$$
\begin{array}{cc}
 & \begin{array}{cccccc} & i & & & n+i & \end{array} \\
A_\xi \equiv & \begin{bmatrix}
\lambda_1 & \lambda_1 & & & & \\
\lambda_1 & \lambda_1 & & & & \\
 & & \ddots & & & \\
 & & & \lambda_n & & \\
 & & & -\lambda_1 & -\lambda_1 & \\
 & & & -\lambda_1 & -\lambda_1 & \\
 & & & & \ddots & \\
 & & & & & -\lambda_n
\end{bmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{cccccc} & i & & & n+i & \end{array} \\
(\tilde\nabla_U A)_\xi \equiv & \begin{bmatrix}
0 & & & & & \\
 & 0 & & & & \\
 & \mu_i & 0 & & & \\
 & & & \ddots & & \\
 & & & 0 & & \\
 & & & 0 & & \\
 & & & -\mu_i & 0 & \\
 & & & & \ddots & \\
 & & & & & 0
\end{bmatrix}
\end{array}
$$

and from (4.2) we obtain $\mu_i = 0$. So, $(\tilde\nabla_U A)_\xi \equiv 0$.

Finally, let ξ, ξ' be two vector fields normal to M such that $A_\xi \not\equiv 0$ and $A_{\xi'} \equiv 0$, then from (4.2)

$$0 = A_{\xi+\xi'}(\tilde\nabla_U A)_{\xi+\xi'} = (\tilde\nabla_U A)_{\xi+\xi'} A_{\xi+\xi'}$$

this implies that

$$0 = A_\xi(\tilde\nabla_U A)_{\xi'} = (\tilde\nabla_U A)_{\xi'} A_\xi$$

and reasoning as above, we obtain that $(\tilde\nabla_U A)_\xi \equiv 0$. This concludes the proof.

It is possible to apply this result to the problem of reduction of codimension, studied by Cecil [2], for this type of immersions.

Corollary. *Let $f : M \to \overline{M}(c)$ be a holomorphic and isometric immersion of a connected, complete, complex n-dimensional Kaehlerian manifold M into an $(n+p)$-dimensional complex-space-form, $\overline{M}(c)$. If the normal curvature tensor is parallel and the first normal space of the immersion has constant dimension k, then there is a totally geodesic $(n+k)$-dimensional submanifold $N(c)$ such that $f(M) \subset N(c)$.*

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A GEOMETRICAL ALMOST AFFINE MODEL

BY

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The notion of almost affine space (over a field) was introduced by R. Miron in [9]. The aim of the present paper is to give a geometrical real model for this structure ; we notice that all the models already known are algebraic. The naturalness of the construction of such a model (construction which can be easily extended to an important class of finite models) may strenghten the interest for the study of the almost affine spaces and proves their geometrical consistency.

1. Almost Affine Spaces. In this section we recall the definition of the K-almost affine spaces, given in [9], and make a brief review of their main properties.

Let K be an arbitrary skew-field and let 1 be its identity. Let $M = \{A, B, C, \ldots\}$ be a set, whose elements are called *points*. The elements of the set $M \times M$ are called *oriented segments* of M, denoted by $(A, B) = \overline{AB}$. Let ρ be an equivalence relation on $M \times M$. We denote by $TM = (M \times M)/\rho$ the quotient set with respect to the relation ρ and by $p : M \times M \to TM$ its natural projection. We denote $p(\overline{AB}) = \overrightarrow{AB}$, and say that $\overrightarrow{AB}$ is the *vector* of the set M determined by the oriented segment $\overline{AB}$ with respect to the equivalence ρ.

Definition 1.1. *On the nonempty set M we have a structure $\mathcal{A}$ of a K-almost affine space if:*

a) *The equivalence relation ρ on $M \times M$ is given, defining the quotient set $TM = (M \times M)/\rho$ of the vectors in M.*

b) *We have a mapping : $*$: $K \times TM \to TM$, $*(\alpha, x) = \alpha * x$, $\forall \alpha \in K$, $\forall x \in TM$.*

c) *The first and the second groups of axioms (see below) are verified.*

The pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a structure of a K-almost affine space on M will be called a K-almsot affine space.

First Group of Axioms

I_1. $\forall A \in M$, $\forall x \in TM$, $\exists B \in M : \overrightarrow{AB} = x$.

I_2. $\forall A, B, A' \in M : \overrightarrow{AB} = \overrightarrow{A'B} \Rightarrow A = A'$,

I_3. $\forall A, B, C, A', B', C' \in M$, $\overrightarrow{AB} = \overrightarrow{A'B'}$, $\overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$ (*Miron - Opaiţ's „triangle" axiom*).

The vector $\overrightarrow{AA}$ is called the *null* vector of M and is denoted by $\vec{0}$. The vector $\overrightarrow{BA}$ is called the *opposite* vector of $\overrightarrow{AB}$ and is denoted by $-\overrightarrow{AB}$. Let $(x, y) \in TM \times TM$ and $A \in M$. Then $\exists_1 B \in M : \overrightarrow{AB} = x$, and $\exists_1 C \in M : \overrightarrow{BC} = y$. The vector $\overrightarrow{AC} = z$ (which does not depend on the point A) is called the *sum* of the vectors x and $y : z = x + y$.

Definition 1.2. *The mapping* $+ : (x, y) \in TM \times TM \to x + y \in TM$, $\forall x, y \in TM$, *is called the addition operation of the vectors of* M.

It is easy to see that the pair $(TM, +)$ is a (generally not Abelian) group.

Second Group of Axioms. *If the addition of vectors is given by the definition 1.2, then :*

II_1. $\forall \alpha \in K$, $\forall x, y \in TM : \alpha * (x + y) = \alpha * x + \alpha * y$.

II_2. $\forall \alpha, \beta \in K$, $\forall x \in TM : (\alpha + \beta) * x = \alpha * x + \beta * x$.

II_3. $\forall \alpha, \beta \in K$, $\forall x \in TM : \alpha * (\beta * x) = (\alpha\beta) * x$.

Definition 1.3. *We call torsion of a K-almost affine space M, the mapping* : $\bar{t} : (x, y) \in TM \times TM \to -(x + y) + (y + x) \in TM$.

The additive group $(TM, +)$ is Abelian if and only if the torsion of M vanishes.

We call a K-Weylean almost affine space structure on the nonempty set M the structure obtained by replacing the axiom I_3 above by Raschevsky's axiom (so called „the parallelogram axiom") :

I'_3. $\forall A, B, A', B' \in M : \overrightarrow{AB} = \overrightarrow{A'B'} \Rightarrow \overrightarrow{AA'} = \overrightarrow{BB'}$.

It is easy to see that any K-Weylean almost affine space is a K-almost affine space, its group $(TM, +)$ being Abelian.

Using Hilbert's method of models, we can prove that the axioms of the first and second group of axioms for a K-almost affine space are noncontradictory and independent. The same is true for the K-Weylean almost affine space structure [8, 9].

Proposition 1.1. *For the K-almost affine space M, the following conditions are equivalent :*

i) $1 * x = x$, $\forall x \in TM$ *and* $x + y = y + x$, $\forall x, y \in TM$ *(that is, TM is a vector space).*

ii) $\forall x \in TM / \{\vec{0}\} : 1 * x \neq \vec{0}$.

iii) The mapping $* : K \times TM \to TM$ *is surjective.*

A K-almost affine space M is called :

a) — K-almost affine *trivial* space if $TM = \{\vec{0}\}$.

b) — K-*zero* almost affine space if $\alpha * x = \vec{0}$ $\forall \alpha \in K$, $\forall x \in TM$.

c) — K-*unitary* almost affine space if $1 * x = x$, $\forall x \in TM$.

Of course, a K-unitary almost affine space is a K-affine space.

The set of vectors TM of a K-almost affine space M is called the *associated* (or *tangent*) *almost linear space* of M.

Let M be a K-almost affine space and TM its associated almost linear space. We denote by zTM the *maximal* zero-almost subspace of TM, and by uTM the *maximal* unitary almost subspace of TM. Then :

Theorem 1.1. (Miron[9]). i) *In TM there exists a unique* $zTM = \{x | x \in TM : 1 * x = \vec{0}\}$.

ii) *In TM there exists a unique* $uTM = \{x | x \in TM : \exists y \in TM,\ x = 1 * y\}$.

iii) *zTM is a normal almost subspace of TM.*

iv) *uTM is a K-linear space.*

v) *The quotient K-almost linear space TM/zTM is a K-linear space isomorphic to uTM.*

vi) $zTM \cap uTM = \{\vec{0}\}$.

vii) *Any* $x \in TM$ *is uniquely written in the form :* $x = x_0 + x_1$, $x_0 \in zTM$, $x_1 \in uTM$.

Theorem 1.2. *TM is the direct sum* $TM = zTM + uTM$ *if and only if uTM is a normal almost linear subspace of TM.*

As an immediate corrollary we have :

Theorem 1.3. (Liebeck [6]). *If the group* $(TM, +)$ *is Abelian, then the sum* $TM = zTM + uTM$ *is direct.*

The following theorem is fundamental, giving a structural characterization of K-almost affine spaces :

Theorem 1.4. (*Radó* [13]). *Let zTM be a K-zero almost linear space and uTM be a K-linear space. Then* $\forall \theta \in \mathrm{Hom}(uTM, \mathrm{Aut}(zTM))$, *there exists the semidirect sum* $zTM +_\theta uTM = TM$, *given by the set* $\{(x, y) | x \in zTM, y \in uTM\}$; *the vector operations are :* $(x, y) + (x', y') = (x + \theta(y)(x'), y + y')$, $\alpha * (x, y) = (\vec{0}, \alpha * y)$. *Defining* $i_1 : zTM \to TM$ *and* $i_2 : uTM \to TM$ *by* $i_1(x) = (x, \vec{0})$, *and* $i_2(y) = (\vec{0}, y)$, *the maximal zero-almost linear subspace of TM is* $i_1(zTM)$, *and the maximal unitary almost linear subspace of TM is* $i_2(uTM)$. *Up to K-isomorphisms, every K-almost linear space can be obtained in this way.*

On the basis of this last theorem, algebraic models for the structure of the K-almost affine space can be obtained.

The notion of almost affine *subspace* is introduced naturally in the following way : let M be a K-almost affine space and M^* be a subset of M ; consider the set of vectors $TM^* = \{\overrightarrow{AB} \in TM : A, B \in M^*\}$.

Definition 1.4. *The subset* M^* *of* M *is called an almost affine space, if the following conditions are satisfied :*

a) The restriction of the mapping $*$ *to* $K \times TM^*$ *is valued in* TM^* ;

b) For the pair (M^*, TM^*) *both groups of axioms are verified.*

Proposition 1.2. *There exists a unique almost affine subspace* M^* *of M, which contains a fixed point* $A_0 \in M$ *and has a given associated almost linear space* $TM^* \subset TM$ *(*TM^* *is called the director almost linear subspace of* M^**).*

Definition 1.5. *We say that the almost affine subspace* $M_1 \neq \varnothing$ *is parallel to the almost affine subspace* M_2 *if* $TM_1 \subset TM_2$, *and we denote this by* $M_1 \| M_2$.

The relation $\|$ is reflexive and transitive. Let $M_2 \neq \varnothing$ be an almost affine subspace of M and $TM_1 \subset TM_2$. Then, through each point $A \in M$ passes a unique almost affine subspace M_1, parallel to M_2, having TM_1 as director subspace.

Definition 1.6. *A family $\mathcal{F} = (M_i)_{i \in I}$ of almost affine subspaces of M is called a foliation of M if:*

a) $\forall i \in I,\ M_i \neq \varnothing$.

b) $\forall A \in M, \exists i_0 \in I : A \in M_{i_0}$.

c) $\forall i,\ j \in I,\ M_i \| M_j$.

An element $M_i \in \mathcal{F}$ is called a leaf of the foliation $\mathcal{F}$.

Proposition 1.3. *There exists a single foliation $\mathcal{F}$ in M with the property that all its leaves have the same director subspace $TM' \subset TM$.*

The two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$, having as director subspaces zTM and uTM are uniquely determined and play an important role in the study of the geometrical structure of the almost affine spaces, as it is obvious from the following fundamental theorems.

Theorem 1.5. (*First fundamental theorem*). *Every K-almost affine space M is generated by the set of points $M_0 \cup M_1$, where $M_0 \in \mathcal{F}_0$, $M_1 \in \mathcal{F}_1$.*

Theorem 1.6. (*Second fundamental theorem*). *The quotient almost affine space $M/\mathcal{F}_0$ is isomorphic to a fixed leaf $M_1 \in \mathcal{F}_1$, and M_1 is a K-affine space.*

Theorem 1.7. (*Third fundamental theorem*). *The K-almost affine space M is the direct sum of a fixed leaf $M_0 \in \mathcal{F}_0$ with a fixed leaf $M_1 \in \mathcal{F}_1$ if and only if $\mathcal{F}_1$ is a normal foliation (that is, uTM is a normal subspace of TM).*

2. The Real Three Dimensional Almost Affine Space. In this section we shall introduce an almost affine structure, starting from a three-dimensional projective space P_3. The construction given below can be extended to projective spaces P_n of any dimension $n \geqslant 3$, and to projective Desarguesian planes. The main idea of this construction is to choose two distinct planes of P_3 as the absolute of the almost affine space to be defined.

Let π, π_0 be two distinct planes of P_3 and d_0 be their common line. It is well known that excluding from P_3 any plane, together with all its points and lines, we obtain the real three-dimensional (Euclidean) affine space P_3/π, which is isomorphic to R^3, having as absolute points (or points at the infinity) the points of the plane π.

We shall consider now the set $A_3 = P_3 \setminus \{\pi \cup \pi_0\}$. The set $\pi \cup \pi_0$ will be called the *absolute* of A_3, its points will be the *absolute points*, and its lines, the *absolute lines* of A_3. The line d_0 will be the line of *double* absolute points.

Let A, $B \in A_3$, $A \neq B$. We shall denote, as usually, by AB the line (uniquely defined in P_3) containing the two points, from which we exclude the absolute points; this will be a *line* of A_3. The following classification of lines in A_3 is obvious:

a) lines which, in P_3, have a double absolute point; they are the lines coplanar with d_0 in P_3. In A_3, we shall call them *z-lines*.

b) the other lines, named *u-lines* in A_3; they are the lines which, in P_3, contain two distinct absolute points (the lines skew to d_0).

The notion of collinearity of the points in A_3 coincides with the collinearity of the same points in P_3.

Let A, B, $C \in A_3$ be three distinct non-collinear points. We denote by ABC the plane of P_3, passing through these points, uniquely determined by them, from which we exclude the absolute lines; this will be a *plane* in A_3. We have two natural classes of planes:

a) *z-planes*, that is, planes which, in P_3, contain d_0. These are the planes whose all lines are z-lines.

b) all the other planes, called *u-planes*; these are the planes which, in P_3, contain two distinct absolute lines, and a single double absolute point.

The notion of coplanarity of the points and lines in A_3 coincides with the coplanarity of the same elements, considered as elements of P_3.

2.1. The First Group of Axioms; Torsion; Translations. In order to introduce an almost affine structure on A_3, we need first (see Definition 1.1), an equivalence relation on the set $A_3 \times A_3$ of the *oriented segments* of A_3. The following two remarks will help us to introduce such an equivalence.

First, let us consider the affine space P_3/π, and the equivalence „$\sim$" of its segments defining the affine vectors (see fig. 1).

Let $\overline{AB} \sim \overline{CD}$ since in an affine space the parallelogram (Raschewsky's) axiom is fulfilled, we have $\overline{AB} \sim \overline{CD} \Leftrightarrow \overline{AC} \sim \overline{BD}$. For the corresponding lines in P_3 this means $AB \cap CD \in \pi$, $AC \cap BD \in \pi$. We call the configuration in fig. 1 a π-parallelogram, and the following result is obvious:

Proposition 2.1. *Two oriented segments are equivalent in the affine sense if and only if they are opposite sides in a π-parallelogram.*

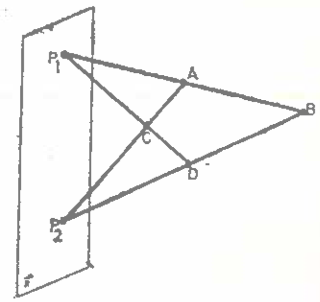

Fig. 1

Second, it is well known a remarkable result of the projective geometry: any projective n-space ($n \geqslant 3$) is Desarguesian, and so are all its subspaces, including *planes*. So, the study of the projective spaces of dimension greater than 2 does not lead to geometrical properties which are fundamentally distinct from the properties of the Desarguesian planes. On the contrary, for non-Desarguesian planes we can obtain very different properties.

If we look for a geometrical configuration defining the almost affine equivalence of the oriented segments (analogous to the π-parallelogram in the affine case), we must notice that Raschewsky's axiom is replaced (see definition 1.1) by the triangle axiom. In this axiom, we are concerned with a pair of triangles, so we are directed to consider a special type of Desargues configuration, which is a natural generalization of the „small Desargues theorem" of the affine case. In fig. 2 we have $AA' \| BB' \| CC'$, $AB \| A'B'$, and $BC \| B'C'$ (where „$\|$" is the affine parallelism). It follows, by the small Desargues theorem, that $AC \| A'C'$.

Definition 2.1. *Let A, B, C, A', B', $C' \in A_3$ be six points, so that the triples (ABC), $(A'B'C')$ form two proper distinct triangles. We say that the six points form a (π_0, π)-Desargues configuration, if in P_3 we have: $AA' \cap BB' \cap CC' = P_0 \in \pi_0$, and $AB \cap A'B' \cap \pi = P_3$, $BC \cap B'C' \cap \pi = P_1$, $CA \cap$*

$\cap C'A'\cap\pi=P_2$, *where* P_1, P_2, P_3 *are three collinear points on* π. *We denote this relation by* $(ABC)\Delta(A'B'C')$ (*see fig.* 3).

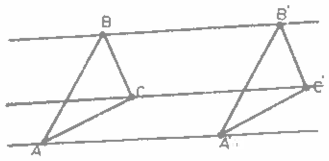

Fig. 2

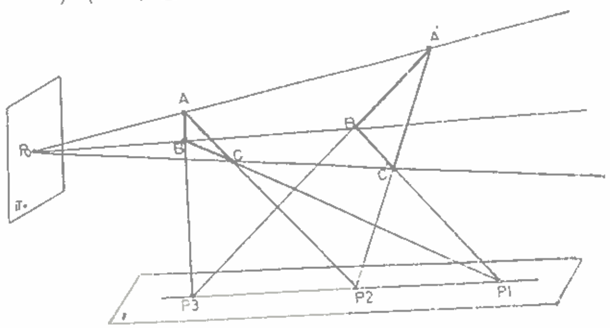

Fig. 3

If we take by definition $(ABC)\Delta(ABC)$, we have the following significant result:

Proposition 2.2. *The relation „Δ" is an equivalence relation on the set of proper triangles of* A_3.

Proof. The reflexivity and the symmetry result straightforward from the definition. In order to prove the transitivity, let us consider $(ABC)\Delta$ $\Delta(A'B'C')$ and $(A'B'C')\Delta(A''B''C'')$ — see fig. 4. We have $AB\cap A'B'\cap$

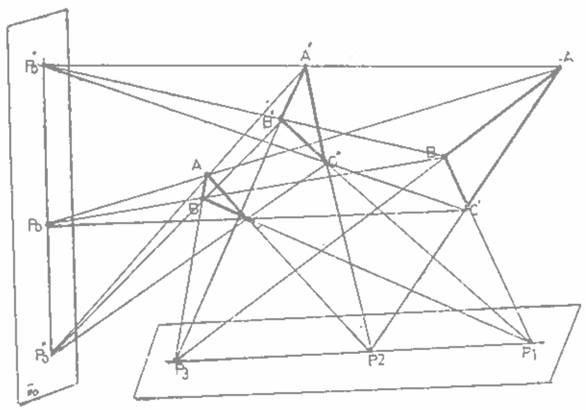

Fig. 4

$\cap A''B''=P_3\in d$, $BC\cap B'C'\cap B''C''=P_1\in d$, and $CA\cap C'A'\cap C''A''=P_2\in$ $\in d$, where the line d belongs to the plane π. We also have $AA'\cap BB'\cap CC'=$ $=P_0\in\pi_0$, and $A'A'\cap B'B''\cap C'C''=P_0'\in\pi_0$. If we consider now the triangles $(AA'A'')$, $(BB'B'')$, and $(CC'C'')$, the conditions of Desargues' theorem are satisfied:

— for $(AA'A'')$ and $(BB'B'')$: $AB\cap A'B'\cap A''B''=P_3$, $AA'\cap BB'=$ $=P_0\in\pi_0$, and $A'A''\cap B'B''=P_0'\in\pi_0$; therefore, $AA''\cap BB''=P_0''\in\pi_0$, with the points P_0, P_0', P_0'' collinear in π_0.

— for $(AA'A'')$ and $(CC'C'')$, it follows in a similar manner that $AA''\cap CC''=P''\in\pi_0$, with the points P_0, P_0', P'' collinear in π_0.

But $P_0''=AA''\cap\pi_0=P''$, and the proof is completed.

Definition 2.2. *Let* A, B, A', $B'\in A_3$ *be four points with* $A\neq B$, $A'\neq B'$. *We say that the oriented segments* $\overline{AB}$, $\overline{A'B'}\in A_3\times A_3$ *are in the relation „δ", if* $\forall C\in A_3$, $C\in AB$, $\exists C'\in A_3$ *so that* $(ABC)\Delta(A'B'C')$. *By definition, we take* $\overline{AA}\ \delta\ \overline{BB}$, $\forall A$, $B\in A_3$.

Proposition 2.3. *The relation* δ *is an equivalence.*

Proof. The reflexivity and the symmetry are immediate. In order to prove the transitivity, see fig. 4. Let us consider $\overline{AB}\,\delta\overline{A'B'}$ and $\overline{A'B'}\,\delta$ $\overline{A''B''}$. If two of the given segments (or all three) are identical, the property $\overline{AB}\delta\overline{A''B''}$ is obvious; else, let $C\in A_3$ be any point not collinear with A, $B\Rightarrow$ $\Rightarrow\exists C'\in A_3$, so that $(ABC)\Delta(A'B'C')$. The point C' is obviously not collinear with $A'B'$, hence $\exists C''\in A_3$, so that $(A'B'C')\Delta(A''B''C'')$. But the relation Δ is transitive, which means $(ABC)\Delta(A''B''C'')$, hence $\overline{AB}\delta\overline{A''B''}$ q.e.d.

We denote now $TA_3=(A_3\times A_3)/\delta$ and let $p: A_3\times A_3\rightarrow TA_3$ be the natural projection. We denote $p(\overline{AB})=\overrightarrow{AB}$ and say that $\overrightarrow{AB}$ is the *vector* determined by the oriented segment $\overline{AB}$ with respect to the equivalence δ.

Obviously: $\overline{AB}\delta\overline{CD}\Leftrightarrow\overrightarrow{AB}=\overrightarrow{CD}$.

Theorem 2.1. *For the set of points* A_3 *and the set of vectors* TA_3, *the first group of axioms of the structure of almost affine space (see definition* 1.1) *is satisfied.*

Proof. — Axiom I_1. Let $A\in A_3$ and $\overrightarrow{A'B'}=x$ (see fig. 4); we denote $P_0=AA'\cap\pi$, and $P_3=A'B'\cap\pi$. The lines P_0B' and AP_3 are coplanar, so they meet in a point B. Let us prove that $\overline{AB}\delta\overline{A'B'}$. We consider $C\in A_3$, $C\notin AB$. We denote $P_2=AC\cap\pi$ and $P_1=BC\cap\pi$; the points P_1, P_2, P_3 are collinear, belonging to the intersection of the planes π and ABC. Then, if $C'=A'P_2\cap P_0C$, by the Desargues' theorem, we have $BC\cap B'C'=P_3$, and this means $(ABC)\Delta(A'B'C')$, q.e.d.

— Axiom I_2. Let A, B, $A'\in A_3$, with $\overrightarrow{AB}=\overrightarrow{A'B}$. By definition, $\overline{AB}\delta$ $\overline{A'B}\Rightarrow\forall C\in A_3$, $C\notin AB$, $\exists C'\in A_3$: $(ABC)\Delta(A'BC')$. But homologous sides in Δ-equivalent triangles meet on π (or are collinear). If A, A', B were not collinear, it would follow $AB\cap A'B=B\in\pi$, which is absurd, because $B\in A_3$. Hence A, B, A' are collinear. In the same manner, from $\overline{BC}\delta\overline{BC'}$, we deduce that B, C, C' are collinear. Now, if $A\neq A'$, it follows $AA'\cap CC'=$ $=B\in\pi_0$, which is impossible, because $B\in A_3$. Hence $A=A'$.

— Axiom I_3. Let us consider A, B, C, A', B', $C'\in A_3$, satisfying $\overrightarrow{AB}=$ $=\overrightarrow{A'B'}$, $\overrightarrow{BC}=\overrightarrow{B'C'}$. If the points A, B, C are not collinear, $\overline{AB}\delta\overline{A'B'}$, $\overline{BC}\delta$ $\overline{B'C'}\Rightarrow(ABC)\Delta(A'B'C')\Rightarrow\overline{AC}\delta\overline{A'C'}\Rightarrow\overrightarrow{AC}=\overrightarrow{A'C'}$. If $C\in AB$, let us take $D\in$ $\in A_3$, $D\in AB\Rightarrow\exists D'$, with $(ABD)\Delta(A'B'D')$. This implies $\overline{BD}\delta\overline{B'D'}$; but we have, also $\overline{BC}\delta\overline{B'C'}$ therefore we have $(BCD)\Delta(B'C'D')\Rightarrow\overline{CD}\delta\overline{C'D'}$;

together with $\overrightarrow{AD}\delta\overrightarrow{A'D'}$, the last relation implies $(ACD)\Delta(A'C'D')\Rightarrow\overrightarrow{AC}\delta\overrightarrow{A'C'}\Rightarrow\overrightarrow{AC}=\overrightarrow{A'C'}$, q.e.d.

Proposition 2.4. *The additive group of the vectors of the set A_3 is not Abelian; the subgroup of the vectors of any z-plane is Abelian.*

Proof. Let $A\in A_3$ and d_B, d_C two distinct u-lines through A. Let $B\in d_B$, $B\neq A$, and $C\in d_C$, $C\neq A$ (see fig. 5). We denote $P_0=AC\cap\pi_0$ and $P=AB\cap\pi$. Let $D=BP_0\cap CP$; it follows that $\overrightarrow{AB}\delta\overrightarrow{CD}$, but it is clear that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are not in the relation δ. If we denote $\overrightarrow{AB}=x$, $\overrightarrow{CA}=y$, we have $\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AB}+\overrightarrow{BD}$, in other words, $y+x=x+\overrightarrow{BD}$, with $\overrightarrow{BD}\neq y$. The second part of the proposition is immediate; we must remember that the z-planes are those planes containing in P_3 the line d_0 of the double absolute points. It is easy to see that the set of the vectors of a z-plane is a subgroup of $(TA_3, +)$. Let π_z be any z-plane; repeating the above construction, we obtain P, $P_0\in d_0$, hence the configuration so obtained is a π-parallelogram; this means that in π_z the relation δ coincides with the affine equivalence $\sim$ of the oriented segments, hence the parallelogram axiom is satisfied. By the proposition given in the first section, stating that for K-Weylean almost affine spaces, the group $(TM, +)$ is Abelian, this implies the commutativity of the addition of vectors.

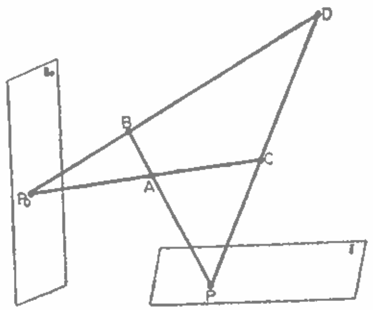

Fig.5

Proposition 2.5. *If $x=\overrightarrow{AB}=\overrightarrow{CD}$, $y=\overrightarrow{AC}=\overrightarrow{BD'}$, then $\tilde{t}(x, y)=\overrightarrow{D'D}$.*

Proof. $\overrightarrow{AD}=\overrightarrow{AC}+\overrightarrow{CD}=y+x$, $\overrightarrow{AD'}=\overrightarrow{AB}+\overrightarrow{BD'}=x+y\Rightarrow\overrightarrow{D'D}=\overrightarrow{D'A}+\overrightarrow{AD}=-\overrightarrow{AD'}+\overrightarrow{AD}=-(x+y)+(y+x)$ (see definition 1.3).

Proposition 2.6. *A subgroup TA_3' of the group $(TA_3, +)$ is Abelian if and only if its torsion vanishes, i.e. $\tilde{t}(x, y)=\vec{0}$, $\forall x, y\in TA_3'$.*

Definition 2.3. *The mapping $t_x: A_3\to A_3$, for which $t_x(A)=A'$, $x=\overrightarrow{AA'}\in TA_3$, is called vector x translation.*

The following results hold in any structure which satisfies the first group of axioms (see [9]):

Proposition 2.7. *Let $\mathcal{T}$ be the set of the translations of A_3; then:*

i) Every $t_x\in\mathcal{T}$ is a bijection.

ii) $\forall t_x, t_y\in\mathcal{T}: t_x(A)=A\Leftrightarrow x=\vec{0}; t_x(A)=A', t_y(A')=A$, $\forall A\in A_3\Leftrightarrow x=-y$

iii) Every $t_x\in\mathcal{T}$ is well determined by a fixed point $A_0\in A_3$ and by its image $A_0'=t_x(A_0)$.

iv) $t_y o t_x=t_{x+y}$; $t_x^{-1}=t_{-x}$, $\forall t_x, t_y\in\mathcal{T}$.

Theorem 2.2. *$(\mathcal{T}, o)$ is a group isomorphic to $(TA_3, +)$.*

Propositions 2.8. *$\overrightarrow{AA'}\delta\overrightarrow{BB'}\Leftrightarrow\overrightarrow{t_x(A)t_x(A')}\,\delta\,\overrightarrow{t_x(B)t_x(B')}\,\forall t_x\in\mathcal{T}$.*

Based on this property, we can define the mapping $Tt_x: TA_3\to TA_3$, given by $Tt_x(\overrightarrow{AA'})=\overrightarrow{t_x(A)t_x(A')}$, $\forall A\in A_3$, $\forall\overrightarrow{AA'}\in TA_3$.

Proposition 2.9. *$Tt_x(y+z)=Tt_x(y)+Tt_x(z)$, $\forall t_x\in\mathcal{T}$, $\forall y, z\in TA_3$.*

Proposition 2.10. *Let TA_3' be a subgroup of TA_3. Then $Tt_x(y)=y$, $\forall x, y\in TA_3$ if and only if $\tilde{t}(x, y)=\vec{0}$ (that is, the subgroup TA_3' is Abelian.*

Corollary 2.1. *On z-planes, the conditions of the above proposition are satisfied.*

§2.2. **The Second Group of Axioms.** We need now a multiplicative operation with real scalars of the vectors defined as in the above paragraph, such that the second group of axioms is satisfied. To this end, let us consider d, a fixed u-line, arbitrarily chosen in A_3. We denote $P_0=\pi_0\cap d$ in the affine space P_3/π. On the affine line $d\cup\{P_0\}$, we choose a point $P_1\neq P_0$; the pair (P_0, P_1) is called an affine coordinate system on the given line and, by a classical construction, we can associate uniquely to every point P_x of the line a real number, x, called the affine coordinate of the point.

If we consider now four points, A, B, C, $D\in d$, having as affine coordinates, respectively, the real numbers a, b, c, and d, distinct from zero, we have $\overrightarrow{AB}\delta\overrightarrow{CD}\Leftrightarrow b/a=d/c$; hence, we can associate uniquely to an oriented segment of the line d an „almost affine measure", equal to the quotient between the affine coordinate of its extremity and the affine coordinate of its origin. Then, to the almost affine vector equality $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ will correspond the relation $b/a.c/b=c/a$ between the almost affine measures of the considered vectors. Therefore:

Theorem 2.3. *The mapping $\varphi: Td\to R/\{O\}$, from the additive group of the almost affine vectors of the line d, to the multiplicative group of the real numbers distinct from zero, defined by $\varphi(\overrightarrow{AB})=b/a$, where a, b, are the affine coordinates of the points A, B, is an isomorphism of groups.*

It is clear that $\varphi(\vec{0})=1$, and $\varphi(-x)=1/\varphi(x)$. Let us notice that the almost affine vectors of d belong to two distinct classes:

— vectors with $\varphi(x)>0$; if $x=\overrightarrow{AB}$ is such a vector, then, on the affine line $d\cup\{P_0\}$, the point P_0 is not between A and B.

— vectors with $\varphi(x)<0$; if $x=\overrightarrow{AB}$ is such a vector, then the affine segment $\overline{AB}$ contains P_0. For such a segment, if M is a point getting over $\overline{AB}$, from A towards B, then $\varphi(\overrightarrow{MB})$ decreases from $\varphi(\overrightarrow{AB})$ to $-\infty$ when M gets near to P_0, and decreases from $+\infty$ to 1, when M moves off P_0.

Let us consider now the mapping $e: R\times Td\to Td$, defined by $e(\alpha, x)=\varphi^{-1}(\mathrm{sign}(\varphi(x)).(|\varphi(x)|)^{\alpha})$, $\forall\alpha\in R$, $\forall x\in Td$.

Proposition 2.11. *The group Td, together with the mapping e, form a structure verifying the axioms of the structure of vector space over R, except the distributivity of the external multiplication with respect to the addition of scalars. If we restrict ourselves to a P_0-half-line of d, we obtain a R-vector space.*

Proof. 1. $\forall\alpha\in R,\ \forall x, y\in Td$: $e(\alpha, x+y)=\varphi^{-1}(\text{sign}\,(\varphi(x+y)).(|\varphi(x+$ $+y)|)^{\alpha})=\varphi^{-1}(\text{sign}\,(\varphi(x)\varphi(y))\,.\,(|\varphi(x)\varphi(y)|)^{\alpha})=\varphi^{-1}(\text{sign}\,(\varphi(x))\,.\,(|\varphi(x)|^{\alpha})\,.$ $.\,\text{sign}(\varphi(y))\,.\,(|\varphi(y)|)^{\alpha})=\varphi^{-1}(\text{sign}(\varphi(x))\,.\,(|\varphi(x)|)^{\alpha})+\varphi^{-1}(\text{sign}(\varphi(y))\,.\,(|\varphi(y)|)^{\alpha})=$ $=e(\alpha, x)+e(\alpha, y)$.

2. Let $\alpha, \beta\in R,\ x\in Td$; then $e(\alpha+\beta, x)=\varphi^{-1}(\text{sign}\,(\varphi(x))\,.\,(|\varphi(x)|)^{\alpha+\beta})=$ $=\varphi^{-1}(\text{sign}\,(\varphi(x))\,.\,(|\varphi(x)|)^{\alpha}\,.\,(|\varphi(x)|)^{\beta})\neq\varphi^{-1}(\text{sign}\,(\varphi(x))\,.\,(|\varphi(x)|)^{\alpha}\,.\,\text{sign}\,(\varphi(x)).$ $.\,(|\varphi(x)|)^{\beta})=\varphi^{-1}(\text{sign}\,(\varphi(x))\,.\,(|\varphi(x)|)^{\alpha})+\varphi^{-1}(\text{sign}\,(\varphi(x))\,.\,(|\varphi(x)|)^{\alpha})=e(\alpha, x)+$ $+e(\beta, x)$. This axiom is not satisfied for those $x\in Td$ with $\varphi(x)<0$. If we restrict ourselves to a P_0-half-line of d (e.g., the half-line containing P_1, denoted by d^+), the axiom is obviously verified ; let us notice that Td^+ is a subgroup of Td.

3. $\forall\alpha,\ \beta\in R,\ \forall x\in Td$, we have : $e(\alpha, e(\beta, x))=e(\alpha, \varphi^{-1}(\text{sign}\,(\varphi(x))\,.$ $(|\varphi(x)|)^{\beta}))=\varphi^{-1}(\text{sign}(\varphi^{-1}(\text{sign}(\varphi(x)).(|\varphi(x)|)^{\beta})).(|\varphi\varphi^{-1}(\text{sign}(\varphi(x))\,.\,(|\varphi(x)|)^{\beta})|^{\alpha})=$ $=\varphi^{-1}(\text{sign}(\varphi(x))\,.\,((|\varphi(x)|)^{\beta})^{\alpha})=\varphi^{-1}(\text{sign}(\varphi(x))\,.\,(|\varphi(x)|)^{\alpha\beta})=e(\alpha\beta, x)$.

4. $e(1, x)=\varphi^{-1}(\text{sign}(\varphi(x))\,.\,|\varphi(x)|)=\alpha^{-1}\varphi(x)=x$, q.e.d.

This last result leads us to the necessity of an important restriction in the construction of the desired R-almost affine structure over A_3. Indeed let us notice that in the considered set of vectors Td, there are vectors such that $x=-x$ (these are the vectors having as representatives the segments $\overline{AB}$, with the points A, B symmetrical relative to the affine origin P_0) ; but such a property holds in a vector space, if and only if this space is constructed over a field which is an extension of Z_2. We wish to introduce a real almost affine structure, but R is not an extension of Z_2. Hence, we must restrict our construction to the „positive" half-line considered above, and, consequently, to the „positive" half-space A_3^+, determined by d^+. We shall see that A_3^+ will have a structure of R-almost affine space, and on the whole set A_3, all the axioms will be verified, except II_2.

Definition 2.4. *Let $A\in A_3^+$. We call the projection of the point A on the u-line d, denoted by* $\text{pr}_d(A)$, *the point in which the z-plane through A cuts d.*

Definition 2.5. *We call the projection of a vector $x=\overrightarrow{AB}\in TA_3^+$, denoted by* $\text{pr}_d(x)$, *the vector* $\overrightarrow{\text{pr}_d(A)\text{pr}_d(B)}\in Td^+$ *(see fig. 6).*

In fig. 6, we use the following notations : let π_A, π_B be the z-planes through A, B ; we denote, as usually, $P_0=d\cap\pi_0$, $p=d\cap\pi$. Then $\text{pr}_d(A)=\pi_A\cap d$; $\text{pr}_d(B)=\pi_B\cap d$; $A_0=PA\cap\pi_B$, $B_0=PB\cap\pi_A$, and we have $\overrightarrow{AA_0}=\overrightarrow{B_0B}=$ $=\overrightarrow{\text{pr}_d(A)\text{pr}_d(B)}=\text{pr}_d(\overrightarrow{AB})$.

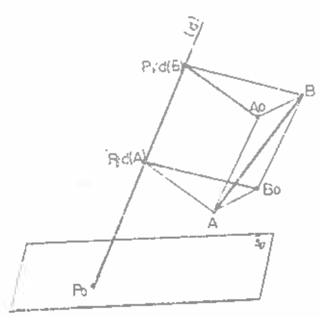

Fig 6

Definition 2.6. *Let $x\in TA_3^+$, $\alpha\in R$; then, the external multiplication* $*:(\alpha, x)\in R\times TA_3^+\to$ $\to\alpha * x\in TA_3^+$ *is defined by* $*(\alpha, x)=\alpha * x=e(\alpha,$ $\text{pr}_d(x))$, *where the mapping e is defined in prop. 2.11.*

Theorem 2.4. *On the set $(TA_3^+, +, *)$, the axioms of the second group of axioms for the structure of R-almost affine space are verified.*

Proof. — Axiom II_1. $\alpha*(x+y)=e(\alpha, \text{pr}_d(x+y))=e(\alpha, \text{pr}_d(x)+\text{pr}_d(y))=$ $=e(\alpha, \text{pr}_d(x))+e(\alpha, \text{pr}_d(y))=\alpha*x+\alpha*y$.

— Axiom II_2. $(\alpha+\beta)*x=e(\alpha+\beta, \text{pr}_d(x))=e(\alpha, \text{pr}_d(x))+e(\beta, \text{pr}_d(x))=$ $=\alpha*x+\beta*x$, because $\varphi(x)>0$.

— Axiom II_3. $\alpha*(\beta*x)=e(\alpha, \text{pr}_d(\beta*x))=e(\alpha, \text{pr}_d(e(\beta, \text{pr}_d(x))))=e(\alpha, e(\beta,$ $\text{pr}_d(x)))=e(\alpha\beta, \text{pr}_d(x))=(\alpha\beta)*x$.

Let us notice that the axiom $1*x=x$ is not verified, because, generally $x\neq\text{pr}_d(x)$.

Now, our three-dimensional model for the structure of the R-almost affine space is completely described. Let us notice that zTA_3^+ is the set of the vectors of an arbitrary z-plane, and uTA_3^+ is the set Td^+ of the vectors of the half-line d^+ considered above. The foliation $\mathcal{F}_0$ is the pencil of all the z-planes of A_3^+, and the foliation $\mathcal{F}_1$ is the pencil of all the u-lines of A_3^+, parallel to the given line d.

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