Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVIII, s. I a, 1982, f. 1

ON THE INDEX OF A CONFORMAL SYMMETRIC RIEMANNIAN MANIFOLD WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION OF K. YANO

BY

P. STAVRE and D. SMARANDA

In the present paper, starting from the results of Levin and Katzin, [2], on the index of a conformal euclidian manifold, we study the index of a conformal symmetric n-dimensional manifold related to the semi-symmetric metric connection of K. Yano, [4]. In §1, besides some important properties of Yano's connection, we formulate some preparatory results which are necessary in the following sections. In § 2 we obtain the solutions of a tensorial equation, and in §3 we establish that, if the manifold is conformally symmetric with respect to Yano's connection and there is a vector field U so that the covariant derivative of the Einstein tensor field on the direction of U vanishes and $d\overline{S}(U)\neq 0$, then the index is 1 ; if the Einstein tensor is covariantly constant and the index is greater than 1, it can not be greater than $n+1$.

Using the results of E. Glodek, [1], we show what the results of this paper become in the case of the Levi-Civita connection.

The case of conformally flat manifolds (with respect to K. Yano's connection) was treated in [3].

§1. **Preliminaries.** Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection, $\mathscr{X}(M)$ the $\mathscr{F}(M)$-module of tangent vector fields, $\Lambda^1(M)$ the $\mathscr{F}(M)$-module of 1-forms and

$$G_* \ \Lambda^1(M)\rightarrow\mathscr{X}(M)$$

the natural isomorphism induced by g. The connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X Y=\nabla_X Y+\omega(Y)X-g(X, Y)G_*\omega \tag{1.1}$$

where $\omega\in\Lambda^1(M)$ is a closed 1-form, is called a special semi-symmetric metric connection, [4].

This connection has the following properties :

$$\overline{K}(X, Y, Z, V)+\overline{K}(Y, X, Z, V)=0, \tag{1.2}$$

$$\overline{K}(X, Y, Z, V) + \overline{K}(X, Y, V, Z)=0, \tag{1.3}$$

$$\overline{K}(X, Y, Z, V) + \overline{K}(Z, V, X. Y)=0, \tag{1.4}$$

(1.5) $$\bar K(X,Y,Z,V)+\bar K(X,Z,V,Y)+\bar K(X,V,Y,Z)=0,$$

(1.6) $$\overline{\mathrm{Ric}}(X,Y)=\overline{\mathrm{Ric}}(Y,X),$$

where

(1.7) $$\bar K(X,Y,Z,V)=g(X,\bar R(Z,V)Y),$$

(1.8) $$\overline{\mathrm{Ric}}(X,Y)=g(\overline{\mathrm{ric}}\,X,Y),$$

(1.9) $$\overline{\mathrm{ric}}\,X=\sum_{i=1}^{n} R(X,X_i)X_i,$$

$\{X_i\}$ being an ortho-normal basis of vector fields.

The fundamental property of the connection (1.1) is that the conformal curvature tensor field is an invariant, [4], i.e.

(1.10) $$\bar C(X,Y)Z=C(X,Y)Z,$$

where

(1.11) $$\bar C(X,Y)Z=\bar R(X,Y)Z-\frac{1}{n-2}[X\bar E(Y,Z)-Y\bar E(X,Z)+g(Y,Z)\bar eX-g(X,Z)\bar eY]-\bar S[Xg(Y,Z)-Yg(X,Z)]/n(n-1)$$

(1.12) $$\bar eX=\overline{\mathrm{ric}}X-XS/n,\quad \bar E(X,Y)=g(\bar eX,Y)$$

If the covariant derivative of $\bar C$ with respect to vanishes identically, we say that (M,g) is $\bar\nabla$-conformally symmetric.

Lemma 1.1. *If (M,g) is an n-dimensional Riemannian $\bar\nabla$-conformally symmetric manifold, $n>2$, then*

(1.13) $$\sum_{i=1}^{n}(\bar\nabla_U\bar E)(X_i,X_i)=0,$$

where $\{X_i\}$ is an ortho-normal basis of vector fields and U is an arbitrary vector field.

Proof. From $\bar\nabla_U\bar C=0$, we have

(1.14) $$(\bar\nabla_U\bar R)(X,Y)Z=\frac{1}{n-2}[X(\bar\nabla_U\bar E)(Y,Z)-Y(\bar\nabla_U\bar E)(X,Z)+g(Y,Z)(\bar\nabla_U\bar e)(X)-g(X,Z)(\bar\nabla_U\bar e)(Y)+d\bar S(U)[Xg(Y,Z)-Yg(X,Z)]/[n(n-1)]].$$

Denoting $Y=Z=X_i$ and taking the sum, we obtain

$$\sum_{i=1}^{n}[\bar R(X,\bar\nabla_U X_i)X_i+\bar R(X,X_i)\bar\nabla_U X_i]=-X\sum_{i=1}^{n}(\bar\nabla_U\bar E)(X_i,X_i)/(n-2).$$

Using (1.2)-(1.5), the above formula becomes

(1.15) $$\sum_{i=1}^{n} g(\bar\nabla_U X_i,\bar R(X_i,X)Y+\bar R(X_i,Y)X)=-g(X,Y)\sum_{i=1}^{n}(\bar\nabla_U E)(X_i,X_i)/(n-1).$$

Taking $X=Y=X_j$ and summing, we find

(1.16) $$-2(n-2)\sum_{j=1}^{n} g(\bar\nabla_U X_j,\overline{\mathrm{ric}}\,X_j)=n\sum_{j=1}^{n}(\bar\nabla_U\bar E)(X_j,X_j).$$

On the other hand, from $(\bar\nabla_U g)(X_i,\mathrm{ric}X_i)=0$, we obtain

(1.17) $$-2\sum_{i=1}^{n} g(\bar\nabla_U X_i,\overline{\mathrm{ric}}\,X_i)=\sum_{i=1}^{n}(\bar\nabla_U\bar E)(X_i,X_i).$$

From (1.16) and (1.17) we get (1.13).

Lemma 1.2. *Let (M,g) be an n-dimensional Riemannian $\bar\nabla$-conformal symmetric manifold, $n>2$. If $\bar\nabla_U Ric=0$, then $\bar\nabla_U\bar E=0$.*

Proof. By lemma 1.1 we have:

(1.18) $$\sum_{i=1}^{n}(\bar\nabla_U\overline{\mathrm{Ric}})(X_i,X_i)=d\bar S(U),$$

and from the hypothesis it follows that $d\bar S=0$. Using (1.12) we obtain $\bar\nabla_U\bar E=0$.

Lemma 1.3. *Let (M,g) be an n-dimensional Riemannian $\bar\nabla$-conformal symmetric manifold, $n>2$, and $\bar\nabla_U\overline{\mathrm{Ric}}\neq 0$. Let Ψ be a non-vanishing differentiable function. If $\Psi\bar\nabla_U\,\overline{\mathrm{Ric}}$ and g are linearly dependent, then $\bar\nabla_U\bar E=0$.*

Proof. By the hypothesis there is $\alpha\in\Lambda^1(M)$ so that:

(1.19) $$\Psi(\bar\nabla_U\overline{\mathrm{Ric}})(X,Y)=\alpha(U)g(X,Y).$$

Using (1.18) it follows that

(1.20) $$\Psi d\bar S=n\,\alpha$$

and the lemma is proved.

§2. The solutions of a tensorial equation. Let (M,g) be an n-dimensional Riemannian manifold, $n>2$, $\bar\nabla$ the special semi-symmetric metric connection of Yano, [4], and take the tensorial equation

(2.1) $$H((\bar\nabla_U\bar R)(X,Y)Z,V)+H(Z,(\bar\nabla_U\bar R)(X,Y)V)=0,$$

where U is an arbitrary vector field and H is a symmetric tensor field. The solutions H of the equation (2.1) are intimately related to the question of the index $i_{\bar\nabla}$ corresponding to $\bar\nabla$-conformal symmetric manifolds.

Theorem 2.1. *If (M,g) is an n-dimensional Riemannian $\bar\nabla$-conformally symmetric manifold, $n>2$, then the equation (2.1) is reducible to the canonical form*

$$(2.2)\qquad \begin{vmatrix} H(X,Z)-\frac{1}{n}\operatorname{tr} H\, g(X,Z) & H(Y,V)-\frac{1}{n}\operatorname{tr} H.g(Y,V) \\ (\overline{\nabla}_U\overline{E})(X,Z) & (\overline{\nabla}_U\overline{E})(Y,V) \end{vmatrix}=0.$$

If

$$(2.3)\qquad \overline{\nabla}_U\overline{E}\neq 0$$

then (2.2) *has the general solution*

$$(2.4)\qquad H_U(X,Y)=f.(\overline{\nabla}_U\overline{\mathrm{Ric}})(X,Y)+\frac{1}{n}\,[trH_U-n.f\,d\,S(U)]g(X,Y)$$

where f is an arbitrary non-vanishing differentiable function.

Proof. Using (1.14) and (2.1) we find

$$(2.5)\qquad \begin{aligned} &H(X,Y)(\overline{\nabla}_U\overline{E})(Y,Z)-H(Y,V)(\overline{\nabla}_U\overline{E})(X,Z)+H(X,Z)(\overline{\nabla}_U\overline{E})(Y,V)-\\ &-H(Y,Z)(\overline{\nabla}_U\overline{E})(X,V)+g(Y,Z)H((\overline{\nabla}_U\overline{e})(X),V)-g(X,Z)H(\overline{\nabla}_U\overline{e})(Y)V)+\\ &+g(Y,V)H((\overline{\nabla}_U\overline{e})(X),Z)-g(X,V)H((\overline{\nabla}_U\overline{e})(Y),Z)=\\ &=\frac{n-2}{n(n-1)}\,d\overline{S}(U)[g(X,Z)H(Y,V)-g(Y,Z)H(X,V)+g(X,V)H(Y,Z)-\\ &\qquad -g(Y,V)H(X,Z)]. \end{aligned}$$

Denoting $X=Z=X_i$ in (2.5), using (1.13) and taking the sum, we get

$$(2.6)\qquad \begin{aligned} &(n-1)H((\overline{\nabla}_U\overline{e})(Y),V)+H(Y,(\overline{\nabla}_U\overline{e})(V))=(\overline{\nabla}_U\overline{E})(Y,V)\operatorname{tr} H-\\ &-\frac{n-2}{n-1}\,d\overline{S}(U)[H(Y,V)-\frac{1}{n}\operatorname{tr} Hg(Y,V)]+g(Y,V)\sum_{i=1}^{n}H(\overline{\nabla}_U\overline{e})(X_i),X_i). \end{aligned}$$

From the symmetry of the right-hand member of (2.6) we have

$$(2.7)\qquad H((\overline{\nabla}_U\overline{e})(Y),V)=H(Y,(\overline{\nabla}_U\overline{e})(V)).$$

Now we replace in (2.5) X by Z, Z by X, Y by V and V by Y, take the sum of the so-obtained formula and (2.5). Then using (2.6) and (2.7) we obtain (2.2). The equation (2.2) depends obviously on the vector field U; that is why we shall denote the solutions by H_U. As $\overline{\nabla}_U\overline{E}\neq 0$, the tensor fields $H_U-\frac{1}{n}\operatorname{tr} H_U.g$ and $\overline{\nabla}_U\overline{E}$ being proportional, (2.4) follows.

Theorem 2.2. *If (M, g) is an n-dimensional Riemannian $\overline{\nabla}$-conformally symmetric manifold, $n>2$, and if there is $U\in\mathcal{X}(M)$ so that*

$$(2.8)\qquad \overline{\nabla}_U\overline{E}=0,\quad d\overline{S}(U)\neq 0,$$

then the solution of (2.1) *is*

$$(2.9)\qquad H=f.g,$$

where f is a differentiable non-vanishing function.

Proof. Using (2.8) the formula (2.5) becomes

$$(2.10)\qquad g(X,Z)H(Y,V)-g(Y,V)H(X,Z)=0,$$

and consequently the tensor fields H and g are proportional and the theorem is proved.

§3. **The index $i_{\overline{\nabla}}$ of (M, g).** By the index $i_{\overline{\nabla}}$ of a manifold M with respect to the connection $\overline{\nabla}$, we understand the greatest number of 2-covariant symmetric $\overline{R}$-independent tensor fields.

As $\overline{\nabla}$ is a metric connection, it follows that $i_{\overline{\nabla}}$ is at least 1.

Theorem 3.1. *Let (M, g) be an n-dimensional Riemannian $\overline{\nabla}$-conformal symmetric manifold, $n>2$. If there is a vector field $U\in\mathcal{X}(M)$ satisfying the conditions* (2.8), *then*

$$i_{\overline{\nabla}}=1.$$

Proof. From theorem 2.2. and from the fact that $\overline{\nabla}_U H=0$ and $\overline{\nabla}_U g==0$, it follows that $f=$constant and the theorem is proved.

Theorem 3.2. *Let (M, g) be an n-dimensional Riemannian $\overline{\nabla}$-conformal symmetric manifold, $n>2$, for which the tensor field $\overline{E}$ is not covariantly constant. If $i_{\overline{\nabla}}>1$, then there is $U\in\mathcal{X}(M)$ so that the equation*

$$(3.1)\qquad \overline{\nabla}_U H=0$$

has the fundamental solutions

$$(3.2)\qquad H_1=g,\quad H_2=\psi\overline{\nabla}_U\overline{\mathrm{Ric}},$$

where ψ is a differentiable non-vanishing function.

Proof. As $\overline{E}$ is not covariantly constant, there is U so that we have (2.3) and then the general solution (2.4) of (2.1) depends on U. As g is obviously a solution of (3.1), it satisfies identically the integrability condition of (3.1), which is exactly (1.3), and also the differential consequence of these (2.1). Therefore g and H_U given by (2.4) are solutions of (3.1), which has at least two solutions as $i_{\overline{\nabla}}>1$. These two solutions are independent. As $\psi\overline{\nabla}_U\overline{\mathrm{Ric}}$ are expressed as a combination of these solutions and by lemma 1.3, $\psi\overline{\nabla}_U\overline{\mathrm{Ric}}$ and g are independent and we get (3.2).

Note. A necessary condition to be satisfied by U is

$$(3.3)\qquad (\overline{\nabla}_U\overline{\mathrm{Ric}})(\overline{R}(X,Y)Z,V)+(\overline{\nabla}_U\overline{\mathrm{Ric}})(Z,\overline{R}(X,Y)V)=0,$$

which is obtained from (2.4) and the integrability condition of (3.1).

Theorem 3.3. *Let (M, g) be an n-dimensional Riemannian $\overline{\nabla}$-conformal symmetric manifold, $n>2$, for which the tensor field $\overline{E}$ is not $\overline{\nabla}$-covariantly constant. If $i_{\overline{\nabla}}>1$, then $i_{\overline{\nabla}}\leqslant n+1$.*

Proof. Let p be the number independent vector fields for which we have

$$\overline{\nabla}_U\overline{E}_i\neq 0,\quad i=1,\ldots,p,$$

and let $\psi_i\overline{\nabla}_{U_i}\overline{\mathrm{Ric}}$ and g be the fundamental solutions of $\overline{\nabla}_{U_i}H=0$. Obviously $p<n$, as U_i are independent. The theorem is completely proved.

Note. If in (1.1) we take $\omega=0$ then $\overline{\nabla}$ is exactly the Levi-Civita connection. In this case by G l o d e k's theorem, [1], the formula $\nabla_U C=0$ gives

$C=0$ or S constant. For $C=0$ the problem is exactly that of L e v i n and K a t z i n, [2]. For $\nabla_U C=0$, $C\neq 0$, the theorems 2.2, respectively 3.1, are not valid, but the theorems 2.1, 3.2 and 3.3 are still valid.

REFERENCES

1. G l o d e k, E.—*Some remarks on conformally symmetric Riemannian spaces*, Colloq. Math., vol. XXIII, 1971, fasc. 1, p. 121—123.
2. L e v i n, J. and K a t z i n, G. H. — *Conformally flat spaces admitting special quadratic first integrals, I, 4*, Tensor N.S. vol. 19 (3), 1968, p. 317—327.
3. S t a v r e, P. and S m a r a n d a, D.— *On the associate index of a semi-symmetric metric connexion of conformally flat Riemannian manifolds*, Colocviul Naţ. de Geom. şi Topologie „Gh. Vrănceanu", Vaslui—Bacău, 1980.
4. Y a n o, K. — *On semi-symmetric metric connection*, Rev. Roumaine de Math. Pures et Appl. (15) 1970, p. 1579—1586.

Received 29.XII.1980

Faculty of Sciences
University of Craiova
Craiova, R. S. Romania

Faculty of Mathematics
University of Bucharest
Bucharest, R. S. Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVIII, s. I a, 1982, f. 1

KAEHLER SUBMANIFOLDS WITH PARALLEL NORMAL CURVATURE TENSOR

BY

FLORENTINO G. SANTOS

1. Introduction. Let M be a Kaehler submanifold of a Kaehler manifold of constant holomorphic sectional curvature c, $\overline{M}(c)$. In [3], C h e n and O g i u e prove that the connection of the normal bundle is flat if and only if $c=0$ and M is totally geodesic.

Thus, in order to study Kaehler submanifolds of the complex projective space we must give weaker conditions about the normal curvature. Some of these conditions were studied by I s h i h a r a, [4], and B a r r o s-S a n t o s, [1].

The purpose of the present paper is to classify Kaehler submanifolds of a Kaehler manifold of constant holomorphic sectional curvature with parallel normal curvature tensor.

We obtain the following

Theorem. *Let M be an n-dimensional Kaehler submanifold of an $(n+p)$-dimensional complex-space-form, $\overline{M}(c)$. Then the normal curvature tensor is parallel if and only if the second fundamental form is parallel.*

The author wishes to express his sincere gratitude to Professor B a n g-y e n C h e n who gave him valuable suggestions.

2. Basic formulas. Let M be an n-dimensional Riemannian submanifold of an $(n+p)$-dimensional Riemannian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ and metric tensor g. Then the second fundamental form σ of M in $\overline{M}$ is given by

$$\overline{\nabla}_X Y=\nabla_X Y+\sigma(X, Y) \tag{2.1}$$

for any vector fields X, Y tangent to M, where ∇ denotes the Levi-Civita connection on M.

If ξ is a normal vector field on M, we write

$$\overline{\nabla}_X\xi=-A_\xi X+\nabla^{\perp}_X\xi, \tag{2.2}$$

where $-A_\xi X$ (respectively, $\nabla^{\perp}_X\xi$) denotes the tangential (respectively, normal) component of $\overline{\nabla}_X\xi$. Then we have $g(A_\xi X, Y)=g(\sigma(X, Y), \xi)$. $\nabla^{\perp}$ is called the normal connection of M in $\overline{M}$.

The normal curvature tensor is given by