$C=0$ or S constant. For $C=0$ the problem is exactly that of L e v i n and K a t z i n, [2]. For $\nabla_U C=0$, $C\neq 0$, the theorems 2.2, respectively 3.1, are not valid, but the theorems 2.1, 3.2 and 3.3 are still valid.

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Faculty of Sciences
University of Craiova
Craiova, R. S. Romania

Faculty of Mathematics
University of Bucharest
Bucharest, R. S. Romania

Analele științifice ale Universității „Al. I. Cuza" din Iași
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KAEHLER SUBMANIFOLDS WITH PARALLEL NORMAL CURVATURE TENSOR

BY

FLORENTINO G. SANTOS

1. Introduction. Let M be a Kaehler submanifold of a Kaehler manifold of constant holomorphic sectional curvature c, $\overline{M}(c)$. In [3], C h e n and O g i u e prove that the connection of the normal bundle is flat if and only if $c=0$ and M is totally geodesic.

Thus, in order to study Kaehler submanifolds of the complex projective space we must give weaker conditions about the normal curvature. Some of these conditions were studied by I s h i h a r a, [4], and B a r r o s-S a n t o s, [1].

The purpose of the present paper is to classify Kaehler submanifolds of a Kaehler manifold of constant holomorphic sectional curvature with parallel normal curvature tensor.

We obtain the following

Theorem. *Let M be an n-dimensional Kaehler submanifold of an $(n+p)$-dimensional complex-space-form, $\overline{M}(c)$. Then the normal curvature tensor is parallel if and only if the second fundamental form is parallel.*

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2. Basic formulas. Let M be an n-dimensional Riemannian submanifold of an $(n+p)$-dimensional Riemannian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ and metric tensor g. Then the second fundamental form σ of M in $\overline{M}$ is given by

$$\overline{\nabla}_X Y=\nabla_X Y+\sigma(X, Y) \tag{2.1}$$

for any vector fields X, Y tangent to M, where ∇ denotes the Levi-Civita connection on M.

If ξ is a normal vector field on M, we write

$$\overline{\nabla}_X \xi=-A_\xi X+\nabla^\perp_X \xi, \tag{2.2}$$

where $-A_\xi X$ (respectively, $\nabla^\perp_X \xi$) denotes the tangential (respectively, normal) component of $\overline{\nabla}_X \xi$. Then we have $g(A_\xi X, Y)=g(\sigma(X, Y), \xi)$. $\nabla^\perp$ is called the normal connection of M in $\overline{M}$.

The normal curvature tensor is given by

$$R^{\perp}(X, Y)=\nabla^{\perp}_X\nabla^{\perp}_Y-\nabla^{\perp}_Y\nabla^{\perp}_X-\nabla^{\perp}_{[X, Y]}.$$

Then $R^{\perp}$ is *parallel* if $\tilde{\nabla}_U R^{\perp}=0$, for any vector field U tangent to M, where

$$(2.3)\qquad (\tilde{\nabla}_U R^{\perp})(X, Y)\xi=\nabla^{\perp}_U(R^{\perp}(X, Y)\xi)-R^{\perp}(\nabla_U X, Y)\xi-R^{\perp}(X, \nabla_U Y)\xi-R^{\perp}(X, Y)\nabla^{\perp}_U\xi.$$

Yano and **K**on, [5], define the covariant derivative of A by

$$(2.4)\qquad (\tilde{\nabla}_U A)_{\xi}X=\nabla_U(A_{\xi}X)-A_{\nabla^{\perp}_U\xi}X-A_{\xi}\nabla_U X,$$

and A is said to be *parallel* if $\tilde{\nabla}_U A\equiv 0$ for any vector field U tangent to M.

It is easy to prove that

$$g((\tilde{\nabla}_U A)_{\xi}X, Y)=g(X, (\tilde{\nabla}_U A)_{\xi}Y).$$

The equations of Gauss, Codazzi and Ricci are given, respectively, by

$$(2.5)\qquad \bar{R}(X, Y, Z, W)=R(X, Y, Z, W)+g(\sigma(X, Z), \sigma(Y, W))-g(\sigma(X, W), \sigma(Y, Z)),$$

$$(2.6)\qquad (\bar{R}(X, Y)Z)^{\perp}=(\bar{\nabla}_X\sigma)(Y, Z)-(\bar{\nabla}_Y\sigma)(X, Z),$$

$$(2.7)\qquad \bar{R}(X, Y, \xi, \eta)=R^{\perp}(X, Y, \xi, \eta)-g([A_{\xi}, A_{\eta}]X, Y),$$

for any vector fields X, Y, Z, W tangent to M and ξ, η normal to M, where $\bar{R}$ (respectively, R) denotes the curvature tensor of $\bar{M}$ (respectively, M), $(\bar{R}(X, Y)Z)^{\perp}$ is the normal component of $\bar{R}(X, Y)Z$ and $\bar{\nabla}_X\sigma$ is given by

$$(2.8)\qquad (\bar{\nabla}_X\sigma)(Y, Z)=\nabla^{\perp}_X(\sigma(Y, Z))-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z).$$

If $\bar{\nabla}_X\sigma=0$ for any vector field X tangent to M, then the second fundamental form is said to be *parallel*. It is easy to see that A is parallel if and only if σ is parallel.

Let $\bar{M}$ be an $(n+p)$-dimensional Kaehler manifold with complex structure J and metric tensor g. $\bar{M}$ is called a *complex-space-form* of constant holomorphic sectional curvature c if the curvature tensor $\bar{R}$ of $\bar{M}$ satisfies

$$(2.9)\qquad \bar{R}(\bar{X}, \bar{Y})\bar{Z}=\frac{c}{4}\{g(\bar{Y}, \bar{Z})\bar{X}-g(\bar{X}, \bar{Z})\bar{Y}+g(J\bar{Y}, \bar{Z})J\bar{X}-g(J\bar{X}, \bar{Z})J\bar{Y}+2g(\bar{X}, J\bar{Y})J\bar{Z}\}$$

for any vector fields $\bar{X}, \bar{Y}, \bar{Z}$ on $\bar{M}$. We denote such a space by $\bar{M}(c)$.

3. Some Lemmas. Lemma 1. *Let M be a Kaehler submanifold of a Kaehler manifold $\bar{M}$. Then*

$$(\tilde{\nabla}_U A)_{J\xi}=J(\tilde{\nabla}_U A)_{\xi},\quad (\tilde{\nabla}_U A)_{\xi}J=-J(\tilde{\nabla}_U A)_{\xi},$$

for any vector fields U tangent to M and ξ normal to M.

Proof. It is known that $A_{J\xi}=JA_{\xi}$ and $A_{\xi}J=-JA_{\xi}$, from which we have that M is minimal. The result follows from (2.4).

Lemma 2. *Let M be a Kaehler submanifold of $\bar{M}(c)$. Then*

$$(\tilde{\nabla}_{JU}A)_{\xi}=-J(\tilde{\nabla}_U A)_{\xi},$$

for any vector fields U tangent to M and ξ normal to M.

Proof. From (2.6) and (2.9) we have

$$g((\bar{\nabla}_{JU}\sigma)(X, Y),\xi)=g((\bar{\nabla}_X\sigma)(JU, Y), \xi).$$

This is equivalent to

$$g((\tilde{\nabla}_{JU}A)_{\xi}X, Y)=g((\tilde{\nabla}_X A)_{\xi}JU, Y).$$

Now, the result follows from Lemma 1.

4. Proof of the Theorem. From (2.7), (2.9), (2.3) and (2.4) we obtain that $\tilde{\nabla}_U R^{\perp}=0$ if and only if

$$(4.1)\qquad [(\tilde{\nabla}_U A)_{\xi}, A_{\eta}]=[(\tilde{\nabla}_U A)_{\eta}, A_{\xi}]$$

for any vector fields U tangent to M and ξ, η normal to M. From Lemma 2, putting $\eta=J\xi$ in (4.1), we have $\tilde{\nabla}_U R^{\perp}=0$ if and only if

$$(4.2)\qquad A_{\xi}(\tilde{\nabla}_U A)_{\xi}=(\tilde{\nabla}_U A)_{\xi}A_{\xi}=0.$$

Then, if the second fundamental form is parallel we know that A is parallel, so from (4.2) the normal curvature tensor is parallel.

Conversely, if the normal curvature tensor is parallel, let ξ be a vector field normal to M such that $A_{\xi}\neq 0$. From (4.2) we can find an orthonormal basis $\{e_i, Je_i\}$, $1\leqslant i\leqslant n$, that diagonalizes A_{ξ} and $(\tilde{\nabla}_U A)_{\xi}$ simultaneously,

$$A_{\xi}=\begin{bmatrix}\lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_i & & & & & \\ & & & \ddots & & & & \\ & & & & \lambda_n & & & \\ & & & & & -\lambda_1 & & \\ & & & & & & \ddots & \\ & & & & & & & -\lambda_i \\ & & & & & & & & \ddots \\ & & & & & & & & & -\lambda_n\end{bmatrix},\quad (\tilde{\nabla}_U A)_{\xi}=\begin{bmatrix}\mu_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \mu_i & & & & & \\ & & & \ddots & & & & \\ & & & & \mu_n & & & \\ & & & & & -\mu_1 & & \\ & & & & & & \ddots & \\ & & & & & & & -\mu_i \\ & & & & & & & & \ddots \\ & & & & & & & & & -\mu_n\end{bmatrix}.$$

If $\lambda_i\neq 0$, $1\leqslant i\leqslant n$, then from (4.2) $\mu_i=0$ for any $1\leqslant i\leqslant n$, that is $(\tilde{\nabla}_U A)_{\xi}\equiv 0$.

Let i, $1\leqslant i\leqslant n$, be an index such that $\lambda_i=0$ and $\lambda_j\neq 0$ for any $j\neq i$, $1\leqslant j\leqslant n$. We can suppose that $i\neq 1$ and then we take the holomorphic basis

$$\{e_1, \ldots, e_1+e_i, \ldots, e_n, Je_1, \ldots, Je_1+Je_i, \ldots, Je_n\}.$$

In this basis

$$A_\xi \equiv \begin{array}{c} \quad\; i \qquad\qquad n+i \\ \begin{bmatrix} \lambda_1 & \lambda_1 & & & & & \\ \lambda_1 & \lambda_1 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_n & & & \\ & & & & -\lambda_1 & -\lambda_1 & \\ & & & & -\lambda_1 & -\lambda_1 & \\ & & & & & & \ddots \\ & & & & & & & -\lambda_n \end{bmatrix} \end{array} \quad (\tilde{\nabla}_U A)_\xi \equiv \begin{array}{c} \quad\; i \qquad\qquad n+i \\ \begin{bmatrix} 0 & & & & & & \\ & 0 & \mu_i & & & & \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & -\mu_i & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{bmatrix} \end{array}$$

and from (4.2) we obtain $\mu_i = 0$. So, $(\tilde{\nabla}_U A)_\xi \equiv 0$.

Finally, let ξ, ξ' be two vector fields normal to M such that $A_\xi \not\equiv 0$ and $A_{\xi'} \equiv 0$, then from (4.2)

$$0 = A_{\xi+\xi'}(\tilde{\nabla}_U A)_{\xi+\xi'} = (\tilde{\nabla}_U A)_{\xi+\xi'} A_{\xi+\xi'}$$

this implies that

$$0 = A_\xi(\tilde{\nabla}_U A)_{\xi'} = (\tilde{\nabla}_U A)_{\xi'} A_\xi$$

and reasoning as above, we obtain that $(\tilde{\nabla}_U A)_\xi \equiv 0$. This concludes the proof.

It is possible to apply this result to the problem of reduction of codimension, studied by C e c i l [2], for this type of immersions.

Corollary. *Let $f: M \to \overline{M}(c)$ be a holomorphic and isometric immersion of a connected, complete, complex n-dimensional Kaehlerian manifold M into an $(n+p)$-dimensional complex-space-form, $\overline{M}(c)$. If the normal curvature tensor is parallel and the first normal space of the immersion has constant dimension k, then there is a totally geodesic $(n+k)$-dimensional submanifold $N(c)$ such that $f(M) \subset N(c)$.*

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Departamento de Geometria
Facultad de Ciencias
Universidad de Granada
Granada, Spain

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A GEOMETRICAL ALMOST AFFINE MODEL

BY

ANDREI RUGINĂ

The notion of almost affine space (over a field) was introduced by R. M i r o n in [9]. The aim of the present paper is to give a geometrical real model for this structure ; we notice that all the models already known are algebraic. The naturalness of the construction of such a model (construction which can be easily extended to an important class of finite models) may strenghten the interest for the study of the almost affine spaces and proves their geometrical consistency.

1. Almost Affine Spaces. In this section we recall the definition of the K-almost affine spaces, given in [9], and make a brief review of their main properties.

Let K be an arbitrary skew-field and let 1 be its identity. Let $M = \{A, B, C, \ldots\}$ be a set, whose elements are called *points*. The elements of the set $M \times M$ are called *oriented segments* of M, denoted by $(A, B) = \overline{AB}$. Let ρ be an equivalence relation on $M \times M$. We denote by $TM = (M \times M)/\rho$ the quotient set with respect to the relation ρ and by $p: M \times M \to TM$ its natural projection. We denote $p(\overline{AB}) = \vec{AB}$, and say that $\vec{AB}$ is the *vector* of the set M determined by the oriented segment $\overline{AB}$ with respect to the equivalence ρ.

Definition 1.1. *On the nonempty set M we have a structure $\mathcal{A}$ of a K-almost affine space if:*

a) *The equivalence relation ρ on $M \times M$ is given, defining the quotient set $TM = (M \times M)/\rho$ of the vectors in M.*

b) *We have a mapping : $*: K \times TM \to TM$, $*(\alpha, x) = \alpha * x$, $\forall \alpha \in K$, $\forall x \in TM$.*

c) *The first and the second groups of axioms (see below) are verified.*

The pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a structure of a K-almost affine space on M will be called a K-almsot affine space.

First Group of Axioms

I_1. $\forall A \in M,\ \forall x \in TM,\ \exists B \in M: \vec{AB} = x.$

I_2. $\forall A, B, A' \in M: \vec{AB} = \vec{A'B} \Rightarrow A = A'$,

I_3. $\forall A, B, C, A', B', C' \in M$, $\vec{AB} = \vec{A'B'}$, $\vec{BC} = \vec{B'C'} \Rightarrow \vec{AC} = \vec{A'C'}$ *(Miron - Opaiţ's „triangle" axiom).*