$\tilde\partial_r = \partial_r x^i \partial_i,$$

which also relates Φ and Ψ coordinates for $[f]_{\approx_x}$ by

$$(\ddot f^i + \dot f^k \dot f^j \Gamma^i_{kj})\partial_i = (\ddot h^r + \dot h^l \dot h^m \tilde\Gamma^r_{lm})\tilde\partial_r,$$

$$\dot f^k = \dot h^l \partial_l x^k, \; \ddot f^i = \ddot h^r \partial_r x^i + \dot h^l \dot h^m \partial_{lm} x^i.$$

It follows that, as required for a connection,

$$\tilde\Gamma^r_{lm} \partial_r x^i = \Gamma^i_{kj} \partial_l x^k \partial_{lm} x^j + \partial_{lm} x^i.$$

Corollary 1. (Cf. [2]). *$T^{(2)}M$ is a $3n$-manifold with atlas $(T^{(2)}U_\alpha, \varphi^\alpha)_{\alpha \in A}$ where*

$$\begin{tikzcd}
\Phi^\alpha : T^{(2)}U_\alpha \arrow[rr] \arrow[dr, "\Phi_\alpha"'] & & \mathbf{R}^{3n} \\
& U_\alpha \times \mathbf{R}^{2n} \arrow[ur, "{(\varphi_\alpha, I)}"'] &
\end{tikzcd}$$

for any given connection ∇ in LM ; then $T^{(2)}M$ is also a vector bundle with fibre $\mathbf{R}^n$ over TM.

Proof. The atlas is clear enough. We have local diffeomorphisms

$$T^{(2)}U_\alpha \equiv TU_\alpha \times \mathbf{R}^n : [f]_{\approx_x} \to ([f]_{\sim_x}, D_x\varphi_\alpha \nabla_{\dot f(0)} \dot f(0))$$

and commutativity with projections because

$$\Pi^{(2,1)}_T : T^{(2)}M \to TM : (f]_{\approx_x} \to [f]_{\sim_x}$$

commutes with them.

Corollary 2. *Charts $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) about $x \in M$ determine a local section*

$$q_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(n\,;R) : x \to q_{\alpha\beta}(x)$$

where

$$D\varphi_\alpha \circ D\bar\varphi_\beta = (I, q_{\alpha\beta}) : TU_\beta \to TU_\alpha.$$

There is a corresponding section

$$Q_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(2n\,;\mathbf{R}) : x \to Q_{\alpha\beta}(x)$$

where

$$D\Phi_\alpha \circ D\Phi_\beta^{-} = (I, Q_{\alpha\beta}) : T^{(2)}U_\alpha \to T^{(2)}U_\beta$$

and

$$Q_{\alpha\beta} = \begin{pmatrix} q_{\alpha\beta} & 0 \\ 0 & q_{\alpha\beta} \end{pmatrix}.$$

Remark. In the rest of this paper we suppose that M has a linear connection ∇; a necessary and sufficient condition is paracompactness (cf. [1] p. 147). Then $T^{(2)}M$ has a structure simply related to that of the tangent bundle: its Whitney sum with itself.

Theorem 1.2. *There is a vector bundle isomorphism*

$$S : T^{(2)}M \equiv TM \oplus TM,\ a \to S_1(a) \oplus S_2(a).$$

Proof. We need a smooth assignation of fibre isomorphisms

$$x \to S_{(x)} : T^{(2)}_x M \equiv T_x M \oplus T_x M.$$

These are obtained from any charts $(U_\alpha, \Phi_\alpha), (U_\beta, \Phi_\beta)$ about $x \in M$ because we have diffeomorphisms

$$\begin{array}{ccc} T^{(2)}U_\alpha & \qquad & TU_\beta \oplus TU_\beta \\ \Phi_\alpha \big\downarrow & & \big\downarrow D\varphi_\beta \oplus D\varphi_\beta \\ U_\alpha \times \mathbf{R}^{2n} & & U_\beta \times \mathbf{R}^n \times U_\beta \times \mathbf{R}^n \end{array}$$

which each yield compatible isomorphisms on fibres by construction because

$$(D_x\varphi_\alpha)^{-1} \circ (D_x\varphi_\beta) \text{ and } \Phi^{-1}_{\alpha(x)} \circ \Phi_{\beta(x)}$$

are isomorphisms.

Corollary 1. *Given a chart (U, φ) about $x \in M$ we obtain frame fields*

$$U \to TU : x \to (\partial_i)_x$$

$$U \to T^2U : x \to (\delta_\lambda)_x$$

where

$$S(\delta_\lambda) = \partial_\lambda \oplus \underline{0},\ \lambda = 1, 2, \ldots, n$$

$$S(\delta_\lambda)\underline{0} = \oplus\, \partial_{\lambda-n},\ \ \lambda = n+1, \ldots, 2n.$$

II. Derivations and Scalar Products on $\mathbf{T^{(2)}M}$. We denote by TM the space of all C^∞ sections of TM and we denote by $T^{(2)}M$ the space of all C^∞ sections of $T^{(2)}M$.

Definition. *A derivation on $T^{(2)}M$ is a map $\bar\nabla$ such that*

$$(\forall x \in M)\ \bar\nabla_x : T_xM \times T^{(2)}M \to T^{(2)}_xM,$$

$$: (v, w) \to \bar\nabla_v w$$

satisfies the following properties

(i) *linearity*: $\forall \lambda, \mu \in \mathbf{R}$, $\bar\nabla_{\lambda v}\mu w = \lambda\mu\bar\nabla_v w$;
(ii) *smoothness*: $v \in TM$, $w \in T^{(2)}M \Rightarrow \bar\nabla_v w \in T^{(2)}M$;
(iii) *derivation*: $\forall C^\infty f : M \to \mathbf{R}$.

$$\bar\nabla_{f.v} w = f.\bar\nabla_v w \ \text{ and } \ \bar\nabla_v(f.w) = v(f).w + f.\bar\nabla_v w.$$

Theorem 2.1. *The connection ∇ in LM determines a derivation on TM, which we also denote by ∇, and a corresponding derivation $\tilde\nabla$ on $T^{(2)}M$.*

Proof. From Theorem 1.2, we obtain an isomorphism on fields;

$$\tilde S : T^{(2)}M \to TM \oplus TM : w \to \tilde S_1(w) \oplus \tilde S_2(w).$$

Hence we put:

$$(\forall x \in M)\ \tilde\nabla_x : T_xM \times T^{(2)}M \equiv T_xM \times TM \oplus TM \to T^{(2)}_xM.$$

$$: (v, w) \equiv (v, \tilde S_1(w) \oplus \tilde S_2(w)) \to S^{-1}(\nabla_v \tilde S_1(w) \oplus \nabla_v \tilde S_2(w)).$$

The desired properties of $\tilde\nabla$ follow from those of ∇. We note that, for a chart (U, φ) about x with respect to which

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k.$$

we find by linearity that

$$\tilde\nabla_{\partial_i}\delta_\lambda = S^{-1}(\nabla_{\partial_i}\partial_\lambda \oplus \underline{0}),\ \lambda = 1, \ldots, n,$$

$$\tilde\nabla_{\partial_i}\delta_\lambda = S^{-1}(\underline{0} + \nabla_{\partial_i}\partial_{\lambda-n}),\ \lambda = n+1, \ldots, 2n.$$

Hence the components of $\tilde\nabla$ are given by

$$\tilde\nabla_{\partial_i}\delta_\lambda = \tilde\Gamma^\tau_{i\lambda}\delta_\tau,\ i = 1, \ldots, n,$$

where

$$\tilde\Gamma^\tau_{i\lambda} = \Gamma^\tau_{i\lambda},\ \lambda, \tau = 1, \ldots, n, \quad \tilde\Gamma^\tau_{i\lambda} = \Gamma^{\tau-n}_{i\,\lambda-n},\ \lambda, \tau = n+1, \ldots, 2n,$$

and

$$\tilde\Gamma^j_{i\lambda} = \Gamma^\lambda_{ij} = 0,\ \lambda = n+1, \ldots, 2n,\ j = 1, \ldots, n.$$

Remark. It is known that every vector bundle admits a scalar product, simply by choosing an inner product on the fibre. For this yields a scalar product on each member of a locally finite open cover and these can be joined together smoothly via a subordinate partition of unity. However in general no unique scalar product is determined in this way.

On the other hand, a Riemannian structure ρ on M is a scalar product on TM and so we expect from Theorem 1.2 that it should fix a scalar product on $T^{(2)}M$. This is indeed the case and moreover ρ determines a derivation on $T^{(2)}M$ that is compatible with this structure.

Theorem 2.2. *If ρ is a Riemannian structure on (M, ∇), then there is a unique scalar product $\bar\rho$ on $T^{(2)}M$, so making $(T^{(2)}M, \bar\rho)$ a Riemannian bundle.*

Proof. We define each fibre inner product $\bar{\rho}_x$ by

$$\begin{tikzcd}[column sep=small] \bar{\rho}_x : T^{(2)}_x M \times T^{(2)}_x M \to \mathbf{R}, & \\ : (a, b) \arrow[r] \arrow[dr, "(S, S)"'] & q_x(a_1, b_1) + \rho_x(a_2, b_2) \\ & (a_1 \oplus a_2, b_1 \oplus b_2) \arrow[u] \end{tikzcd}$$

The required properties of bilinearity, symmetry and positive definiteness of $\bar{\rho}_x$ follow directly from that of ρ_x. Since S is a bundle isomorphism and $\bar{\rho}$ is a smooth section of the bundle of bilinear forms it follows that $\bar{\rho}$ is smooth and hence a scalar product on $T^{(2)}M$.

Theorem 2.3. *A scalar product $\bar{\rho}$ on $T^{(2)}M$ determines a unique compatible and $\oplus$-symmetric derivation $\bar{\nabla}$ on $T^{(2)}M$.* (By $\oplus$-symmetry we mean the natural symmetry arising from $T^{(2)}M \equiv \equiv TM \oplus TM$).

Proof. Any scalar product $\bar{\rho}$ determines a derivation $\bar{\nabla}$ satisfying the compatibility condition

$$(\forall v \in TM)\ (\forall a, b \in T^{(2)}M),\ v(\bar{\rho}(a, b)) = \bar{\rho}(\bar{\nabla}_v a, b) + \bar{\rho}(a, \bar{\nabla}_v b).$$

In components this becomes

$$\partial_i \bar{\rho}_{\lambda\mu} = \bar{\Gamma}^{\tau}_{i\lambda} \bar{\rho}_{\tau\mu} + \bar{\Gamma}^{\tau}_{i\mu} \bar{\rho}_{\tau\lambda}.$$

This is insufficient to fix $\bar{\nabla}$ uniquely but by Theorem 1.2 we know that the components must be of the form

$$[\bar{\Gamma}^{\tau}_{i\lambda}] = [A^{\tau} B^{\tau}] \text{ for each } \tau = 1, \ldots, 2n.$$

where A^{τ} and B^{τ} are $n \times n$ matrices. Hence a natural condition to impose on the $\bar{\Gamma}^{\tau}_{i\lambda}$ is symmetry in each of its component matrices A^{τ}, B^{τ}. This does determine the $\bar{\Gamma}^{\tau}_{i\lambda}$ uniquely and it can be checked that they in turn yield a derivation $\bar{\nabla}$ on $T^{(2)}M$.

Corollary 1. *If ∇^{ρ} is the Levi-Civita connection on a Riemannian manifold (M, ρ) then the derivation $\bar{\nabla}$ induced on $T^{(2)}M$ by $\bar{\rho}$ coincides with $\tilde{\nabla}^{\rho}$, as given in Theorem 2.1. That is, with slight abuse of domains,*

$$\bar{\nabla} = \tilde{S}^{-1} \circ (\nabla^{\rho} \oplus \nabla^{\rho}) \circ \tilde{S} = \tilde{\nabla}^{\rho}.$$

Corollary 2. The constructions above are easily adapted to a pseudo-Riemannian structure g on M. Then $(T^{(2)}M, \bar{g})$ is a pseudo-Riemannian bundle and the derivation $\bar{\nabla}$ determined on $T^{(2)}M$ coincides with that induced there by the Levi-Civita connection on (M, g).

Definition. *The torsion map of a derivation $\hat{\nabla}$ on $T^{(2)}M$ is the map*

$$T_{\hat{\nabla}} : TM \times T^{(2)}M \to T^{(2)}M$$

$$: (X, A) \to \hat{\nabla}_X A - \tilde{S}^{-1}(\Delta_{\tilde{S}_1(A)} X), \Delta_{\tilde{S}_2(A)} X)$$
$$- \tilde{S}^{-1}([X, S_1(A)], [X, \tilde{S}_2(A)]).$$

Theorem 2.4. *A derivation $\hat{\nabla}$ induced on $T^{(2)}M$ by a scalar product ρ on M coincides with $\tilde{\nabla}^{\rho}$ if and only if the torsion map $T_{\hat{\nabla}}$ is trivial (i.e. the zero map).*

Proof. The Levi-Civita connection ∇^{ρ} in LM is torsion-free and hence satisfies the symmetry condition

$$\nabla^{\rho}_X Y = \nabla^{\rho}_Y X - [X, Y] \quad (\forall X, Y \in TM).$$

From Corollary 1 to Theorem 2.3 we can write

$$\tilde{\nabla}^{\rho}_X A = \tilde{S}^{-1}(\nabla^{\rho}_X \tilde{S}_1(A), \nabla^{\rho}_X \tilde{S}_2(A)).$$

Hence, as expected $\tilde{\nabla}^{\rho}$ has trivial torsion map because ∇^{ρ} is torsion free. Conversely, if $\hat{\nabla}$ is induced by a scalar product ρ and it has trivial torsion map then by Theorem 2.3 it coincides with $\tilde{\nabla}^{\rho}$.

III. The frame bundle of order two. For any smooth n-manifold M, TM is a vector bundle with fibre $\mathbf{R}^n$ over M. It is an associated bundle of LM, the principal fibre bundle of frames for TM with structure group $Gl(n; \mathbf{R})$. A similar situation arises for $T^{(2)}M$.

Theorem 3.1. *The bundle $L^{(2)}M$ of all linear frames for the tangent spaces of order two over (M, ∇) is a principal fibre bundle over M with structure group $Gl(2n; \mathbf{R})$ and $T^{(2)}M$ is an associated bundle to $L^{(2)}M$.*

Corollary. *LM is isomorphic to a subbundle $L^{(1)}M$ of $L^{(2)}M$.*

Proof. Maps on local triviality neighbourhoods are given by

$$\begin{tikzcd} LU \arrow[d, "L_\alpha"'] & L^{(2)}U \arrow[d, "L^{(2)}_\beta"] \\ U \times Gl(n; \mathbf{R}) \arrow[r, "i"] & U \times Gl(2n; \mathbf{R}) \end{tikzcd}$$

for $U = U_\alpha \cap U_\beta$ and charts (U_α, Φ_α), (U_β, Φ_β) about $x \in U$ where

$$i : (x, B) \to \left(x, \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}\right).$$

Each L_α, $L^{(2)}_\beta$ is a diffeomorphism, well-behaved on fibres under change of chart. Hence there is a well-defined principal bundle morphism

$$l : LM \to L^{(2)}M,$$

and by construction LM is isomorphic to $L^{(1)}M = l(LM)$ which has structure group $Gl(n; \mathbf{R}) \oplus I$.

Remark. Corresponding to the canonical or basic 1-form θ on LM we have $\bar{\theta}$, where for all $\bar{u} \in L^{(2)}M$ we define

$$\begin{tikzcd} \tilde{\theta} : T_{\tilde{u}} L^{(2)}M \times T_{\tilde{u}} L^{(2)}M \arrow[rr] \arrow[d, "D\Pi^{(2)}_L \oplus D\Pi^{(2)}_L"'] & & \mathbf{R}^{2n} \\ TM \oplus TM \arrow[rr, "S^{-1}"] & & T^{(2)}M \arrow[u, "\Pi_{\tilde{u}} : X^i \tilde{u}_i \to (X^i)"'] \end{tikzcd}$$

Through expression in terms of matrices, we see the correspondence:

$$\theta : TLM \to \mathbf{R}^n : (x, b, X, A) \to b^{-1}X \quad (\text{cf. } [1], \text{ p. } 95)$$

$$\tilde{\theta} : ((x, b, X, A), (x, b, Y, B)) \to b^{-1} \begin{vmatrix} X \\ Y \end{vmatrix}.$$

Using the injection $l : LM \to L^{(2)}M$ of the foregoing Corollary.

$$\tilde{\theta} \circ (Dl, 0) = (I, 0) \circ \theta.$$

Theorem 2.3. *Every connection in LM determines a unique connection in $L^{(2)}M$ and so $T^{(2)}L^{(2)}M$ is a vector bundle if $T^{(2)}M$ is.*

Proof. The natural candidate is that connection in $L^{(2)}M$ which determines the derivation on $T^{(2)}M$ (from Theorem 2.1) arising from the connection in LM which puts the vector bundle structure on $T^{(2)}M$. We obtain it as follows:

Given a connection ∇ in LM there is a unique derivation $\tilde{\nabla}$ on $T^{(2)}M$ induced by the isomorphism

$$\tilde{S} : T^{(2)}M \to TM \oplus TM : w \to \tilde{S}_1(w) \oplus \tilde{S}_2(w).$$

We define $w \in T_{\tilde{u}} L^{(2)}M$ to be horizontal if there is a curve

$$\tilde{c} : (-\varepsilon, \varepsilon) \to L^{(2)}M : t \to (\tilde{c}_\alpha(t)), \text{ with } (\dot{\tilde{c}}(0) = w,$$

such that its $\tilde{\nabla}$-derivation is zero along the projection

$$\Pi^{(2)}_L \circ \tilde{c} = c : (-\varepsilon, \varepsilon) \to M : t \to c(t).$$

That is:

$$\tilde{\nabla}_{\dot{c}} \tilde{c}_\alpha = 0, \quad \alpha = 1, \ldots, 2n, \text{ and } \dot{\tilde{c}}(0) = w.$$

Now we put, for all $\tilde{u} \in L^{(2)}M$

$$G_{\tilde{u}} = \ker D\Pi^{(2)}_L |_{\tilde{u}}$$

$$H_{\tilde{u}} = \{w \in T_{\tilde{u}} L^{(2)}M \mid w \text{ is horizontal}\}.$$

Then it follows that

(i) $(\forall \tilde{u} \in L^{(2)}M)$ $T_{\tilde{u}} L^{(2)}M \equiv H_{\tilde{u}} \oplus G_{\tilde{u}}$,

(ii) $(\forall \tilde{u} \in L^{(2)}M)$ $D\Pi^{(2)}_L |_{H_{\tilde{u}}} : H_{\tilde{u}} \to T_{\Pi^{(2)}_L \tilde{u}} M$ is an isomorphism,

(iii) $(\forall g \in Gl(2n ; \mathbf{R}),\ \forall \tilde{u} \in L^{(2)}M)$ $DR_g H_{\tilde{u}} = H_{R_g \tilde{u}}$.

Also, the splitting in (i) is smooth over $\tilde{u} \in L^{(2)}M$ and hence we have a connection in $L^{(2)}M$, which we denote by $\tilde{\nabla}$.

Corollary 1. At any $\tilde{u}^{(2)} \in L^{(2)}M$ the splitting of $w \in T_{\tilde{u}} L^{(2)}M$ into horizontal and vertical components can be displayed as follows.

$$w = ((x^i), (b^\alpha_\beta \delta_\alpha), X^i \partial_i, (B^\alpha_\beta)) = w_H \oplus w_G,$$

$$w_H = ((x^i), (b^\alpha_\beta \delta_\alpha), X^i \partial_i, (-b^\lambda_\beta \tilde{\Gamma}^\alpha_{i\lambda} X^i)),$$

$$w_G = ((x^i), (b^\alpha_\beta \delta_\alpha), 0, (B^\alpha_\beta + b^\lambda_\beta \tilde{\Gamma}^\alpha_{i\lambda} X^i)).$$

Corollary 2. *Every vector field $v : M \to TM$ has:*

(i) *a unique horizontal lift $v^\uparrow : LM \to TLM$ through any given frame $u \in LM$*

(ii) *a unique second horizontal lift $v^\Uparrow : L^{(2)}M \to TL^{(2)}M$ through any given frame $\tilde{u} \in L^{(2)}M$.*

Similarly, every curve c in M has a unique ∇-horizontal curve through any given $u \in L_{c(0)}M$ and a unique $\tilde{\nabla}$-horizontal curve $c^\Uparrow$ through any given $\tilde{u} \in L^{(2)}_{c(0)}M$.

Corollary 3. The connection form $\tilde{\omega}$ of $\tilde{\nabla}$ is the map given by

$$\tilde{\omega} : TL^{(2)}M \to \mathfrak{gl}(2n ; \mathbf{R}) : (x, b, X, B) \to (B + b\tilde{\Gamma}X)b^{-1}$$

in matrix form (cf. [1] p. 118), where as usual we identify the Lie algebra $\mathfrak{gl}(2n ; \mathbf{R})$ with $\mathbf{R}^{4n^2}$. Also, from the injections

$$l : LM \to L^{(2)}M \text{ and } J : \mathfrak{gl}(2n ; \mathbf{R}), \text{ we have}$$

$$\tilde{\omega} \circ Dl = J \circ \omega$$

where ω is the connection form of ∇.

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REFERENCES

1. Dodson, C. T. J. — *Categories, Bundles and Spacetime Topology*. Shiva Publishing, Orpington, Kent, (1980).
2. Radivoiovici, M. S. — *On the geometry of the tangent bundle of order two*. Proc. Bolyai Math. Soc. (1979) (in press).
3. Yano, K. and Ishihara, S. — *Tangent and Cotangent Bundles* Dekker. New York (1973).

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University of Lancaster and
University of Bucharest